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On the level statistics problem for the one-dimensional Schrödinger operator with random decaying potential

By

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Abstract

In this note, we review our recent work [2] on the level statistics problem of the one-dimensional Schrödinger operator with random potential decaying like $x^{-\alpha}$ at infinity. We proved that (i)(ac spectrum case) for $\alpha > \frac{1}{2}$, the point process ξ_L consisting of the rescaled eigenvalues converges to a clock process, and the fluctuation of the eigenvalue spacing converges to Gaussian. (ii)(critical case) for $\alpha = \frac{1}{2}$, ξ_L converges to the limit of the circular β-ensemble.

§ 1. Introduction

In this paper, we study the following Schrödinger operator

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on } L^2(\mathbf{R})$$

where $a \in C^\infty$ is real valued, $a(-t) = a(t)$, non-increasing for $t \geq 0$, and satisfies

$$C_1 t^{-\alpha} \leq a(t) \leq C_2 t^{-\alpha}$$

for some positive constants C_1, C_2. F is a real-valued, smooth, non-constant function on a compact Riemannian manifold M. $\{X_t\}$ is a Brownian motion on M. Since the potential $a(t)F(X_t)$ is $-\frac{d^2}{dt^2}$-compact, we have $\sigma_{ess}(H) = [0, \infty)$. For the nature of the spectrum of H in $[0, \infty)$, Kotani-Ushiroya[4] proved that

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(1) for $\alpha < \frac{1}{2}$: the spectrum on $[0, \infty)$ is pure point with exponentially decaying eigenfunctions,
(2) for $\alpha = \frac{1}{2}$: the spectrum on is pure point on $[0, E_c]$ and purely singular continuous on $[E_c, \infty)$ with some explicitly computable E_c,
(3) for $\alpha > \frac{1}{2}$: if we furthermore assume $\langle F \rangle := \int_M F(x)dx = 0$, the spectrum on $[0, \infty)$ is purely absolutely continuous.

In this note we report our results on the level statistics on this operator. For that purpose, let $H_L := H|_{[0,L]}$ be the local Hamiltonian with Dirichlet boundary condition and let $\{E_n(L)\}_{n=1}^{\infty}$ be its eigenvalues in the increasing order. Let $n(L) \in \mathbf{N}$ be s.t. $\{E_n(L)\}_{n \geq n(L)}$ coincides with the set of non-negative eigenvalues of H_L. We arbitrarily take the reference energy $E_0 > 0$ and consider the following point process

$$\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}$$

in order to study the local fluctuation of eigenvalues near E_0. Our aim is to identify the limit of ξ_L as $L \to \infty$.

This problem was first studied by Molchanov[7]. He proved that, when $\alpha = 0$, ξ_L converges to the Poisson process. Killip-Stoiciu [3] studied the CMV matrices whose matrix elements decay like $n^{-\alpha}$. They showed that (i) for $\alpha > \frac{1}{2}$: ξ_L converges to the clock process, (ii) for $\alpha = \frac{1}{2}$: ξ_L converges to the limit of the β-ensemble, (iii) for $0 < \alpha < \frac{1}{2}$: ξ_L converges to the Poisson process. Krichevski-Valko-Virag[6] studied the one-dimensional discrete Schrödinger operator with the random potential decaying like $n^{-1/2}$, and proved that ξ_L converges to the Sine$_\beta$-process. For the multidimensional Anderson-type model, we refer to [8] who proved the convergence to the Poisson process in the localized regime. The aim of our work is to do the analog of that by Killip-Stoiciu[3] for the one-dimensional Schrödinger operator in the continuum.

In section 2(resp. section 3), we state our results for ac-case : $\alpha > \frac{1}{2}$ (resp. critical-case : $\alpha = \frac{1}{2}$)[1].

§ 2. AC-case

Definition 2.1. Let μ be a probability measure on $[0, \pi)$. We say that ξ is the clock process with spacing π w.r.t. μ if and only if

$$\mathbf{E}[e^{-\xi(f)}] = \int_0^{\pi} d\mu(\phi) \exp\left(-\sum_{n \in \mathbf{Z}} f(n\pi - \phi) \right)$$

where $f \in C_c(\mathbf{R})$ and $\xi(f) := \int_{\mathbf{R}} f d\xi$.

[1] We have not obtained results for pp-case : $\alpha < \frac{1}{2}$.

We set

$$(x)_{\pi\mathbf{Z}} := x - [x]_{\pi\mathbf{Z}}, \quad [x]_{\pi\mathbf{Z}} := \max\{k\pi \mid k\pi \le x, k \in \mathbf{Z}\}.$$

In order that ξ_L converge to a point process, we need to take subsequences. Thus we assume

(A)
(1) $\alpha > \frac{1}{2}$, $\langle F \rangle = 0$.
(2) A sequence $\{L_k\}_{k=1}^{\infty}$ satisfies $\lim_{k\to\infty} L_k = \infty$ and

$$(\sqrt{E_0}L_k)_{\pi\mathbf{Z}} = \beta + o(1), \quad k \to \infty$$

for some $\beta \in [0, \pi)$.

In fact, if $a \equiv 0$, we need to take a subsequence satisfying (A)(2) in order that the limit exists, and the limit depends on β.

Theorem 2.2. Assume (A). Then ξ_{L_k} converges in distribution to the clock process with spacing π w.r.t. a probability measure μ_β on $[0, \pi)$.

Remark 2.3. Let x_t be the solution to the eigenvalue equation : $H_L x_t = \kappa^2 x_t$. If we set

$$\begin{pmatrix} x_t \\ x_t'/\kappa \end{pmatrix} = \begin{pmatrix} r_t \sin\theta_t \\ r_t \cos\theta_t \end{pmatrix}, \quad \theta_t = \kappa t + \tilde{\theta}_t,$$

then $\tilde{\theta}_t$ has a limit as t goes to infinity[4] : $\lim_{t\to\infty} \tilde{\theta}_t = \tilde{\theta}_\infty$, a.s. μ_β is the distribution of the random variable $(\beta + \tilde{\theta}_\infty(\sqrt{E_0}))_{\pi\mathbf{Z}}$.

Remark 2.4. We can consider point processes w.r.t. two reference energies $E_0, E_0'(E_0 \neq E_0')$ at the same time : suppose a sequence $\{L_k\}_{k=1}^{\infty}$ satisfies

$$(\sqrt{E_0}L_k)_{\pi\mathbf{Z}} = \beta + o(1), \quad (\sqrt{E_0'}L_k)_{\pi\mathbf{Z}} = \beta' + o(1), \quad k \to \infty$$

for some $\beta, \beta' \in [0, \pi)$. We set

$$\xi_L := \sum_{n \ge n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}, \quad \xi_L' := \sum_{n \ge n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0'})}.$$

Then the joint distribution of ξ_{L_k}, ξ_{L_k}' converges, for $f, g \in C_c(\mathbf{R})$,

$$\lim_{k\to\infty} \mathbf{E}\left[\exp\left(-\xi_{L_k}(f) - \xi_{L_k}(g)\right)\right]$$
$$= \int_0^{\pi} d\mu(\phi, \phi') \exp\left(-\sum_{n\in\mathbf{Z}} (f(n\pi - \phi) + g(n\pi - \phi'))\right)$$

where $\mu(\phi,\phi')$ is the joint distribution of $(\beta+\tilde{\theta}_\infty(\sqrt{E_0}))_{\pi\mathbf{Z}}$ and $(\beta'+\tilde{\theta}_\infty(\sqrt{E_0'}))_{\pi\mathbf{Z}}$. We are unable to identify $\mu(\phi,\phi')$ but it may be possible that ϕ,ϕ' are correlated.

Remark 2.5. If we rearrange eigenvalues

$$\cdots < E'_{-2}(L) < E'_{-1}(L) < E_0 \le E'_0(L) < E'_1(L) < E'_2(L) < \cdots$$

near the reference energy E_0, then for any $j \in \mathbf{Z}$ we have

$$\lim_{L\to\infty} L\left(\sqrt{E'_{j+1}(L)} - \sqrt{E'_j(L)}\right) = \pi, \quad a.s. \tag{2.1}$$

which is called the strong clock behavior [1].

We next study the finer structure of the eigenvalue spacing under the following assumption.

(B)
(1) $\frac{1}{2} < \alpha < 1$, $\langle F \rangle = 0$,
(2) A sequence $\{L_k\}_{k=1}^{\infty}$ satisfies $\lim_{k\to\infty} L_k = \infty$ and

$$\sqrt{E_0}L_k = a_k\pi + \beta + o(1), \quad k \to \infty$$

for some $\{a_k\}_{k=1}^{\infty}(\subset \mathbf{N})$ and $\beta \in [0,\pi)$,
(3) $a(t) = t^{-\alpha}(1+o(1))$, $t \to \infty$.

Roughly speaking, $E_{a_k}(L)$ is the eigenvalue closest to E_0. In view of (2.1), we set

$$X_k(n) := \left\{ \left(\sqrt{E_{a_k+n+1}(L_k)} - \sqrt{E_{a_k+n}(L_k)} \right) L_k - \pi \right\} L_k^{\alpha-\frac{1}{2}}, \quad n \in \mathbf{Z}.$$

Theorem 2.6. Assume (B). Then $\{X_k(n)\}_{n\in\mathbf{Z}}$ converges in distribution to the Gaussian system with covariance

$$C(n,n') = \frac{C(E_0)}{8E_0} Re \int_0^1 s^{-2\alpha} e^{i(n-n')\pi s} 2(1-\cos\pi s)ds, \quad n,\ n' \in \mathbf{Z},$$
$$\text{where } C(E) := \int_M \left| \nabla (L + 2i\sqrt{E})^{-1} F \right|^2 dx$$

L is the generator of (X_t).

Remark 2.7. Suppose we consider two reference energies $E_0, E_0' (E_0 \neq E_0')$ at the same time and suppose a sequence $\{L_k\}_{k=1}^{\infty}$ satisfies $\lim_{k\to\infty} L_k = \infty$ and

$$\sqrt{E_0}L_k = a_k\pi + \beta + o(1), \quad \sqrt{E_0'}L_k = b_k\pi + \beta' + o(1), \quad k \to \infty$$

for some $a_k, b_k \in \mathbf{N}$, and $\beta, \beta' \in [0,\pi)$. Then $\{X_k(n)\}_n, \{X'_k(m)\}_m$ converge jointly to the mutually independent Gaussian systems.

§ 3. Critical-case

Definition 3.1. The circular β-ensemble with n-points is given by

$$\mathbf{E}_n^\beta[G] := \frac{1}{Z_{n,\beta}} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_n}{2\pi} G(\theta_1, \cdots, \theta_n) |\triangle(e^{i\theta_1}, \cdots, e^{i\theta_n})|^\beta$$

where $Z_{n,\beta}$ is the normalization constant, $G \in C(\mathbf{T}^n)$ is bounded and $\triangle$ is the Vandermonde determinant. The limit ξ_β of the circular β-ensemble is defined

$$\mathbf{E}[e^{-\xi_\beta(f)}] = \lim_{n\to\infty} \mathbf{E}_n^\beta \left[\exp\left(-\sum_{j=1}^n f(n\theta_j) \right) \right], \quad f \in C_c(\mathbf{R})$$

whose existence and characterization is given by [3]. We set the following assumption.

(C) $\langle F \rangle = 0$ and $a(t) = t^{-\frac{1}{2}}(1 + o(1)), \quad t \to \infty.$

Theorem 3.2. Assume (C). Writing $\xi_\beta = \sum_j \delta_{\lambda_j}$, let $\xi'_\beta := \sum_j \delta_{\lambda_j/2}$. Then $\xi_L \stackrel{d}{\to} \xi'_\beta$ with $\beta = \frac{8E_0}{C(E_0)^2}$.

Remark 3.3. The corresponding $\beta = \beta(E_0) = \frac{8E_0}{C(E_0)}$ depends on the reference energy E_0, so that the spacing distribution may change if we look at the different region in the spectrum. To see how β changes, we recall some results in [4]. Let $\sigma_F(\lambda)$ be the spectral measure of the generator L of $\{X_t\}$ with respect to F. Then

$$\gamma(E) := -\frac{1}{4E} \int_{-\infty}^0 \frac{\lambda}{\lambda^2 + 4E} \, d\sigma_F(\lambda), \quad E > 0$$

is the Lyapunov exponent in the sense that any generalized eigenfunction ψ_E of H satisfies

$$\lim_{|t|\to\infty} (\log t)^{-1} \log \left\{ \psi_E(t)^2 + \psi'_E(t)^2 \right\}^{1/2} = -\gamma(E), \quad a.s..$$

Moreover $E < E_c$ (resp. $E > E_c$) if and only if $\gamma(E) > \frac{1}{2}$ (resp. $\gamma(E) < \frac{1}{2}$) and $\gamma(E_c) = \frac{1}{2}$. Since $C(E) = 8E \cdot \gamma(E)$, we have $\beta(E) = \frac{1}{\gamma(E)}$. It then follows that $E < E_c$ (resp. $E > E_c$) if and only if $\beta(E) < 2$ (resp. $\beta(E) > 2$) and $\beta(E_c) = 2$ (Figure 1.). Similar statement also holds for discrete Hamiltonian and CMV matrices. This is consistent with our general belief that in the point spectrum (resp. in the continuous spectrum) the level repulsion is weak (resp. strong).

We note that, for $\beta = 2$, the circular β-ensemble with n-points coincides with the eigenvalue distribution of the unitary ensemble with the Haar measure on $U(n)$. In [9], Valkó-Virág showed that Sine$_\beta$ process has a phase transition at $\beta = 2$.

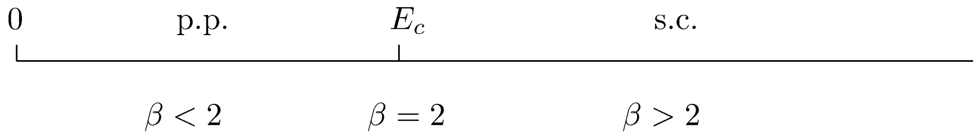

Figure 1. Spectrum and corresponding β.

Remark 3.4. If we consider two reference energies $E_0, E_0' (E_0 \neq E_0')$, then the corresponding point process ξ_L, ξ_L' converges jointly to the independent $\xi_\beta, \xi'_{\beta'}$.

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