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Width of resonances created by homoclinic orbits - isotropic fixed point case -

By

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Abstract

We give a resonance free domain near an energy trapped by homoclinic orbits associated with a hyperbolic fixed point. We study the case where the fixed point is isotropic and the outgoing manifold and the incoming one intersect tangentially, and show that, under some geometrical assumption, the resonance width is larger than a multiple of $h/|\log h|$.

§ 1. Introduction

We consider semiclassical Schrödinger operators $-h^2\Delta + V(x)$ with a smooth potential $V(x)$ decaying at infinity. We suppose that there is a positive energy E_0 such that the trapped trajectories of the corresponding classical dynamics on its energy surface consist of a hyperbolic fixed point and some associated homoclinic trajectories. We will give a semiclassical estimate from below for the imaginary part of resonances near this energy. This note is a brief summary of the forthcoming paper [3].

A typical example is two bumps potential, one of which has a non-degenerate maximum of value E_0, the other one being of higher height. Depending on the shape of the potential, the set of homoclinic trajectories has different geometrical properties, and the resonances have different imaginary parts.

In [2], we have considered the case where the fixed point is anisotropic. For the above example, it means that the eigenvalues of the Hessian of the potential at the lowest maximum are not equal. Under some generic conditions, there exists a constant

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$\delta > 0$ such that, for any positive constant $C > 0$, there is no resonance in the box $]E_0 - Ch, E_0 + Ch[-i[0, \delta h[$ for sufficiently small $h > 0$. This estimate remains true even in the isotropic case if the dimension of the set of homoclinic trajectories is smaller than the space dimension.

In this report, we restrict ourselves to the complementary case, i.e. the isotropic fixed point case with homoclinic trajectories of maximal dimension. Compared with the former case, the trapping is stronger and the resonances are expected to have smaller imaginary parts. We will show that there is no resonance in the box $]E_0 - \delta h, E_0 + \delta h[-i[0, \delta h/|\log h|[$ for some $\delta > 0$ and sufficiently small h, under the condition that the "measure" of the set of homoclinic trajectories at the fixed point (see (2.4)) is small enough (Theorem 2.3). As a matter of fact, we will obtain a more precise result in larger intervals for the real part of size Ch (Theorem 2.7). In particular, we can observe a transition for the imaginary part of the resonances from the trapping energy $E < E_0$ to the non-trapping one $E > E_0$.

The proof is essentially based on the connection formula of microlocal solutions at a hyperbolic fixed point established in [1]. In this formula, the solution, microlocally on the outgoing stable manifold, is given by the action of a Fourier integral operator on the solution microlocalized on the incoming stable manifold. We make use of the explicit expression of the first term of the symbol.

After stating the results in the next section, we review the connection formula in Section 3 restricting ourselves to the isotropic case, and give the outline of the proof in Section 4.

§ 2. Results

We consider the semiclassical Schrödinger operator

$$P := -h^2\Delta + V(x) = -h^2 \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + V(x), \tag{2.1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, h is a small positive parameter and $V(x)$ is a potential satisfying

(A1) $V(x) \in C^\infty(\mathbb{R}^n; \mathbb{R})$ and extends holomorphically in a sector

$$\mathcal{S} = \big\{x \in \mathbb{C}^n;\ |\operatorname{Im} x| \le (\tan\theta_0)\langle \operatorname{Re} x\rangle \text{ and } |\operatorname{Re} x| > C\big\},$$

for some positive constants θ_0 and C. Moreover $V(x)$ tends to 0 as x tends to ∞ in $\mathcal{S}$.

To the self-adjoint operator P on $L^2(\mathbb{R}^n)$ with $\sigma_{\rm ess}(P) = \mathbb{R}_+$, we associate a *distorted* operator $\widetilde{P}_\mu = U_\mu P U_{-\mu}$, where $(U_\mu f)(x) := (\det(\mathrm{Id} + \mu dF))^{1/2} f(x + \mu F(x))$ for small

$|\mu|$, and $F \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ with $F = 0$ on $|x| < R$, $F = x$ on $|x| > R+1$ for large R. The complex eigenvalues of $P_\theta := \widetilde{P}_{i\theta}$ for $0 < \theta < \theta_0$ in $\{E \in \mathbb{C}\setminus\{0\};\ -2\theta < \arg E < 0\}$ are independent of θ and called the *resonances* of P, see [8].

(A2) The origin is a non-degenerate isotropic maximal point with maximal value $E_0 > 0$:

$$V(x) = E_0 - \frac{\lambda^2}{4}x^2 + \mathcal{O}(x^3) \quad \text{as } x \to 0, \tag{2.2}$$

for a positive constant $\lambda > 0$, where $x^2 = x_1^2 + \cdots + x_n^2$.

Let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ denote the momentum and $p(x,\xi) = \xi^2 + V(x)$ be the classical Hamiltonian corresponding to P. The assumption (A2) means that the origin $(0,0)$ in the phase space $T^*\mathbb{R}^n = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ is a hyperbolic fixed point for the Hamiltonian vector field $H_p = \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi = 2\xi \cdot \partial_x - \partial_x V(x) \cdot \partial_\xi$. The stable manifold theorem shows that the outgoing stable manifold Λ_+ and the incoming stable manifold Λ_-

$$\Lambda_\pm := \big\{(x,\xi) \in p^{-1}(E_0);\ \exp(tH_p)(x,\xi) \to (0,0) \text{ as } t \to \mp\infty\big\},$$

are Lagrangian manifolds with generating functions $\phi_\pm(x) = \pm\frac{\lambda}{4}x^2 + \mathcal{O}(x^3)$, i.e.

$$\Lambda_\pm = \{(x,\xi);\ \xi = \partial_x \phi_\pm(x)\}.$$

Let $K(E_0)$ be the set of trapped trajectories on the energy surface $p^{-1}(E_0)$:

$$K(E_0) = \big\{(x,\xi) \in p^{-1}(E_0);\ t \mapsto \exp(tH_p)(x,\xi) \text{ is bounded}\big\}.$$

We assume that $K(E_0)$ consists of the fixed point $(0,0)$ and of homoclinic trajectories associated with this point. More precisely,

(A3) $K(E_0) = \Lambda_+ \cap \Lambda_-$ and $\mathcal{H} := \Lambda_+ \cap \Lambda_- \setminus \{(0,0)\}$.

The results below are of interest only in the case $\mathcal{H} \neq \emptyset$, otherwise see [5] and [10].

Lemma 2.1. *For any $\alpha \in \mathbb{S}^{n-1}$, there exists a unique Hamiltonian curve $\rho_+(t,\alpha) = (x_+(t,\alpha), \xi_+(t,\alpha))$ on Λ_+ such that, for any $\varepsilon > 0$,*

$$x_+(t,\alpha) = e^{\lambda t}\alpha + \mathcal{O}\big(e^{(2\lambda-\varepsilon)t}\big) \quad \textit{as } t \to -\infty.$$

Then, we define

$$\mathcal{H}_{\mathrm{tang}} := \{\rho \in \mathcal{H};\ T_\rho\Lambda_+ = T_\rho\Lambda_-\},$$

the set of the points at which Λ_+ and Λ_- are tangent, and

$$\mathcal{H}^\infty := \{\alpha \in \mathbb{S}^{n-1};\ \rho(\cdot,\alpha) \in \mathcal{H}\},$$
$$\mathcal{H}^\infty_{\mathrm{tang}} := \{\alpha \in \mathbb{S}^{n-1};\ \rho(\cdot,\alpha) \in \mathcal{H}_{\mathrm{tang}}\},$$

the asymptotic directions of the Hamiltonian curves in $\mathcal{H}$ and $\mathcal{H}_{\mathrm{tang}}$. Note that these two sets are compact. We assume

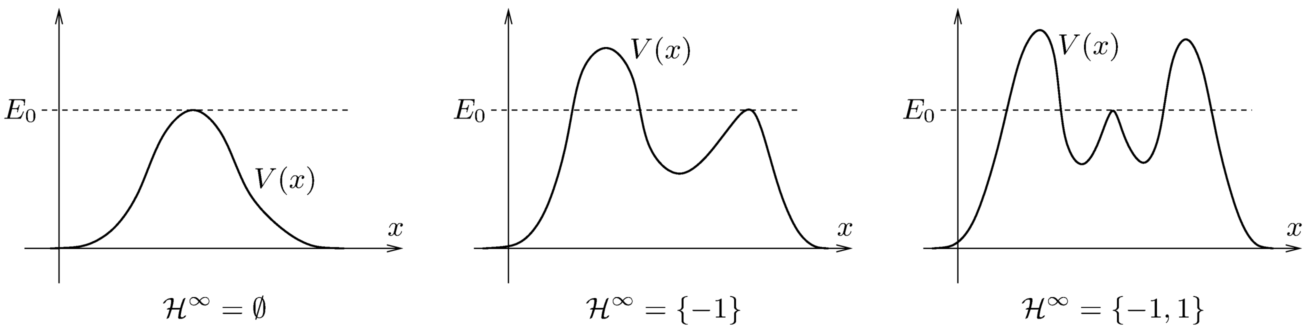

Figure 1. The three possible cases in 1D.

(A4) $\alpha \cdot \beta \neq 0$ for any $\alpha, \beta \in \mathcal{H}^\infty$.

Let $\alpha \in \mathcal{H}^\infty_{\rm tang}$. For any sufficiently small ε, there exist unique times $t^\varepsilon_\pm(\alpha)$ satisfying $|x_+(t^\varepsilon_\pm(\alpha), \alpha)| = \varepsilon$ and $t^\varepsilon_\pm(\alpha) \to \mp\infty$ as $\varepsilon \to 0$. Then, it is well known that the quantity

$$\mathcal{M}_\varepsilon(\alpha) = \frac{\mathcal{D}(t^\varepsilon_+(\alpha), \alpha)}{\mathcal{D}(t^\varepsilon_-(\alpha), \alpha)} \quad \text{with} \quad \mathcal{D}(t, \alpha) = \sqrt{\Big| \det \frac{\partial x_+(t, \alpha)}{\partial(t, \alpha)} \Big|},$$

represents the evolution of the amplitude of WKB solutions along the curve $x_+(t, \alpha)$ from the time $t^\varepsilon_+(\alpha)$ to the time $t^\varepsilon_-(\alpha)$, see for example [6]. This function $\mathcal{M}_\varepsilon(\alpha)$ has a positive limit $\mathcal{M}_0(\alpha)$ as ε tends to 0

$$\mathcal{M}_0(\alpha) := \lim_{\varepsilon \to 0} \mathcal{M}_\varepsilon(\alpha), \tag{2.3}$$

which is continuous with respect to $\alpha \in \mathcal{H}^\infty_{\rm tang}$ and hence bounded. We also define a constant associated with the quantum propagation through the fixed point:

$$\mathcal{J}_0(\alpha) := (2\pi)^{-n/2} \Gamma\Big(\frac{n}{2}\Big) \int_{\mathcal{H}^\infty_{\rm tang}} |\alpha \cdot \omega|^{-n/2} d\omega. \tag{2.4}$$

The amplification around the trapped set is then controlled by the quantity

$$\mathcal{A}_0 := \max_{\alpha \in \mathcal{H}^\infty_{\rm tang}} \mathcal{M}_0(\alpha) \mathcal{J}_0(\alpha) \in [0, +\infty[. \tag{2.5}$$

Remark 2.2. *In the one-dimensional case, $\mathcal{H}^\infty = \mathcal{H}^\infty_{\rm tang} \subset \{-1, 1\}$ and, for each $\alpha \in \mathcal{H}^\infty$, one has*

$$\mathcal{M}_0(\alpha) = 1, \quad \mathcal{J}_0(\alpha) = \begin{cases} 0 & \text{if } \mathcal{H}^\infty = \emptyset, \\ 1/\sqrt{2} & \text{if } \mathcal{H}^\infty = \{1\} \text{ or } \{-1\}, \\ \sqrt{2} & \text{if } \mathcal{H}^\infty = \{-1, 1\}. \end{cases} \tag{2.6}$$

We state our main result in two steps: in Theorem 2.3, we consider the case where the real part of the energy lies in a small interval around E_0. Then, the width of resonances is estimated from below uniformly with respect to the real part. In Theorem 2.7, the real part is extended to a wider interval of $\mathcal{O}(h)$, and we can observe the transition of the width from the trapping energy $E < E_0$ to the non-trapping energy $E > E_0$, under suitable assumptions.

Theorem 2.3. *Assume* (A1), (A2), (A3), (A4) *and*

$$\mathcal{A}_0 < 1. \tag{2.7}$$

Then, for all $\delta > 0$, *there exists* $\nu > 0$ *such that* P *has no resonance in the box*

$$[E_0 - \nu h, E_0 + \nu h] + i\Big[\big(\lambda \log(\mathcal{A}_0) + \delta\big)\frac{h}{|\log h|}, 0\Big], \tag{2.8}$$

for sufficiently small h. *Moreover, for all* $\chi \in C_0^\infty(\mathbb{R}^n)$, *there exists a positive constant* N *such that, for any* E *in this domain, one has*

$$\big\|\chi(P - E)^{-1}\chi\big\| \leq h^{-N}, \tag{2.9}$$

for sufficiently small h.

When $\mathcal{A}_0 = 0$, we use the convention that $\log(\mathcal{A}_0)$ appearing in (2.8) can be taken as any arbitrary large negative constant.

Example 2.4. Consider the case $n = 1$. Due to Remark 2.2, the condition (2.7) is satisfied if $\operatorname{card}(\mathcal{H}^\infty) \leq 1$ but not satisfied if $\mathcal{H}^\infty = \{-1, 1\}$. When $\mathcal{H}^\infty = \{1\}$ or $\mathcal{H}^\infty = \{-1\}$, the precise location of the resonances was studied in [7], and this result implies that our estimate (2.8) from below of the imaginary part of the resonances is optimal. When $\mathcal{H}^\infty = \{-1, 1\}$, on the contrary, we are in the well in an island situation, and the resonances are exponentially close to the real axis.

Example 2.5. In dimension $n = 2$, let (r, θ) be the polar coordinates. We consider

$$V(x) = q_0(r) + q_1(r - a)\psi(\theta),$$

where the $q_\bullet(r)$'s are even functions in $C_0^\infty(\mathbb{R})$ with $rq_\bullet'(r) < 0$ for $r \neq 0$ and $E_0 = q_0(0) < q_1(0)$, a is a sufficiently large constant such that $\operatorname{supp} q_0(r) \cap \operatorname{supp} q_1(r - a) = \emptyset$ and $\psi(\theta) \in C_0^\infty([-\theta_1 - \varepsilon, \theta_1 + \varepsilon])$ is equal to 1 for $|\theta| \leq \theta_1$ and $\theta\psi'(\theta) < 0$ for $\theta_1 < |\theta| < \theta_1 + \varepsilon$ for $\theta_1 < \pi/4$ and small enough $\varepsilon > 0$. It can be checked that the conditions (A1) to (A4) are all satisfied, and moreover $\mathcal{H}^\infty = \mathcal{H}^\infty_{\rm tang} = [-\theta_1, \theta_1]$ and $\mathcal{M}_0(\alpha) = 1$. $\mathcal{J}_0(\alpha)$ can also be computed explicitly, and the condition (2.7) is satisfied if $\sin(2\theta_1) < \tanh(2\pi)$.

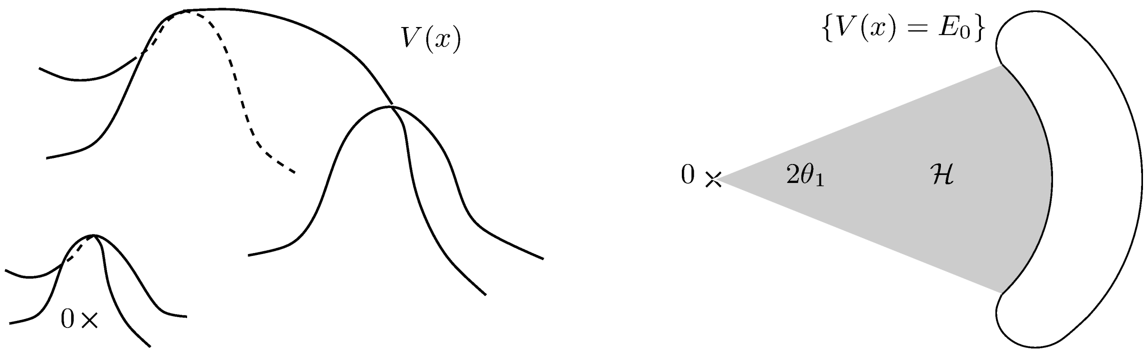

Figure 2. The geometrical setting of Example 2.5.

The resolvent estimate (2.9), together with the estimate $\|\chi(P-E)^{-1}\chi\| \lesssim |\operatorname{Im} E|^{-1}$ in the upper half plane with $\operatorname{Im} E = \mathcal{O}(h/|\log h|^2)$, leads us to the following resolvent estimate on the real axis by the maximum principle for holomorphic functions:

Corollary 2.6. *Under the assumptions of the previous theorem, we have*

$$\big\|\chi(P-E)^{-1}\chi\big\| \lesssim h^{-1}|\log h|^2,$$

for E in the real interval $]E_0 - \nu h, E_0 + \nu h[$.

For a more precise version of Theorem 2.3, we need to define a function $\mathcal{J}_0(\alpha, s)$ which depends on the real variable $s = (\operatorname{Re} E - E_0)/(\lambda h)$.

$$\mathcal{J}_0(\alpha, s) := (2\pi)^{-n/2}\Big|\Gamma\Big(\frac{n}{2} - is\Big)\Big| \int_{\mathcal{H}^\infty_{\rm tang}} e^{-\frac{\pi}{2} s \operatorname{sign} \alpha\cdot\omega}|\alpha\cdot\omega|^{-n/2} d\omega,$$

where sign stands for the sign of a real number. As in (2.5), we set

$$\mathcal{A}_0(s) = \max_{\alpha\in\mathcal{H}^\infty_{\rm tang}} \mathcal{M}_0(\alpha)\mathcal{J}_0(\alpha, s). \tag{2.10}$$

Theorem 2.7. *Assume* (A1), (A2), (A3) *and* (A4). *Then, for any large $C > 0$ and for any small $\delta > 0$, P has no resonance in the domain*

$$\begin{cases} E_0 - Ch \leq \operatorname{Re} E \leq E_0 + Ch, \\ \Big(\lambda \log\Big(\mathcal{A}_0\Big(\dfrac{\operatorname{Re} E - E_0}{\lambda h}\Big)\Big) + \delta\Big)\dfrac{h}{|\log h|} \leq \operatorname{Im} E \leq 0, \end{cases} \tag{2.11}$$

for sufficiently small h. Moreover, for $\chi \in C_0^\infty(\mathbb{R}^n)$, there exists a positive constant N such that, for any E in this domain, one has

$$\big\|\chi(P-E)^{-1}\chi\big\| \leq h^{-N},$$

for sufficiently small h.

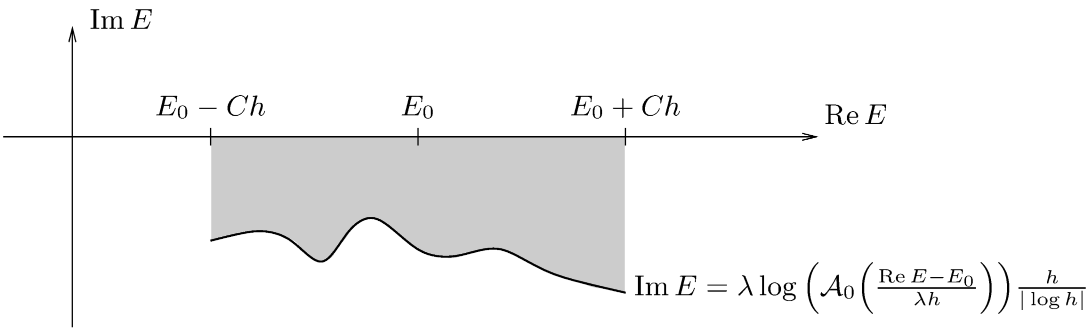

Figure 3. The resonance free zone given by Theorem 2.7.

Remark 2.8. *Note that in the region where* $\mathcal{A}_0((\operatorname{Re} E - E_0)/(\lambda h)) \geq 1$, *the above theorem is empty.*

When $|\operatorname{Re} E - E_0| \leq \delta h$, we have $\mathcal{J}_0(\alpha, (\operatorname{Re} E - E_0)/(\lambda h)) = \mathcal{J}_0(\alpha) + \mathcal{O}(\delta)$ where $\mathcal{J}_0(\cdot)$ is given by (2.4). Thus, we can recover Theorem 2.3.

Let us now look at the behavior of $-\log \mathcal{A}_0(s)$ when $s \to \pm\infty$. To make the discussion clearer, we assume that $\operatorname{sign} \alpha \cdot \omega = 1$ for all $\alpha, \omega \in \mathcal{H}^{\infty}_{\mathrm{tang}}$. Note that this is the case in Example 2.4 and Example 2.5. We have, as $s \to \pm\infty$,

$$(2\pi)^{-n/2}\Big|\Gamma\Big(\frac{n}{2} - is\Big)\Big|e^{-\pi s/2} = \begin{cases} e^{-\frac{\pi}{2}s(1+\operatorname{sign} s)}\big(1 - e^{-2\pi|s|}/2\big) & \text{if } n = 1, \\ e^{-\frac{\pi}{2}s(1+\operatorname{sign} s)}|s|^{(n-1)/2} & \text{if } n \geq 2. \end{cases}$$

Hence, as $s \to +\infty$, $-\log \mathcal{A}_0(s)$ is linearly growing for any $n \geq 1$. This means that the resonance free domain has larger and larger imaginary part as the energy grows to the non trapping region. As $s \to -\infty$, on the contrary, $-\log \mathcal{A}_0(s)$ is exponentially decaying with respect to s for $n = 1$. This reflect the fact that we are in the well in an island situation for energies below E_0. However for $n \geq 2$, it becomes negative for sufficiently large negative s and this theorem says nothing for such energies.

§ 3. Microlocal connection formula at a hyperbolic fixed point

The proofs of our theorems are essentially based on the microlocal connection formula (3.2) at the fixed point $(0, 0)$, which was obtained in [1]. Here we give a short survey of the results there under the isotropic condition (2.2).

We say that u is microlocally 0 in an open set $\omega \subset T^*\mathbb{R}^n$ if

$$\| \operatorname{Op}_h(\psi)u\| = \mathcal{O}(h^{\infty}),$$

for some $\psi \in C_0^{\infty}(T^*\mathbb{R}^n)$ with $\psi = 1$ in ω, where $\operatorname{Op}_h(\psi)$ is the h-pseudodifferential operator with symbol ψ given by

$$\operatorname{Op}_h(\psi)u = \frac{1}{(2\pi h)^n}\iint e^{i(x-y)\cdot\xi/h}\psi\Big(\frac{x+y}{2}, \xi\Big)u(y)\, dy\, d\xi.$$

Let $\Omega \subset T^*\mathbb{R}^n$ be a small neighborhood of $(0,0)$ and $\varepsilon > 0$ be small enough. We consider the microlocal Cauchy problem, with $E = E_0 + hz$,

$$\begin{cases} Pu = Eu & \text{microlocally in } \Omega, \\ u = u_0(x) & \text{microlocally on } \mathcal{C} := \Lambda_- \cap \{|x| = \varepsilon\}. \end{cases} \tag{3.1}$$

Remark that the initial surface $\mathcal{C}$ is transversal to the Hamiltonian vector field for sufficiently small ε.

Theorem 3.1. *There exists a constant $\mu > 0$ such that if $u_0 = 0$ and if $z(h)$ is in $]-C, C[-i[0,\mu[$ for any constant $C > 0$, then any solution $u \in L^2(\mathbb{R}^n)$ of* (3.1) *with $\|u\| \leq 1$, is microlocally* 0 *in a neighborhood Ω' of the origin.*

Remark 3.2. *More precisely, Theorem 3.1 holds for $z(h)$ outside any small neighborhood of size h of some discrete set. On the other hand, Theorem 3.1 holds also in the analytic category, changing of course the notion of C^∞-microsupport to that of analytic microsupport and in this case the exceptional discrete set is known to be $-i\mathcal{E}_0$ where*

$$\mathcal{E}_0 = \Big\{\lambda\Big(\alpha + \frac{n}{2}\Big);\ \alpha \in \mathbb{N} = \{0, 1, \dots\}\Big\},$$

is the set of eigenvalues of the isotropic harmonic oscillator $-\Delta + \frac{\lambda^2}{4}x^2$. In particular, μ can be taken to be $\frac{n}{2}\lambda - \delta$, for any small $\delta > 0$.

Theorem 3.1 says that the data u_0 given on $\Lambda_- \cap \{|x| = \varepsilon\}$ uniquely determines the solution u at any point $\rho_F = (x, \xi)$ on Λ_+ (if it exists). Next theorem enables us to represent u near ρ_F in terms of u_0 which, restricted to the initial surface $\mathcal{C}$, has its support in a small neighborhood of a point $\rho_I = (y, \eta) \in \mathcal{C}$.

We make an assumption on the initial point $\rho_I = (y, \eta) \in \mathcal{C}$ and the final point $\rho_F = (x, \xi) \in \Lambda_+$. The integral curve of H_p starting from ρ_I and ρ_F tends to $(0,0)$ as t tends to $+\infty$ and $-\infty$ respectively. More precisely, they have the following asymptotic expansion

$$\exp(tH_p)(\rho_I) \sim \sum_{k=1}^{\infty} \gamma_k^-(t, y, \eta) e^{-k\lambda t} \quad \text{as } t \to +\infty,$$
$$\exp(tH_p)(\rho_F) \sim \sum_{k=1}^{\infty} \gamma_k^+(t, x, \xi) e^{k\lambda t} \quad \text{as } t \to -\infty.$$

The coefficients $\gamma_k^-(t, y, \eta)$ and $\gamma_k^+(t, x, \xi)$ are vector valued polynomials in t, and in particular γ_1 is independent of t. In our Schrödinger case, the x-space projection of $\gamma_1^-(y, \eta)$ and $\gamma_1^+(x, \xi)$ coincide if $y = x$, and we will denote it simply by $g(x)$. We assume

(H) $g(x) \cdot g(y) \neq 0$.

Remark 3.3. *Let $\phi_1(x)$ be the function defined from ρ_I by*

$$\begin{cases} 2\nabla\phi_+ \cdot \nabla\phi_1 - \lambda\phi_1 = 0, \\ \nabla\phi_1(0) = -\lambda g(y). \end{cases}$$

Then, (H) *implies $\phi_1(x) \neq 0$.*

We assume, without loss of generality, that $g(y)$ is parallel to x_1-axis. Since p is of real principal type near ρ_I, we can modify the initial surface $\mathcal{C}$ so that it is given by $\{x_1 = \varepsilon\} \cap \Lambda_-$ near ρ_I. Hence, denoting $y = (\varepsilon, y')$, the initial data u_0 on $\mathcal{C}$ is a function of y' localized in a small neighborhood of x'_I.

Let $\psi(x) = \psi_{\varepsilon,y'}(x)$ be the solution to the Cauchy problem for the eikonal equation

$$\begin{cases} |\nabla\psi|^2 + V(x) = E_0, \\ \psi|_{x_1=\varepsilon} = \eta' \cdot x' \text{ where } \eta' = \dfrac{\partial\phi_-(\varepsilon, y')}{\partial y'}. \end{cases}$$

With the notations $(x(t), \xi(t)) = \exp(tH_p)(x, \xi)$ and $(y(t), \eta(t)) = \exp(tH_p)(y, \eta)$, the integrals

$$I_+^\infty(x) := \int_0^{-\infty} \Big(\Delta\phi_+(x(t)) - \frac{n\lambda}{2}\Big)\, dt,$$
$$I_-^\infty(y) := \int_0^{+\infty} \Big(\Delta\psi(y(t)) - \frac{(n-2)\lambda}{2}\Big)\, dt,$$

converge.

Theorem 3.4. *Assume* (H) *and $z \in]-C, C[-i[0, \mu[$. Then, the microlocal Cauchy problem* (3.1) *has a solution u which, microlocally near $\rho_F = (x, \xi)$, has the following integral representation*

$$u(x, h) = \frac{h^{-iz/\lambda}}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} e^{i(\phi_+(x) - \phi_-(\varepsilon, y'))/h} d(x, y', h) u_0(\varepsilon, y')\, dy'. \tag{3.2}$$

Here, the symbol $d \in S_h^0(1)$ has the asymptotic expansion

$$d(x, y', h) \sim \sum_{k=0}^{\infty} d_k(x, y', \log h) h^k, \tag{3.3}$$

where $d_k(x, y', \log h)$ are polynomials in $\log h$. In particular, d_0 is independent of $\log h$ and given by

$$\begin{aligned} d_0(x, y') = {} & e^{-i\pi n(1+\sigma)/4} \lambda^{iz/\lambda - (n-1)/2} \exp\Big(-\frac{\pi\sigma}{2\lambda} z\Big) \Gamma\Big(\frac{n}{2} - \frac{iz}{\lambda}\Big) \\ & \times e^{I_+^\infty(x) - I_-^\infty(y)} \sqrt{|\det \nabla_{y'}^2 \phi_-(y)|} \frac{|g(y)|}{|g(x) \cdot g(y)|^{n/2 - iz/\lambda}}, \end{aligned} \tag{3.4}$$

where $\sigma = \operatorname{sign}(g(x) \cdot g(y))$.

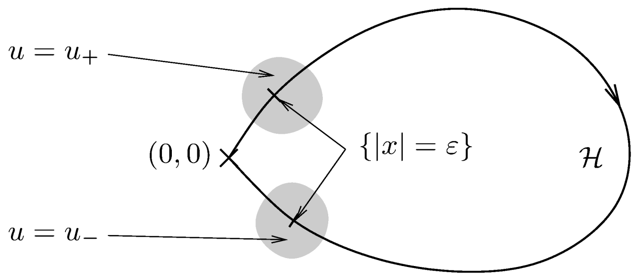

Figure 4. The geometrical setting for the proof of Theorem 2.7.

§ 4. Sketch of the proof of Theorem 2.7

It is enough to prove that $\|(P_\theta - E)^{-1}\| \leq h^{-N}$ for E satisfying (2.11). Recall that P_θ is the distorted operator of P defined in Section 2. For this, we proceed by contradiction. If this estimate did not hold, then there would exist $u = u(x, h)$ with $\|u\| = 1$ and E satisfying (2.11) such that

$$(P_\theta - E)u = \mathcal{O}(h^\infty). \tag{4.1}$$

Let us look at u microlocally in the phase space.

By the ellipticity of P, u is microlocally 0 outside the energy surface $p^{-1}(E_0)$. Furthermore, using the ellipticity of the distorted operator P_θ at infinity, it is enough to consider u in a compact set in x. Moreover, (4.1) and $\|u\| \leq 1$ imply that u is microlocally 0 in the incoming region

$$\Big\{(x,\xi) \in p^{-1}(E_0);\ |x| \gg 1 \text{ and } \cos(x,\xi) := \frac{x \cdot \xi}{|x||\xi|} < -\frac{1}{2}\Big\},$$

see [4, Theorem 2]. Then, with the standard propagation of singularities, it turns out that u is microlocally 0 outside Λ_+, see for example [9]. In particular, u vanishes microlocally in $\Lambda_- \setminus \mathcal{H}$.

In what follows, we mean, by the notations $\Lambda_\pm$, their restriction to a small neighborhood of $(0,0)$ such that $\exp(tH_p)(x,\xi) \in \Lambda_\pm$ for all $\mp t > 0$ if $(x,\xi) \in \Lambda_\pm$ respectively. We denote by $\widetilde{\Lambda}_+$ the global evolution of Λ_+.

Let $u_\pm$ be the restriction of u on $\mathcal{H} \cap \Lambda_\pm$ respectively. We express each of these two microlocal solutions in term of the other one. First, we use the results of [1], recalled briefly in the previous section, to compute u_+ from u_- passing through the fixed point. Then, we use the standard Maslov theory (see [6]) to compute u_- from u_+ following the homoclinic trajectories.

Now, we apply Theorem 3.1 and Theorem 3.4 in order to obtain u_+ from u_-. It is possible because the imaginary part of E, which is of order $h/|\log h|$, is smaller than μh. Moreover, the assumption (H) needed for Theorem 3.4 is guaranteed by (A4).

By assumption, we have $\|u_-\| \leq 1$. By a first use of those theorems, we see that u_+ is a Lagrangian distribution of order h^{-C} for some C,

$$u_+ \in \mathcal{I}\big(\mathcal{H} \cap \Lambda_+, h^{-C}\big), \quad u_+(x,h) = h^{-C} a_+(x,h) e^{i\phi_+(x)/h}, \tag{4.2}$$

for some symbol $a_+(x,h) \in S_h^0(1)$, the space of uniformly bounded functions together with their derivatives.

Next, using the Maslov theory along $\mathcal{H}$, we see that u is everywhere on $\mathcal{H}$ a Lagrangian distribution of constant order. In particular,

$$u_- \in \mathcal{I}\big(\widetilde{\Lambda}_+, h^{-C}\big), \quad u_-(y,h) = h^{-C} a_-(y,h) e^{i\widetilde{\phi}_+(y)/h}, \tag{4.3}$$

where the phase $\widetilde{\phi}_+(y)$ is a generating function of $\widetilde{\Lambda}_+$ near $\mathcal{H} \cap \Lambda_-$ and satisfies

$$\partial_y \widetilde{\phi}_+(y) = \partial_y \phi_+(y) \text{ on } \mathcal{H}, \quad \partial_y^2 \widetilde{\phi}_+(y) = \partial_y^2 \phi_+(y) \text{ on } \mathcal{H}_{\mathrm{tang}}.$$

Moreover, the symbol $a_-(y,h) \in S_h^0(1)$ is obtained by solving the transport equation and satisfies, modulo lower order terms,

$$a_-(y,h) = \mathcal{M}_\varepsilon(\alpha) a_+(x,h), \tag{4.4}$$

if $y = x_+(t_-^\varepsilon(\alpha), \alpha)$ and $x = x_+(t_+^\varepsilon(\alpha), \alpha)$.

Now we use Theorem 3.4 for a second time. Here we suppose that the homoclinic trajectories are transversal to the plane $y_1 = \varepsilon$ for small enough ε. Then we get, on $\mathcal{H} \cap \Lambda_+$,

$$u_+(x,h) = \frac{h^{-C-iz/\lambda} e^{i\phi_+(x)/h}}{(2\pi)^{n/2}} \int_{y_1=\varepsilon} e^{i(\widetilde{\phi}_+(y) - \phi_-(y))/h} d(x,y',h) a_-(y,h)\, dy',$$

with $z = (E - E_0)/h$. In view of the stationary and non-stationary method, the integral is of $\mathcal{O}(h^\infty)$ outside $\mathcal{H}$ and of $\mathcal{O}(h^{1/2})$ outside $\mathcal{H}_{\mathrm{tang}}$. Hence the principal contribution comes from $\mathcal{H}_{\mathrm{tang}} \cap \{y_1 = \varepsilon\}$. Then, modulo lower order terms, $|a_+(x,h)|$ satisfies

$$|a_+(x,h)| \leq \frac{h^{-|\operatorname{Im} z|/\lambda}}{(2\pi)^{n/2}} \int_{\mathcal{H}_{\mathrm{tang}} \cap \{y_1=\varepsilon\}} |d_0(x,y')| |a_-(y,h)|\, dy'.$$

We estimate the right hand side modifying the domain of integration to $\mathcal{H}_{\mathrm{tang}}^\infty \cap \{|y| = \varepsilon\}$. We see that, when $x = \varepsilon\alpha$, $y = \varepsilon\omega$ and $\varepsilon \to 0$,

$$I_+^\infty(x) = \mathcal{O}(\varepsilon), \qquad I_-^\infty(y) = -\frac{n-1}{2} \log 2 + \mathcal{O}(\varepsilon),$$
$$\sqrt{|\det \nabla_{y'}^2 \phi_-(y)|} = \Big(\frac{\lambda}{2}\Big)^{\frac{n-1}{2}} + \mathcal{O}(\varepsilon), \qquad g(y) = y(1 + \mathcal{O}(\varepsilon)),$$

and hence, we have, since $|\operatorname{Im} z| = \mathcal{O}(1/|\log h|)$,

$$|d_0(x,y)| = |\alpha\cdot\omega|^{-n/2}\varepsilon^{1-n}\exp\Big(-\frac{\pi\sigma}{2}s\Big)\Gamma\Big(\frac{n}{2}-is\Big)\big(1+\mathcal{O}(\varepsilon)\big),$$

uniformly with respect to α and ω. Recall that $\sigma = \operatorname{sign}(\alpha\cdot\omega)$, $s = (\operatorname{Re} E - E_0)/(\lambda h)$. It follows that

$$|a_+(\varepsilon\alpha,h)| \leq \frac{h^{-|\operatorname{Im} z|/\lambda}}{(2\pi)^{n/2}}\Big|\Gamma\Big(\frac{n}{2}-is\Big)\Big|\int_{\mathcal{H}^\infty_{\rm tang}}\exp\Big(-\frac{\pi\sigma}{2}s\Big)|\alpha\cdot\omega|^{-n/2}\big(1+\mathcal{O}(\varepsilon)\big)|a_-(\varepsilon\omega,h)|\,d\omega.$$

Combining with (4.4), we obtain

$$\begin{aligned}|a_-(\varepsilon\beta,h)| \leq \mathcal{M}_\varepsilon(\alpha)&\frac{h^{-|\operatorname{Im} z|/\lambda}}{(2\pi)^{n/2}}\Big|\Gamma\Big(\frac{n}{2}-is\Big)\Big|\\ &\times\int_{\mathcal{H}^\infty_{\rm tang}}\exp\Big(-\frac{\pi\sigma}{2}s\Big)|\alpha\cdot\omega|^{-n/2}\big(1+\mathcal{O}(\varepsilon)\big)|a_-(\varepsilon\omega,h)|\,d\omega,\end{aligned}$$

where $\beta=\beta(\alpha)$ is the asymptotic direction of $x_+(t,\alpha)$ as $t\to+\infty$. Using the definition (2.10), we get

$$|a_-(\varepsilon\beta,h)| \leq h^{-|\operatorname{Im} z|/\lambda}\mathcal{A}_0(s)\big(1+o_{\varepsilon\to0}(1)\big)\max_{\omega\in\mathcal{H}^\infty_{\rm tang}}|a_-(\varepsilon\omega,h)|.$$

Taking the maximum of the left hand side over $\beta\in\mathcal{H}^\infty_{\rm tang}$, it yields

$$\max_{\omega\in\mathcal{H}^\infty_{\rm tang}}|a_-(\varepsilon\omega,h)| \leq h^{-|\operatorname{Im} z|/\lambda}\mathcal{A}_0(s)\big(1+o_{\varepsilon\to0}(1)\big)\max_{\omega\in\mathcal{H}^\infty_{\rm tang}}|a_-(\varepsilon\omega,h)|.$$

On the other hand, for E satisfying (2.11), we have

$$h^{-|\operatorname{Im} z|/\lambda}\mathcal{A}_0(s) \leq e^{-\delta}.$$

Thus, we eventually obtain

$$\max_{\omega\in\mathcal{H}^\infty_{\rm tang}}|a_-(\varepsilon\omega,h)| \leq e^{-\delta}\big(1+o_{\varepsilon\to0}(1)\big)\max_{\omega\in\mathcal{H}^\infty_{\rm tang}}|a_-(\varepsilon\omega,h)|.$$

Taking $\varepsilon>0$ small enough so that $e^{-\delta}(1+o_{\varepsilon\to0}(1))<1$, it implies that $a_-=0$ and then $\|u\|=\mathcal{O}(h^\infty)$ which is a contradiction with the assumption $\|u\|=1$.

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