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Absence of embedded eigenvalues for the Schrödinger operator on manifold

By

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Abstract

We discuss the absence of eigenvalues above some critical energy for the Schrödinger operator on a manifold with asymptotically Euclidean and/or hyperbolic ends. The critical energy can be computed only by the geometry of the ends. The main ingredients of proof are the super-exponential decay estimates for eigenfunctions and the absence of super-exponentially decaying eigenfunctions.

§ 1. Main result

This article is based on authors' recent work [IS]. Let (M, g) be a non-compact connected Riemannian manifold of dimension $d \geq 1$. We discuss the absence of eigenvalues above some constant E_0 for the Schrödinger operator H on $\mathcal{H} = L^2(M)$:

$$H = H_0 + V; \quad H_0 = -\tfrac{1}{2}\triangle = \tfrac{1}{2}p_i^* g^{ij} p_j, \quad p_i = -\mathrm{i}\partial_i.$$

We impose the four conditions listed below.

We will denote, in local coordinates, for $r \in C^1(M)$ and $f \in C^1(M)$

$$\partial^r f = (\partial_i r) g^{ij} (\partial_j f),$$

and for $f \in C^2(M)$

$$(\nabla^2 f)_{ij} = \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f; \quad \Gamma^k_{ij} = \tfrac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}).$$

These differential operators ∂^r and ∇^2 are the gradient vector field for r and the geometric Hessian, respectively.

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Condition 1.1. There exist an unbounded real-valued function $r \in C^4(M)$, $r(x) \geq 1$, constants $c_1 > c_2 > 0$ and a decomposition $\triangle r^2 = \rho_1 + \rho_2 + \rho_3$ such that:

1. There exists a constant $r_0 \geq 1$ such that, as quadratic forms on TM,

$$\nabla^2 r^2 \geq (c_1 + \tfrac{1}{2}\rho_1)g \text{ and } \rho_1 \geq 0 \text{ for } r \geq r_0. \tag{1.1}$$

   Moreover,

$$\liminf_{r\to\infty}(r\partial^r|\mathrm{d}r|^2 + (c_2 + \tfrac{1}{2}\rho_1)|\mathrm{d}r|^2) > 0, \quad \limsup_{r\to\infty}|\mathrm{d}r| < \infty. \tag{1.2}$$

2. The following bounds hold

$$\limsup_{r\to\infty}|r^{-1}\triangle r^2| < \infty, \tag{1.3}$$

$$\limsup_{r\to\infty}\rho_1 < \infty, \quad \limsup_{r\to\infty}|\mathrm{d}\rho_2| < \infty, \quad \limsup_{r\to\infty}\triangle\rho_3 < \infty. \tag{1.4}$$

The inequality (1.1) implies the convexity $\nabla^2 r^2 \geq c_1 g > 0$ for $r \geq r_0$ and guarantees the existence of "expanding end" with lower growth rate $c_1 > 0$. Due to the equality

$$(\nabla^2 r^2)^{ij}(\partial_i r)(\partial_j r) = 2|\mathrm{d}r|^4 + 2r(\nabla^2 r)^{ij}(\partial_i r)(\partial_j r) = 2|\mathrm{d}r|^4 + r\partial^r|\mathrm{d}r|^2,$$

(1.2) imposes a further lower bound for the $\mathrm{d}r \otimes \mathrm{d}r$ component of $\nabla^2 r^2$. On the other hand, combining $\triangle = \operatorname{tr}\nabla^2$ and the positivity of $\nabla^2 r^2$, we can think of (1.3) and (1.4) as upper bounds for the growth rate.

Condition 1.2. There exists a decomposition $V = V_1 + V_2$, $V_1 \in L^2_{\mathrm{loc}}(M)$, $V_2 \in C^1(M)$, such that V_1 and V_2 are real-valued and

$$\limsup_{r\to\infty}|V| < \infty, \quad \limsup_{r\to\infty} r|V_1| < \infty, \quad \limsup_{r\to\infty} r\partial^r V_2 < \infty. \tag{1.5}$$

Under Condition 1.2 the Schrödinger operator H is defined at least on $C_c^\infty(M)$. However it is not necessarily essentially self-adjoint, since (M, g) is allowed to be incomplete and that V is allowed to be unbounded. To fix a self-adjoint extension we choose a non-negative $\chi \in C^\infty(\mathbb{R})$ with

$$\chi(r) = \begin{cases} 0 & \text{for } r \leq 1, \\ 1 & \text{for } r \geq 2, \end{cases}$$

and then set

$$\chi_\nu(r) = \chi(r/\nu), \quad \nu \geq 1.$$

We consider the function χ_ν as being composed with the function r from Condition 1.1.

Condition 1.3. The operator H defined on $C_{\mathrm{c}}^{\infty}(M)$ has a self-adjoint extension, denoted by H again, such that for any $\psi \in \mathcal{D}(H)$ there exists a sequence $\psi_n \in C_{\mathrm{c}}^{\infty}(M)$ such that for all large $\nu \geq 1$

$$\|\chi_\nu(\psi - \psi_n)\| + \|\chi_\nu(H\psi - H\psi_n)\| \to 0 \quad \text{as } n \to \infty.$$

Finally we impose for this self-adjoint extension the *unique continuation property*:

Condition 1.4. If $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$ for some $E \in \mathbb{R}$, and $\phi(x) = 0$ in some open subset, then $\phi(x) = 0$ in M.

In Section 2 we shall give some criteria for Conditions 1.1–1.4.

Theorem 1.5. Suppose Conditions 1.1–1.4, and define $E_0 \in \mathbb{R}$ by

$$E_0 = \inf_{c \in (0, c_1 - c_2]} \limsup_{r \to \infty} \Big(V + \tfrac{|\beta|^2 - c\gamma}{2c\alpha_c} \Big),$$

where

$$\alpha_c = c_1 - c + \tfrac{1}{2}\rho_1, \quad \beta = \tfrac{1}{4}\mathrm{d}\rho_2 + V_1 \mathrm{d}r^2, \quad \gamma = -\tfrac{1}{4}\triangle\rho_3 + (\triangle r^2)V_1 - 2r\partial^r V_2. \tag{1.6}$$

Then the eigenvalues of H are absent above E_0, i.e., $\sigma_{\mathrm{pp}}(H) \cap (E_0, \infty) = \emptyset$.

The proof of Theorem 1.5, the detail of which we omit in this article, follows the scheme of [FHH2O, FH, DG, MS]. The proof employs, in particular, a Mourre-type commutator estimate with respect to the "conjugate operator"

$$A = \mathrm{i}[H_0, r^2] = \tfrac{1}{2}\{(\partial_i r^2) g^{ij} p_j + p_i^* g^{ij} (\partial_j r^2)\} = r p^r + (p^r)^* r; \quad p^r = -\mathrm{i}\partial^r,$$

where the function r is that of Condition 1.1. Here we only note that the quantities in (1.6) indeed appear in the Mourre-type commutator computation:

Lemma 1.6. As a quadratic form on $C_{\mathrm{c}}^{\infty}(M)$,

$$\mathrm{i}[H, A] = p_i^* (\nabla^2 r^2 - \alpha_c g)^{ij} p_j + 2\,\mathrm{Re}(\alpha_c H_0) - 2\,\mathrm{Im}(\beta^i p_i) + \gamma,$$

where α_c, β, γ are defined in (1.6).

Besides Theorem 1.5 itself, we also generalize [VW], see Section 3.

§ 2. Examples satisfying Conditions 1.1–1.4

§ 2.1. Global conditions

The following criterion regarding the essential self-adjointness for H is well-known:

Proposition 2.1. Let (M, g) be a complete Riemannian manifold of dimension $d \geq 1$. Then the free Schrödinger operator H_0 is essentially self-adjoint on $C_{\mathrm{c}}^\infty(M)$. Suppose V is real-valued, measurable, bounded outside a compact set and in addition: $V \in L^2_{\mathrm{loc}}(M)$ for $d = 1, 2, 3$, $V \in L^p_{\mathrm{loc}}(M)$ for some $p > 2$ if $d = 4$ while $V \in L^{d/2}_{\mathrm{loc}}(M)$ for $d \geq 5$. Then V is infinitesimally relatively small. In particular H is essentially self-adjoint on $C_{\mathrm{c}}^\infty(M)$.

As for the unique continuation property, there is an extensive literature, and we only refer to [Wo] and references therein, quoting here the following sufficient conditions supplementing connectivity and the conditions in Proposition 2.1: 1) $d = 2, 3, 4$ and V is globally bounded, or 2) $d \geq 5$. One could, of course, add 3) $d = 1$.

§ 2.2. Conditions inside an end

We consider a connected and complete (M, g) of dimension $d \geq 2$ and take $V = 0$ for simplicity. We shall investigate the meaning of Condition 1.1 in the case where, (M, g) has the following explicit *end* structure of warped-product type: There exists an open subset $E \subset M$ such that the closure $\bar{E}$ is isometric to $[0, \infty) \times S$ endowed with a metric of the form

$$g = \mathrm{d}r \otimes \mathrm{d}r + f(r) h_{\alpha\beta}(\sigma)\, \mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta, \qquad (r, \sigma) \in [0, \infty) \times S, \quad \alpha, \beta = 2, \dots, d,$$

where S is a $(d-1)$-dimensional manifold. Then r is a distance function from $\{0\} \times S$ and smoothly defined in E. In particular we have $|\mathrm{d}r| = 1$ which obviously implies (1.2) for any $c_2 > 0$. Notice here that Condition 1.1 involves only the part of the function r at large values, so in agreement with Condition 1.1 we can cut and extend it to a smooth function on M obeying $r \geq 1$. This is tacitly understood below. To examine the remaining statements (1.1), (1.3) and (1.4) of Condition 1.1 we compute

$$\nabla^2 r^2 = 2\, \mathrm{d}r \otimes \mathrm{d}r + r f'(r) h_{\alpha\beta}(\sigma)\, \mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta, \quad \triangle r^2 = g^{ij} (\nabla^2 r^2)_{ij} = 2 + r f'(r).$$

Then we have

Examples 2.2.

1. Let $f = r^{2a}$ with $a > 0$. Then (1.1), (1.3) and (1.4) hold with $c_1 = \min\{2, 2a\}$ and $\rho_1 = 0$, $\rho_2 = 2 + 2a(d-1)$, $\rho_3 = 0$, and $E_0 = 0$.
2. Let $f = \exp(2\kappa r^q)$ with $\kappa > 0$ and $q \in (0, 1)$. Then (1.1), (1.3) and (1.4) hold with $c_1 = 2$ and $\rho_1 = 0$, $\rho_2 = 2 + 2\kappa q(d-1) r^q$, $\rho_3 = 0$, and $E_0 = 0$.
3. Let $f = \exp(2\kappa r)$ with $\kappa > 0$. Then (1.1), (1.3) and (1.4) hold with $c_1 = 2$ and $\rho_1 = 0$, $\rho_2 = 2 + 2\kappa(d-1) r$, $\rho_3 = 0$, and $E_0 = \kappa^2 (d-1)^2 / 8$.

For all of these examples it is easy to compute that the essential spectrum $\sigma_{\mathrm{ess}}(H_0) \supseteq [E_0, \infty)$. If in addition $M \setminus E$ and S are compact then we have $\sigma_{\mathrm{ess}}(H_0) = [E_0, \infty)$. Whence indeed the absence of eigenvalues in (E_0, ∞) as stated in Theorem 1.5 is optimal under these additional conditions for the above examples.

The bounds in Condition 1.1 can be viewed as those for the principal curvatures of angular manifolds $S_r = \{r\} \times S$, and we can obtain corollaries for Theorem 1.5 in terms of these geometric quantities, recovering and extending various results of [K1, K2]. We refer to [IS] for the detail.

§ 3. Absence of super-exponentially decaying eigenfunctions

The proof of Theorem 1.5 is done combining the following two propositions, a priori super-exponential decay estimates for eigenfunctions and the absence of super-exponentially decaying eigenfunctions:

Proposition 3.1. Suppose Conditions 1.1–1.3. If $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$ for some $E > E_0$, then $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$ for any $\sigma \geq 0$.

Proposition 3.2. Suppose Conditions 1.1–1.4. If $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$ for some $E \in \mathbb{R}$ and $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$ for any $\sigma \geq 0$, then $\phi(x) = 0$ in M.

We do not prove these propositions in this article, but here we note that we can actually prove a little generalized version of Proposition 3.2. This generalized version recovers the result of [VW]. Let us replace Conditions 1.1 and 1.2 by the following ones stated in terms of a parameter $\tau \leq 1$:

Condition 3.3. There exist an unbounded real-valued function $r \in C^4(M)$, $r(x) \geq 1$, constants $c_1 > c_2 > 0$ and a decomposition $\triangle r^2 = \rho_1 + \rho_2 + \rho_3$ such that:

1. There exist constants $r_0 \geq 1$ and $C > 0$ such that

$$\nabla^2 r^2 \geq (c_1 r^\tau + \tfrac{1}{2}\rho_1)g - Cr^\tau \mathrm{d}r \otimes \mathrm{d}r \text{ and } \rho_1 \geq 0 \text{ for } r \geq r_0. \tag{3.1}$$

   Moreover,

$$\liminf_{r\to\infty} r^{-\tau}(r\partial^r |\mathrm{d}r|^2 + (c_2 r^\tau + \tfrac{1}{2}\rho_1)|\mathrm{d}r|^2) > 0, \quad \limsup_{r\to\infty} |\mathrm{d}r| < \infty. \tag{3.2}$$

2. The following bounds hold

$$\limsup_{r\to\infty} |r^{-1}\triangle r^2| < \infty, \tag{3.3}$$

$$\limsup_{r\to\infty} r^{-\tau}\rho_1 < \infty, \quad \limsup_{r\to\infty} r^{-\tau}|\mathrm{d}\rho_2| < \infty, \quad \limsup_{r\to\infty} r^{-\tau}\triangle\rho_3 < \infty. \tag{3.4}$$

Condition 3.4. There exists a decomposition $V = V_1 + V_2$, $V_1 \in L^2_{\mathrm{loc}}(M)$, $V_2 \in C^1(M)$ such that V_1 and V_2 are real-valued and

$$\limsup_{r\to\infty} |V| < \infty, \quad \limsup_{r\to\infty} r^{1-\tau}|V_1| < \infty, \quad \limsup_{r\to\infty} r^{1-\tau}\partial^r V_2 < \infty. \tag{3.5}$$

The case $\tau = 0$ corresponds to Conditions 1.1 and 1.2 although even in this case (3.1) is weaker than (1.1) since now possibly some negativity of $\nabla^2 r^2$ along the $\mathrm{d}r \otimes \mathrm{d}r$ component occurs. The weakening of these conditions will be compensated by the assumption of super-exponential decay for the considered eigenfunction. Another remark here is that the negative case, $\tau < 0$, is also allowed. With Examples 2.2 in mind, this means that an end of very slow expansion, which is so slow that the end might be asymptotic to a straight cylinder, could be treated. In the other extreme case $\tau = 1$ the bounds (3.4) and (3.5) are relaxing (1.4) and (1.5), respectively.

Under these conditions we prove

Proposition 3.5. Suppose Conditions 3.3 and 3.4 for some $\tau \leq 1$. Suppose Conditions 1.3 and 1.4. If $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$ for some $E \in \mathbb{R}$ and $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$ for any $\sigma \geq 0$, then $\phi(x) = 0$ in M.

Proposition 3.5 generalizes [VW] when $\tau = 1$ while Proposition 3.2 does not. This is because a manifold of bounded geometry and pinched negative curvature is always endowed with an end with a metric of the form

$$g = \mathrm{d}r \otimes \mathrm{d}r + g_{\alpha\beta}(r, \sigma)\, \mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta,$$

uniformly and strictly positive $\nabla^2 r_{|S_r}$ and bounded derivatives of $\triangle r$ for r large. Then the verification of Condition 3.3 is straightforward. For these geometric terminologies we refer to [VW] and references therein.

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