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Spectral properties of massless Dirac operators with real-valued potentials

By

KARL MICHAEL SCHMIDT* and TOMIO UMEDA**

Abstract

We prove that a Schrödinger-type theorem holds for massless Dirac operators under minimal assumptions on the potential, and apply this result to conclude that the spectrum of a certain class of such operators covers the whole real line. We also discuss embedded eigenvalues of massless Dirac operators with suitable scalar potentials.

§ 1. Introduction

This paper is an announcement of results on spectral properties of Dirac operators with real-valued potentials and will be followed by a complete treatment in which all proofs are given.

The Dirac operators to be considered in this paper are

(1.1) \[ H_2 = -i \sigma \cdot \nabla + q(x) \quad \text{in} \ L^2(\mathbb{R}^2; \mathbb{C}^2) \]

and

(1.2) \[ H_3 = -i \alpha \cdot \nabla + q(x) \quad \text{in} \ L^2(\mathbb{R}^3; \mathbb{C}^4). \]

Here \( \sigma = (\sigma_1, \sigma_2) \) and \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) are given as follows:

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

and

\[ \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad (j \in \{1, 2, 3\}) \quad \text{with} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
The dot products are to be read as
\[
\sigma \cdot \nabla = \sigma_1 \frac{\partial}{\partial x_1} + \sigma_2 \frac{\partial}{\partial x_2}
\]
in (1.1) and
\[
\alpha \cdot \nabla = \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}
\]
in (1.2). The potential \( q \) is a real-valued function on \( \mathbb{R}^d \), where \( d = 2 \) or \( d = 3 \), respectively. The operators \( H_2, H_3 \) differ from the standard Dirac operator in that they lack a mass term, usually represented by an additional anti-commuting matrix: \( \sigma_3 \) for the two-dimensional case and
\[
\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\]
for the three-dimensional case, where \( I \) is a \( 2 \times 2 \) identity matrix.

The purpose of the present paper is to show that \( \sigma(H_d) = \mathbb{R} \) under minimal assumptions on \( q \). In particular, we shall not require any restriction on the growth or decay of the potential \( q \) at infinity.

We have two motivations. First, the spectrum of the one-dimensional massless Dirac operator
\[
H_1 = -i\sigma_2 \frac{d}{dx} + q(x) \quad \text{in } L^2(\mathbb{R}; \mathbb{C}^2)
\]
covers the whole real axis and is purely absolutely continuous whenever \( q \in L_{loc}^1(\mathbb{R}, \mathbb{R}) \). This surprising fact was first pointed out by one of the authors in [8]. By separation in spherical polar coordinates, this result also implies that \( \sigma(H_d) = \mathbb{R} \) if \( q \) is rotationally symmetric; see [9]. Second, it is believed that the energy spectrum of graphene, in which electron transport is governed by Dirac equations in two dimensions without a mass term, has no bandgap (zero bandgap); see [2], [4], [7]. For these reasons, it is natural to make an attempt to show that \( \sigma(H_d) = \mathbb{R} \) under minimal assumptions on \( q \).

\section{Embedded eigenvalues}

It is difficult to imagine that the spectra of \( H_2 \) and \( H_3 \) are always purely absolutely continuous regardless of \( q \). Actually, in the three-dimensional case, we have an example of \( q \) which gives rise to a zero mode of \( H_3 \), \( i.e. \) an example of \( q \) for which \( H_3 \) has the embedded eigenvalue 0.

Example 1. Let \( q(x) = -3/\langle x \rangle^2 \), where \( \langle x \rangle = \sqrt{1 + |x|^2} \). Then there exists a unique self-adjoint realization of \( H_3 \) in \( L^2(\mathbb{R}^3; \mathbb{C}^4) \) with \( \text{Dom}(H_3) = H^1(\mathbb{R}^3; \mathbb{C}^4) \), the
Sobolev space of order 1. If one puts

\[ f(x) = \langle x \rangle^{-3}(I_4 + i\alpha \cdot x)\phi_0 \]

with \( \phi_0 \) a unit vector in \( \mathbb{C}^4 \), then a direct calculation shows that \( H_3 f = 0 \). This implies that \( 0 \in \sigma_p(H_3) \), because \( f \in \text{Dom}(H_3) \). Thus \( H_3 \) has a zero mode. Since \( \sigma(H_3) = \mathbb{R} \), the energy 0 is an embedded eigenvalue of \( H_3 \).

The potential \( q \) and the zero mode \( f \) in Example 1 were motivated by [6].

Example 2 below indicates that there is a difference in spectral property between \( H_3 \) and \( H_2 \). In fact, quite a similar construction in Example 2 below gives a zero resonance of \( H_2 \), not a zero mode of \( H_2 \). On the other hand, we do not know if the potential \( q \) in Example 2 gives rise to a zero mode of \( H_2 \).

**Example 2.** Let \( q(x) = -2/\langle x \rangle^2 \). Then there exists a unique self-adjoint realization of \( H_2 \) in \( L^2(\mathbb{R}^2;\mathbb{C}^2) \) with \( \text{Dom}(H_2) = H^1(\mathbb{R}^2;\mathbb{C}^2) \), and \( \sigma(H_2) = \mathbb{R} \). If

\[ \psi(x) = \langle x \rangle^{-2}(I_2 + i\sigma \cdot x)\phi_0, \]

\( \phi_0 \) a unit vector in \( \mathbb{C}^2 \), then one sees that \( H_2 \psi = 0 \). However, it is clear that \( \psi \not\in L^2(\mathbb{R}^2;\mathbb{C}^2) \). Therefore, \( \psi \) is not a zero mode of \( H_2 \). On the other hand, one finds that \( \psi \in L^{2,-s}(\mathbb{R}^2;\mathbb{C}^2) \) for \( \forall s > 0 \), where

\[ L^{2,-s}(\mathbb{R}^2;\mathbb{C}^2) = \{ \varphi \mid \|\langle x \rangle^{-s}\varphi\|_{L^2} < \infty \}. \]

This means that \( \psi \) is a zero resonance of \( H_2 \).

It is not an easy task to clarify whether \( H_d \) has embedded eigenvalues for general potentials. However, we have a good control of the embedded eigenvalues of \( H_d \) if \( q(x) \) is rotationally symmetric. To formulate a result, we need to introduce the definition of the limit range \( \mathcal{R}_\infty(q) \) of \( q \):

\[ \mathcal{R}_\infty(q) = \bigcap_{r>0} \overline{\{q(x)\mid |x| \geq r\}}, \]

where \( \overline{A} \) denotes the closure of a subset \( A \subset \mathbb{R} \).

**Theorem 2.1 (Schmidt[9]).** Let \( q(x) = \eta(\langle x \rangle) \) and let \( \eta \in L^1_{loc}(0, \infty) \). Suppose that there exists a real number \( E \in \mathbb{R} \setminus \mathcal{R}_\infty(q) \) such that

\[ \frac{1}{r(E - \eta(r)) - 1} \in BV(r_0, \infty) \]

for some \( r_0 > 0 \), where \( BV(r_0, \infty) \) denotes the set of functions of bounded variations on the interval \( (r_0, \infty) \). Then \( \sigma_p(H_d) \subset \mathcal{R}_\infty(q) \) for \( d \in \{2, 3\} \).
Theorem 2.1 is a direct consequence of [9, Corollary 1]. If \( q \) is not rotationally symmetric, we can prove the following.

**Theorem 2.2.** Let \( q \in C^{1}(\mathbb{R}^{d};\mathbb{R}) \), \( d \in \{2, 3\} \), and suppose that both \( q \) and \((x \cdot \nabla)q)\) are bounded functions. Then \( \sigma_{p}(H_{d}) \subset [m_{q}, M_{q}] \), where
\[
m_{q} = \inf_{x}\{q(x) + (x \cdot \nabla)q(x)\}, \quad M_{q} = \sup_{x}\{q(x) + (x \cdot \nabla)q(x)\}.
\]

The proof of Theorem 2.2 is based on a virial theorem in an abstract setting; see [1, Lemma 2.1].

§ 3. Schnol’s theorem for Dirac operators

We now prepare a Schnol’ theorem for Dirac operators. As for Schnol’s theorem, we refer the reader [3, p.21, Theorem 2.9] which is a characterization of the spectra of Schrödinger operators in terms of polynomially bounded eigenfunctions. In the three-dimensional Dirac operators, the theorem can be stated as follows:

**Theorem 3.1.** Let \( q \in L_{\text{loc}}^{2}(\mathbb{R}^{3};\mathbb{R}) \), and let \( E \) be a real number. Suppose \( f \) is a polynomially bounded measurable function on \( \mathbb{R}^{3} \), not identically 0, and satisfies the equation
\[
(-i \alpha \cdot \nabla + q)f = Ef
\]
in the distribution sense. Then \( E \in \sigma(H_{3}) \) for any self-adjoint realization \( H_{3} \) such that \( \text{Dom}(H_{3}) \supset H^{1}(\mathbb{R}^{3};\mathbb{C}^{4}) \cap \text{Dom}(q) \).

**Outline of the proof of Theorem 3.1.** We follow the line of the proof of [3, p. 21, Theorem 2.9].

We may suppose \( f \not\in L^{2}(\mathbb{R}^{3};\mathbb{C}^{4}) \) without loss of generality. Let \( \varphi \in C_{0}^{\infty}(\mathbb{R}^{3}) \) such that \( \varphi(x) = 1 \ (|x| \leq 1) \) and \( \varphi(x) = 0 \ (|x| \geq 2) \), and put \( \varphi_{n}(x) = \varphi(x/n) \) for \( n \in \mathbb{N} \). Then introducing a monotonically increasing sequence \( (M(n))_{n\in\mathbb{N}} \) by
\[
M(n) = \int_{|x|\leq n} |f(x)|^{2} \, dx,
\]
and defining \( f_{n} := \varphi_{n}f/\|\varphi_{n}f\|_{L^{2}} \), we can show that there exists a positive constant \( C \) such that
\[
\|(H_{3} - E)f_{n}\|_{L^{2}}^{2} \leq C \frac{M(2n) - M(n)}{n^{2}M(n)}.
\]
for all \( n \).
On the other hand, we can deduce that
\[
\lim_{n \to \infty} \inf_{\infty} \frac{M(2n) - M(n)}{n^2 M(n)} = 0,
\]
which implies that there exists a subsequence \((M(n_k))_{k \in \mathbb{N}}\) such that
\[
(3.3) \quad \lim_{k \to \infty} \frac{M(2n_k) - M(n_k)}{n_k^2 M(n_k)} = 0.
\]
Combining (3.2) with (3.3), we can conclude that \(E \in \sigma(H_3)\). ■

We give an application of the Schnol’ theorem, and show that the spectra of massless Dirac operators with real-valued potentials always coincide with the whole real axis, provided that the potentials are of the form specified in Theorem 3.2 below.

**Theorem 3.2.** Let \(\eta \in C^1(\mathbb{R};\mathbb{R})\) and define \(q(x) := \eta(x \cdot k)\) on \(\mathbb{R}^d\), \(d \in \{2, 3\}\), where \(k \in \mathbb{R}^d\) is a unit vector. Then \(\sigma(H_d) = \mathbb{R}\).

**Outline of the proof of Theorem 3.2.** We only give the proof for \(d = 3\). Put
\[
\xi(t) = \int_0^t \eta(\tau) d\tau.
\]
Choose a unit vector \(\phi_0 \in \mathbb{C}^4, \neq 0\), so that \((\alpha \cdot k)\phi_0 = \phi_0\). For a given \(E \in \mathbb{R}\), define
\[
f(x) = e^{-i(\alpha \cdot k)\xi(x \cdot k)} e^{iEx \cdot k}\phi_0.
\]
Then \(f\) satisfies the equation (3.1). Moreover, \(f \in C^1(\mathbb{R}^3;\mathbb{C}^4)\), and \(|f(x)|_{\mathbb{C}^4} = 1\) for all \(x \in \mathbb{R}^3\). It follows from Theorem 3.1 that \(E \in \sigma(H_3)\). Since \(E\) is an arbitrary real number, we can conclude that \(\sigma(H_3) = \mathbb{R}\). ■

§ 4. The main result

We now state the main theorem, which greatly generalizes Theorem 3.2.

**Theorem 4.1.** Let \(q \in L^2_{\text{loc}}(\mathbb{R}^3;\mathbb{R})\). Suppose that there is a sequence \((k_n)_{n \in \mathbb{N}}\) of unit vectors in \(\mathbb{R}^3\), a sequence \((B_{r_n}(a_n))_{n \in \mathbb{N}}\) of balls with centre \(a_n \in \mathbb{R}^3\) and radius \(r_n \to \infty\) \((n \to \infty)\), and a sequence of square-integrable functions \(q_n : (-r_n, r_n) \to \mathbb{R}\) \((n \in \mathbb{N})\) such that
\[
r_n^{-3} \int_{B(a_n, r_n)} \left| q(x) - q_n((x - a_n) \cdot k_n) \right|^2 dx \to 0
\]
as \(n \to \infty\). Then \(\sigma(H_3) = \mathbb{R}\) for any self-adjoint extension \(H_3\) of
\[
(-i\alpha \cdot \nabla + q)|_{C_0^\infty(\mathbb{R}^3)^4}.
\]
The two dimensional analogue of the statements above holds true.
Outline of the proof of Theorem 4.1. For a given energy $E \in \mathbb{R}$, we shall construct a singular sequence $(h_n)_{n \in \mathbb{N}}$, thus showing that $E \in \sigma(H_3)$. To this end, we first choose a function $\tilde{\eta}_n \in C^\infty(-r_n, r_n)$ so that

$$\frac{1}{2r_n} \int_{-r_n}^{r_n} |q_n(t) - \tilde{\eta}_n(t)|^2 \, dt \to 0$$

as $n \to \infty$. Then following the idea in the proof of Theorem 3.2, we put

$$\xi_n(t) = \int_0^t \tilde{\eta}_n(\tau) \, d\tau,$$

and choose a sequence of unit vectors $\phi_n \in \mathbb{C}^4$, $\neq 0$, so that $(\alpha \cdot k_n)\phi_n = \phi_n$, and define a sequence of functions $(f_n)_{n \in \mathbb{N}}$ by

$$f_n(x) = e^{-i(\alpha \cdot k_n)\xi_n((x-a_n) \cdot k_n)}e^{iE x \cdot k_n} \phi_n : B_{r_n}(a_n) \to \mathbb{C}^4.$$

We now choose a function $\chi \in C^\infty_0(\mathbb{R}^3)$ such that $\text{supp}\chi \subset B_1(0)$ and that $\|\chi\|_{L^2} = 1$. Putting

$$h_n(x) = r_n^{-3/2} \chi\left(\frac{x-a_n}{r_n}\right) f_n(x),$$

we can obtain the desired singular sequence. ■

References