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Supercoset construction of Yang–Baxter-deformed $AdS_5 \times S^5$ backgrounds

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We study Yang–Baxter deformations of the $AdS_5 \times S^5$ superstring with the classical Yang–Baxter equation. We make a general argument on the supercoset construction and present a formula to describe the dilaton in terms of classical r-matrices. The supercoset construction is explicitly performed for some classical r-matrices, and the full backgrounds including the Ramond–Ramond (R–R) sector and dilaton are derived. Within the class of Abelian r-matrices, perfect agreement is shown for well-known examples including gravity duals of non-commutative gauge theories, γ-deformations of S^5 and Schrödinger spacetimes. It is remarkable that the supercoset construction works well, even if the resulting backgrounds are not maximally supersymmetric. In particular, three-parameter γ-deformations of S^5 and Schrödinger spacetimes do not preserve any supersymmetries. As for non-Abelian r-matrices, we will focus upon a specific example. The resulting background does not satisfy the equation of motion of the Neveu–Schwarz–Neveu–Schwarz two-form because the R–R three-form is not closed.

Subject Index B20, B21, B81

1. Introduction

The Yang–Baxter deformation [1–3] is a systematic way to study integrable deformations of non-linear sigma models in two dimensions. Given a classical r-matrix satisfying the classical Yang–Baxter equation (CYBE), an integrable deformation is determined and the associated Lax pair follows automatically. This correspondence between a deformed geometry and a classical r-matrix indicates a profound connection between a differential geometry and a finite-size matrix. Hence it is significant to make the understanding of the Yang–Baxter deformation much deeper from the viewpoints of theoretical physics and pure mathematics.

The Yang–Baxter deformation was originally invented for principal chiral models with the modified classical Yang–Baxter equation (mCYBE). Now that it is generalized to symmetric cosets [4] and the homogeneous CYBE [5], one can study Yang–Baxter deformations of symmetric coset sigma models with a lot of examples of classical r-matrices. For the related affine algebras, see the series of works [6–14].

The most interesting coset sigma model is type IIB string theory on $AdS_5 \times S^5$ in the context of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [15]. The classical string action has been constructed in the Green–Schwarz formulation based on a supercoset [16]

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}.$$

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This coset enjoys the $\mathbb{Z}_4$-grading property and ensures classical integrability [17] (for a nice review, see [18]). The integrability plays an important role in checking the conjectured relation in AdS/CFT (for a comprehensive review, see [19]).

By employing the Yang–Baxter deformation, Delduc, Magro, and Vicedo constructed the classical action of a q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring [20,21]. This deformation comes from the classical r-matrix of Drinfel'd–Jimbo type satisfying the mCYBE [22–24]. The string frame metric and Neveu–Schwarz–Neveu–Schwarz (NS–NS) two-form were derived by Arutyunov, Borsato, and Frolov [25]. Then they performed the supercoset construction and derived the remaining sector [26] (for earlier attempts, see [27,28]). As a result, the full background does not satisfy the equations of motion of type IIB supergravity, although it is related to a complete solution [29] via T-dualities, apart from the dilaton part. In particular, the dilaton cannot be separated so that the Ramond–Ramond (R–R) flux should satisfy the Bianchi identity. Recently, Arutyunov et al. proposed an exciting conjecture that type IIB supergravity itself would get deformed; for example, the definition of R–R field strength may be modified [30]. This "modified gravity conjecture" may be connected to our result presented here.

One may also consider Yang–Baxter deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring with classical r-matrices satisfying the homogeneous CYBE [31]. A strong advantage in this case is that partial deformations of $\mathrm{AdS}_5 \times \mathrm{S}^5$ are possible. In fact, for well-known backgrounds including gravity duals of non-commutative gauge theories [32,33], γ-deformations of S^5 [34,35], and Schrödinger spacetimes [36–38], the associated classical r-matrices have been identified in a series of works [39–45] (for short summaries, see [46,47]). However, the analysis has been limited to the bosonic sector so far, and it is still necessary to confirm the R–R sector and dilaton by performing the supercoset construction explicitly.

The goal of this present work is to perform the supercoset construction and present the resulting backgrounds for some classical r-matrices. We will first give a general treatment basically by following the seminal paper by Arutyunov, Borsato, and Frolov [26]. Then we derive the backgrounds for some classical r-matrices. As a byproduct, we present the master formula to describe the dilaton in terms of classical r-matrices.

Within the class of Abelian classical r-matrices, the perfect agreement is shown for well-known examples including gravity duals of non-commutative gauge theories [32,33], γ-deformations of S^5 [34,35], and Schrödinger spacetimes [36–38]. It is worth noting that the supercoset construction works well, even though the resulting backgrounds are not maximally supersymmetric. More strikingly, three-parameter γ-deformations of S^5 and Schrödinger spacetimes do not preserve any supersymmetries [36–38]. Hence it seems likely that the supercoset construction works well with the class of the Abelian classical r-matrices. It is consistent with the interpretation as TsT transformations [12–14,35,45,48–50].

As for non-Abelian classical r-matrices, we will focus on a specific example discussed in [42,43]. The resulting background does not satisfy the equation of motion of the NS–NS two-form because the Bianchi identity of the R–R three-form is broken, namely the field strength is not closed. It is also remarkable that this background is different from the one proposed in [42,43], and hence the identification made in [42,43] was not correct. Anyway, this result indicates that there would be some potential problems in the non-Abelian cases. This is really intriguing, but just an example. It is of importance to study extensively other non-Abelian r-matrices.

This paper is organized as follows. Section 2 introduces the classical action of the Yang–Baxter-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring based on the CYBE. In Sect. 3, we discuss the supercoset construction, following the procedure of [26]. Most of the argument does not rely on specific expressions of classical r-matrices and is quite general. We present the conjectured master formula to describe the dilaton in terms of classical r-matrices. In Sect. 4, we present the resulting backgrounds for concrete examples of classical r-matrices. Section 5 is devoted to the conclusion and discussion. Appendix A provides a matrix representation of the superalgebra $\mathfrak{su}(2,2|4)$.

2. Yang–Baxter-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring

In this section, we give a short introduction to the classical action of Yang–Baxter-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring based on the homogeneous CYBE [31]. This construction basically follows from the work with the mCYBE [20,21].

The deformed classical action of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring is given by

$$S = -\frac{\sqrt{\lambda_c}}{4}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\, (\gamma^{ab}-\epsilon^{ab})\,\mathrm{STr}\Big[A_a\, d\circ \frac{1}{1-\eta R_g\circ d}(A_b)\Big], \tag{2.1}$$

where the left-invariant one-form A_a is defined as

$$A_a \equiv -g^{-1}\partial_a g, \qquad g\in SU(2,2|4), \tag{2.2}$$

with the world-sheet index $a=(\tau,\sigma)$. Here, the conformal gauge is supposed and the world-sheet metric is taken to be the diagonal form $\gamma^{ab}=\mathrm{diag}(-1,+1)$. Hence there is no coupling of the dilaton to the world-sheet scalar curvature. The anti-symmetric tensor ϵ^{ab} is normalized as $\epsilon^{\tau\sigma}=+1$. The constant λ_c in front of the action (2.1) is the 't Hooft coupling. The deformation is measured by a constant parameter η, and the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ action [16] is reproduced when $\eta=0$.

A key ingredient in Yang–Baxter deformations is the operator R_g defined as

$$R_g(X) \equiv g^{-1}R(gXg^{-1})g, \qquad X\in\mathfrak{su}(2,2|4), \tag{2.3}$$

where a linear operator $R:\mathfrak{su}(2,2|4)\to\mathfrak{su}(2,2|4)$ is a solution of the CYBE,[1]

$$[R(X),R(Y)]-R([R(X),Y]+[X,R(Y)])=0. \tag{2.4}$$

This R-operator is connected to a *skew-symmetric* classical r-matrix in the tensorial notation through the following formula:

$$R(X)=\mathrm{STr}_2[r(1\otimes X)]=\sum_i\left(a_i\,\mathrm{STr}[b_iX]-b_i\,\mathrm{STr}[a_iX]\right). \tag{2.5}$$

Here, r is represented by

$$r=\sum_i a_i\wedge b_i\equiv\sum_i(a_i\otimes b_i-b_i\otimes a_i) \qquad \text{with} \qquad a_i,\ b_i\in\mathfrak{su}(2,2|4). \tag{2.6}$$

The projection operator d is defined as

$$d\equiv P_1+2P_2-P_3, \tag{2.7}$$

where P_ℓ ($\ell=0,1,2,3$) are projections to the $\mathbb{Z}_4$-graded components of $\mathfrak{su}(2,2|4)$. In particular, $P_0(\mathfrak{su}(2,2|4))$ is a local symmetry of the classical action $\mathfrak{so}(1,4)\oplus\mathfrak{so}(5)$. The numerical coefficients in the linear combination (2.7) are fixed by requiring kappa symmetry [16,31].

[1] In the original work [31], a wider class of R-operators is discussed and their image is given by $\mathfrak{gl}(4|4)$. The $\mathfrak{gl}(4|4)$ image is restricted on $\mathfrak{su}(2,2|4)$ under the coset projection d, as pointed out in [44].

3. Supercoset construction

In this section, we shall consider the supercoset construction, starting from the deformed action (2.1). The following argument will be undertaken without fixing a specific expression of classical r-matrices and hence will be quite general. Our purpose here is to extract the R–R fluxes and dilaton, and hence we will investigate the deformed action at the quadratic level of fermions.

3.1. The $\mathfrak{su}(2,2|4)$ superalgebra

For the subsequent argument, it is necessary to determine our convention and notation for the $\mathfrak{su}(2,2|4)$ superalgebra. Hereafter, we will work with the following algebra [16]:

$$
\begin{aligned}
&[\mathbf{P}_{\check{m}},\mathbf{P}_{\check{n}}]=\mathbf{J}_{\check{m}\check{n}}, \quad [\mathbf{P}_{\hat{m}},\mathbf{P}_{\hat{n}}]=-\mathbf{J}_{\hat{m}\hat{n}},\\
&\left[\mathbf{P}_{\check{m}},\mathbf{J}_{\check{n}\check{p}}\right]=\eta_{\check{m}\check{n}}\,\mathbf{P}_{\check{p}}-\eta_{\check{m}\check{p}}\,\mathbf{P}_{\check{n}}, \quad \left[\mathbf{P}_{\hat{m}},\mathbf{J}_{\hat{n}\hat{p}}\right]=\eta_{\hat{m}\hat{n}}\,\mathbf{P}_{\hat{p}}-\eta_{\hat{m}\hat{p}}\,\mathbf{P}_{\hat{n}},\\
&\left[\mathbf{J}_{\check{m}\check{n}},\mathbf{J}_{\check{p}\check{q}}\right]=\eta_{\check{n}\check{p}}\,\mathbf{J}_{\check{m}\check{q}}+(3\text{ terms}), \quad \left[\mathbf{J}_{\hat{m}\hat{n}},\mathbf{J}_{\hat{p}\hat{q}}\right]=\eta_{\hat{n}\hat{p}}\,\mathbf{J}_{\hat{m}\hat{q}}+(3\text{ terms}),\\
&\left[\mathbf{Q}^I,\mathbf{P}_{\check{m}}\right]=-\frac{i}{2}\epsilon^{IJ}\,\mathbf{Q}^J\gamma_{\check{m}}, \quad \left[\mathbf{Q}^I,\mathbf{P}_{\hat{m}}\right]=\frac{1}{2}\epsilon^{IJ}\,\mathbf{Q}^J\gamma_{\hat{m}},\\
&\left[\mathbf{Q}^I,\mathbf{J}_{\check{m}\check{n}}\right]=-\frac{1}{2}\delta^{IJ}\,\mathbf{Q}^J\gamma_{\check{m}\check{n}}, \quad \left[\mathbf{Q}^I,\mathbf{J}_{\hat{m}\hat{n}}\right]=-\frac{1}{2}\delta^{IJ}\,\mathbf{Q}^J\gamma_{\hat{m}\hat{n}},\\
&\left\{(\mathbf{Q}^{\check{\alpha}\hat{\alpha}})^I,(\mathbf{Q}^{\check{\beta}\hat{\beta}})^J\right\}=\delta^{IJ}\left[iK^{\check{\alpha}\check{\gamma}}K^{\hat{\alpha}\hat{\beta}}(\gamma^{\check{m}})_{\check{\gamma}}{}^{\check{\beta}}\,\mathbf{P}_{\check{m}}-K^{\check{\alpha}\check{\beta}}K^{\hat{\alpha}\hat{\gamma}}(\gamma^{\hat{m}})_{\hat{\gamma}}{}^{\hat{\beta}}\,\mathbf{P}_{\hat{m}}-\frac{i}{2}K^{\check{\alpha}\check{\beta}}K^{\hat{\alpha}\hat{\beta}}\,\mathbf{1_8}\right]\\
&\qquad -\frac{1}{2}\epsilon^{IJ}\left[K^{\check{\alpha}\check{\gamma}}K^{\hat{\alpha}\hat{\beta}}(\gamma^{\check{m}\check{n}})_{\check{\gamma}}{}^{\check{\beta}}\,\mathbf{J}_{\check{m}\check{n}}-K^{\check{\alpha}\check{\beta}}K^{\hat{\alpha}\hat{\gamma}}(\gamma^{\hat{m}\hat{n}})_{\hat{\gamma}}{}^{\hat{\beta}}\,\mathbf{J}_{\hat{m}\hat{n}}\right].
\end{aligned}
\tag{3.1}
$$

The generators $\mathbf{P}_m$ are translations and the index $m=(\check{m},\hat{m})$ ($\check{m}=0,\ldots,4$; $\hat{m}=5,\ldots,9$) describes the ten-dimensional spacetime, where the indices $\check{m}$ and $\hat{m}$ are for AdS_5 and S^5, respectively. Then $\mathbf{J}_{\check{m}\check{n}}$ and $\mathbf{J}_{\hat{m}\hat{n}}$ describe rotations in AdS_5 and S^5, respectively. The supercharges $\mathbf{Q}^I$ ($I=1,2$) are written as $\mathbf{Q}^I=(\mathbf{Q}^{\check{\alpha}\hat{\alpha}})^I$ ($\check{\alpha}=1,\ldots,4$; $\hat{\alpha}=1,\ldots,4$). The anti-symmetric tensor ϵ^{IJ} ($I,J=1,2$) is normalized as $\epsilon^{12}=+1$. The constant matrices $K^{\check{\alpha}\check{\beta}}$ and $K^{\hat{\alpha}\hat{\beta}}$ are charge conjugation matrices in AdS_5 and S^5, respectively.

3.2. A group parametrization and the left-invariant current

Then let us introduce a parametrization of the group element $g\in SU(2,2|4)$ as follows:

$$
g=g_{\mathrm{b}}\,g_{\mathrm{f}}. \tag{3.2}
$$

Here, g_{b} is a bosonic element and parametrized with an appropriate coordinate system, depending on the backgrounds we are concerned with, as in the previous works [39–41]. We assume that the bosonic element is parametrized as

$$
\begin{aligned}
g_{\mathrm{b}}&={g_{\mathrm{b}}}^{\mathrm{AdS}_5}\,{g_{\mathrm{b}}}^{\mathrm{S}^5},\\
{g_{\mathrm{b}}}^{\mathrm{AdS}_5}&=\exp\!\left[x^0\,P_0+x^1\,P_1+x^2\,P_2+x^3\,P_3\right]\exp\!\left[(\log z)\,D\right],\\
{g_{\mathrm{b}}}^{\mathrm{S}^5}&=\exp\!\left[\frac{i}{2}(\phi_1\,h_1+\phi_2\,h_2+\phi_3\,h_3)\right]\exp\!\left[\zeta\,\mathbf{J}_{68}\right]\exp\!\left[-i\,r\,\mathbf{P}_6\right].
\end{aligned}
\tag{3.3}
$$

Note here that the translations P_μ, the dilatation D, and the Cartan generators of $\mathfrak{su}(4)$ h_i $(i = 1, 2, 3)$ are embedded into 8×8 matrices as

$$\begin{pmatrix} P_\mu & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix}, \quad \begin{pmatrix} D & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & h_i \end{pmatrix}. \tag{3.4}$$

The coordinates x^μ and z describe the Poincaré AdS_5, and r, ζ, $\phi_{i=1,2,3}$ parametrize the round S_5. Then g_f is a group element generated by the supercharges as follows:

$$\begin{aligned} g_\mathrm{f} &= \exp(\mathbf{Q}^I \theta_I) \qquad (I = 1, 2), \\ \text{where} \qquad \mathbf{Q}^I \theta_I &\equiv (\mathbf{Q}^{\check{\alpha}\hat{\alpha}})^I \, (\theta_{\check{\alpha}\hat{\alpha}})_I \qquad (\check{\alpha} = 1, \ldots, 4;\ \hat{\alpha} = 1, \ldots, 4). \end{aligned} \tag{3.5}$$

Here, $\theta_I = (\theta_{\check{\alpha}\hat{\alpha}})_I$ are Grassmann-odd coordinates and correspond to a couple of 16-component Majorana–Weyl spinors satisfying the Majorana condition:

$$\bar{\theta}_I \equiv \theta_I^\dagger \gamma^0 = {}^t\theta_I (K \otimes K)\,. \tag{3.6}$$

Then the left-invariant one-form A can be expanded as [16]

$$\begin{aligned} A = (e^m + \frac{i}{2}\bar{\theta}_I \gamma^m D^{IJ}\theta_J)\,\mathbf{P}_m - \mathbf{Q}^I\, D^{IJ}\theta_J + \frac{1}{2}\omega^{mn}\,\mathbf{J}_{mn} \\ - \frac{1}{4}\epsilon^{IJ}\bar{\theta}_I(\gamma^{\check{m}\check{n}}\,\mathbf{J}_{\check{m}\check{n}} - \gamma^{\hat{m}\hat{n}}\,\mathbf{J}_{\hat{m}\hat{n}})D^{JK}\theta_K, \end{aligned} \tag{3.7}$$

where the covariant derivative for θ is defined as

$$D^{IJ}\theta_J = \delta^{IJ}\left(d\theta_J - \frac{1}{4}\omega^{mn}\gamma_{mn}\theta_J\right) + \frac{i}{2}\epsilon^{IJ}\, e^m\,\gamma_m\theta_J. \tag{3.8}$$

Here, the last term represents the contribution of the R–R five-form field strength.

For later convenience, it is helpful to rearrange the above expansion of A with respect to the order of θ as follows:

$$A = A_{(0)} + A_{(1)} + A_{(2)}.$$

Here, $A_{(p)}$ is the p th order of θ and the explicit expressions of $A_{(p)}$ are given by

$$\begin{aligned} A_{(0)} &= e^m\,\mathbf{P}_m + \frac{1}{2}\omega^{mn}\,\mathbf{J}_{mn}, \\ A_{(1)} &= -\mathbf{Q}^I\, D^{IJ}\,\theta_J, \\ A_{(2)} &= \frac{i}{2}\bar{\theta}_I\gamma^m D^{IJ}\theta_J\,\mathbf{P}_m - \frac{1}{4}\epsilon^{IJ}\bar{\theta}_I(\gamma^{\check{m}\check{n}}\,\mathbf{J}_{\check{m}\check{n}} - \gamma^{\hat{m}\hat{n}}\,\mathbf{J}_{\hat{m}\hat{n}})D^{JK}\theta_K. \end{aligned} \tag{3.9}$$

Thus we have prepared to write down the undeformed action of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring at the quadratic order of θ.

3.3. Decomposing the deformation operator

In the case of the deformed action (2.1), it is further necessary to expand the deformation operator in terms of θ because R_g contains the adjoint operation with g as denoted in (2.3). We will basically follow the strategy of [26] hereafter.

Let us introduce the following operator $\mathcal{O}$ and expand it in terms of θ as

$$\begin{aligned}\mathcal{O} &\equiv 1 - \eta R_g \circ d \\ &= \mathcal{O}_{(0)} + \mathcal{O}_{(1)} + \mathcal{O}_{(2)} + \mathcal{O}(\theta^3).\end{aligned} \tag{3.10}$$

Then the inverse operator $\mathcal{O}^{\text{inv}}$ can also be expanded as

$$\begin{aligned}\mathcal{O}^{\text{inv}} &\equiv \frac{1}{1 - \eta R_g \circ d} \\ &= \mathcal{O}^{\text{inv}}_{(0)} + \mathcal{O}^{\text{inv}}_{(1)} + \mathcal{O}^{\text{inv}}_{(2)} + \mathcal{O}(\theta^3).\end{aligned} \tag{3.11}$$

Here, due to the relation $\mathcal{O} \circ \mathcal{O}^{\text{inv}} = 1$, each of the components $\mathcal{O}^{\text{inv}}_{(p)}$ $(p = 0, 1, 2)$ can be expressed as follows:

$$\begin{aligned}\mathcal{O}^{\text{inv}}_{(0)} &= \frac{1}{1 - \eta R_{g_b} \circ d}, \\ \mathcal{O}^{\text{inv}}_{(1)} &= -\mathcal{O}^{\text{inv}}_{(0)} \circ \mathcal{O}_{(1)} \circ \mathcal{O}^{\text{inv}}_{(0)}, \\ \mathcal{O}^{\text{inv}}_{(2)} &= -\mathcal{O}^{\text{inv}}_{(0)} \circ \mathcal{O}_{(2)} \circ \mathcal{O}^{\text{inv}}_{(0)} - \mathcal{O}^{\text{inv}}_{(1)} \circ \mathcal{O}_{(1)} \circ \mathcal{O}^{\text{inv}}_{(0)}.\end{aligned} \tag{3.12}$$

In the following, for simplicity, we will concentrate only on the bosonic deformations[2] generated by bosonic generators a_i, b_i like

$$a_i,\ b_i\ \in \mathfrak{su}(2,2) \oplus \mathfrak{su}(4). \tag{3.13}$$

Then the action of $R_{g_b} \circ d$ can be evaluated as

$$\begin{aligned}R_{g_b} \circ d(\mathbf{P}_m) &= 2\left(\lambda_m{}^n \mathbf{P}_n + \frac{1}{2}\lambda_m{}^{np}\mathbf{J}_{np}\right), \\ R_{g_b} \circ d(\mathbf{J}_{mn}) = 0, &\qquad R_{g_b} \circ d(\mathbf{Q}^I) = 0.\end{aligned} \tag{3.14}$$

Here, from the relation in (2.5), $\lambda_m{}^n$ and $\lambda_m{}^{np}$ are expressed as

$$\begin{aligned}\lambda_m{}^n &\equiv (a_i^{g_b})^n\,(b_i^{g_b})_m - (b_i^{g_b})^n\,(a_i^{g_b})_m, \\ \lambda_m{}^{np} &\equiv (a_i^{g_b})^{np}\,(b_i^{g_b})_m - (b_i^{g_b})^{np}\,(a_i^{g_b})_m,\end{aligned} \tag{3.15}$$

where $(a_i^{g_b})^m$, $(a_i^{g_b})^{mn}$, $(b_i^{g_b})^m$, and $(b_i^{g_b})^{mn}$ are defined as

$$\begin{aligned}a_i^{g_b} &\equiv g_b^{-1}\,a_i\,g_b = (a_i^{g_b})^m\,\mathbf{P}_m + \frac{1}{2}(a_i^{g_b})^{mn}\,\mathbf{J}_{mn}, \\ b_i^{g_b} &\equiv g_b^{-1}\,b_i\,g_b = (a_i^{g_b})^m\,\mathbf{P}_m + \frac{1}{2}(b_i^{g_b})^{mn}\,\mathbf{J}_{mn}.\end{aligned} \tag{3.16}$$

Now the action of $\mathcal{O}^{\text{inv}}_{(0)}$, $\mathcal{O}^{\text{inv}}_{(1)}$, and $\mathcal{O}^{\text{inv}}_{(2)}$ can be examined as follows.

The action of $\mathcal{O}^{\text{inv}}_{(0)}$ is given by

$$\begin{aligned}\mathcal{O}^{\text{inv}}_{(0)}(\mathbf{P}_m) &\equiv k_m{}^n\,\mathbf{P}_n + \frac{1}{2}l_m{}^{np}\,\mathbf{J}_{np}, \\ \mathcal{O}^{\text{inv}}_{(0)}(\mathbf{J}_{mn}) = \mathbf{J}_{mn}, &\qquad \mathcal{O}^{\text{inv}}_{(0)}(\mathbf{Q}^I) = \mathbf{Q}^I,\end{aligned} \tag{3.17}$$

[2] It would also be interesting to consider fermionic deformations. For such an attempt, see [31].

where ${k_m}^n$ is determined by the following relation:

$$
{k_m}^n = (\delta - 2\eta\lambda)^{-1}{}_m{}^n. \tag{3.18}
$$

When $\eta = 0$, ${k_m}^n$ is reduced to ${\delta_m}^n$. Here we have not displayed the explicit form of ${l_m}^{np}$, because it does not appear in the final expression due to the presence of the projection operators.

Then the action of $\mathcal{O}^{\text{inv}}_{(1)}$ is written as

$$
\begin{aligned}
&\mathcal{O}^{\text{inv}}_{(1)}(\mathbf{P}_m) = i\epsilon^{IJ}{k_m}^n\eta{\lambda_n}^p\,\mathbf{Q}^J\,\gamma_p\theta_I + \frac{1}{2}\delta^{IJ}{k_m}^n\eta{\lambda_n}^{pq}\,\mathbf{Q}^J\,\gamma_{pq}\theta_I,\\
&\mathcal{O}^{\text{inv}}_{(1)}(\mathbf{J}_{mn}) = 0,\\
&\mathcal{O}^{\text{inv}}_{(1)}(\mathbf{Q}^I) = i\sigma_3^{IJ}\,{k_m}^p\eta\lambda^{nm}\,\bar{\theta}_J\gamma_n\,\mathbf{P}_p + \frac{1}{2}\sigma_1^{IJ}\,{k_m}^q\eta\lambda^{m,np}\,\bar{\theta}_J\gamma_{np}\,\mathbf{P}_q + \text{terms with } \mathbf{J}.
\end{aligned} \tag{3.19}
$$

Here, the terms proportional to $\mathbf{J}_{mn}$ are not explicitly written down because they do not contribute to the final expression.

Finally, the action of $\mathcal{O}^{\text{inv}}_{(2)}$ is evaluated as

$$
\begin{aligned}
\mathcal{O}^{\text{inv}}_{(2)}(\mathbf{P}_m) = \bar{\theta}_I&\left[\delta^{IJ}{(\mathcal{M}^{P\delta}_{(2)})_m}^n + \epsilon^{IJ}{(\mathcal{M}^{P\epsilon}_{(2)})_m}^n + \sigma_1^{IJ}{(\mathcal{M}^{P\sigma_1}_{(2)})_m}^n + \sigma_3^{IJ}{(\mathcal{M}^{P\sigma_3}_{(2)})_m}^n\right]\theta_J\,\mathbf{P}_n\\
&+\ \text{terms with } \mathbf{J},\\
\mathcal{O}^{\text{inv}}_{(2)}(\mathbf{J}_{mn}) = 0,&\qquad \mathcal{O}^{\text{inv}}_{(2)}(\mathbf{Q}^I) = \text{irrelevant terms},
\end{aligned} \tag{3.20}
$$

where $\mathcal{M}^{P\delta}_{(2)}$, $\mathcal{M}^{P\epsilon}_{(2)}$, $\mathcal{M}^{P\sigma_1}_{(2)}$, and $\mathcal{M}^{P\sigma_3}_{(2)}$ are defined as

$$
\begin{aligned}
{(\mathcal{M}^{P\delta}_{(2)})_m}^n &\equiv -\frac{i}{4}\left[({k_r}^n\gamma^r)({k_m}^s\eta{\lambda_s}^{pq}\gamma_{pq}) - ({k_r}^n\eta\lambda^{r,pq}\gamma_{pq})({k_m}^s\gamma_s)\right],\\
{(\mathcal{M}^{P\epsilon}_{(2)})_m}^n &\equiv -\frac{1}{2}\left[({k_p}^n\gamma^p)({k_m}^s\eta{\lambda_s}^q\gamma_q) - ({k_p}^n\eta\lambda^{pq}\gamma_q)({k_m}^s\gamma_s)\right],\\
{(\mathcal{M}^{P\sigma_1}_{(2)})_m}^n &\equiv ({k_s}^n\eta\lambda^{rs}\gamma_r)({k_m}^q\eta{\lambda_q}^p\gamma_p) + \frac{1}{4}({k_s}^n\eta\lambda^{s,rt}\gamma_{rt})({k_m}^u\eta{\lambda_u}^{pq}\gamma_{pq}),\\
{(\mathcal{M}^{P\sigma_3}_{(2)})_m}^n &\equiv \frac{i}{2}\left[({k_s}^n\eta\lambda^{rs}\gamma_r)({k_m}^t\eta{\lambda_t}^{pq}\gamma_{pq}) + ({k_s}^n\eta\lambda^{s,rt}\gamma_{rt})({k_m}^q\eta{\lambda_q}^p\gamma_p)\right].
\end{aligned} \tag{3.21}
$$

Here, the terms proportional to $\mathbf{J}_{mn}$ have not been written down on the same reasoning. Furthermore, the explicit expression of $\mathcal{O}^{\text{inv}}_{(2)}(\mathbf{Q}^I)$ is not necessary for our argument because it always leads to higher-order contributions with $\mathcal{O}(\theta^4)$ in the resulting Lagrangian.

Next is to evaluate the Lagrangian using the formulae obtained above.

3.4. *The deformed Lagrangian at order θ^2*

Let us now examine the deformed action at the second order of θ.

Now the Lagrangian in (2.1) can be rewritten as

$$
\mathcal{L} = -\frac{\sqrt{\lambda_c}}{4}\left(\gamma^{ab} - \epsilon^{ab}\right)\mathrm{STr}\left[\tilde{d}(A_a)\mathcal{O}^{\text{inv}}(A_b)\right], \tag{3.22}
$$

where $\tilde{d}$ is defined as

$$
\tilde{d} \equiv -P_1 + 2P_2 + P_3. \tag{3.23}
$$

This Lagrangian can be expanded in terms of θ as

$$\mathcal{L} = \mathcal{L}_{(0)} + \mathcal{L}_{(2,0,0)} + \mathcal{L}_{(0,0,2)} + \mathcal{L}_{(1,1,0)} + \mathcal{L}_{(0,1,1)} + \mathcal{L}_{(0,2,0)} + \mathcal{L}_{(1,0,1)} + \mathcal{O}(\theta^4). \tag{3.24}$$

Here, $\mathcal{L}_{(0)}$ does not include any θ. The second-order term $\mathcal{L}_{(l,m,n)}$ contains two θs. The set of subscripts (l, m, n) indicates the numbers of θ included in $\tilde{d}(A_a)$, $\mathcal{O}^{\text{inv}}$, and A_b, respectively. For example, in the case of $\mathcal{L}_{(2,0,0)}$, the two θs are included in $\tilde{d}(A_a)$, and there is no θ in $\mathcal{O}^{\text{inv}}$ and A_b. That is, $\mathcal{L}_{(2,0,0)}$ is given by

$$\mathcal{L}_{(2,0,0)} = -\frac{\sqrt{\lambda_c}}{4}\left(\gamma^{ab} - \epsilon^{ab}\right) \mathrm{STr}\left[\tilde{d}((A_{(2)})_a)\mathcal{O}^{\text{inv}}_{(0)}((A_{(0)})_b)\right]. \tag{3.25}$$

In the following, let us see each term of the expansion (3.24). The first one is $\mathcal{L}_{(0)}$ and does not contain any fermions. This can be rewritten into the standard form as follows:

$$\begin{aligned}
\mathcal{L}_{(0)} &= -\frac{\sqrt{\lambda_c}}{4}(\gamma^{ab} - \epsilon^{ab})\, \mathrm{STr}\left[\tilde{d}((A_{(0)})_a)\mathcal{O}^{\text{inv}}_{(0)}((A_{(0)})_b)\right] \\
&= -\frac{\sqrt{\lambda_c}}{2}(\gamma^{ab} - \epsilon^{ab})\, e^m_a e^n_b\, k_{nm} \\
&= -\frac{\sqrt{\lambda_c}}{2}\left[\gamma^{ab} e^m_\mu e^n_\nu\, k_{(mn)}\, \partial_a X^\mu \partial_b X^\nu - \epsilon^{ab} e^m_\mu e^n_\nu\, k_{[nm]}\, \partial_a X^\mu \partial_b X^\nu\right].
\end{aligned} \tag{3.26}$$

Here we have used the relation $e^m_a = e^m_\mu \partial_a X^\mu$, and the X^μs are the target-spacetime coordinates. The last expression (3.26) should be compared with the standard bosonic string action

$$\mathcal{L}^{\mathrm{b}} = -\frac{\sqrt{\lambda_c}}{2}\left[\gamma^{ab}\, \widetilde{G}_{MN}\, \partial_a X^M \partial_b X^N - \epsilon^{ab}\, B_{MN}\, \partial_a X^M \partial_b X^N\right] \tag{3.27}$$

with the spacetime metric $\widetilde{G}$ and NS–NS two-form B. Then one can obtain the following relations:

$$\begin{aligned}
\widetilde{G}_{MN} &\equiv e^m_M e^n_N\, k_{(mn)} = \tilde{e}^m_M \tilde{e}_{mN}, \\
B_{MN} &\equiv e^m_M e^n_N\, k_{[nm]}.
\end{aligned} \tag{3.28}$$

Here we have introduced the vielbeins $\tilde{e}^m_M$ for the deformed metric for our later convenience. Note that the index M is raised and lowered by $\widetilde{G}^{MN}$ and $\widetilde{G}_{MN}$, respectively.

Then let us evaluate the combination $\mathcal{L}_{(2,0,0)} + \mathcal{L}_{(0,0,2)}$. From the point of view of symmetry, this combination is convenient and can be evaluated as

$$\begin{aligned}
&\mathcal{L}_{(2,0,0)} + \mathcal{L}_{(0,0,2)} \\
&= -\frac{\sqrt{\lambda_c}}{4}(\gamma^{ab} - \epsilon^{ab})\, \mathrm{STr}\big[i\bar{\theta}_I \gamma^m D^{IJ}_a \theta_J\, \mathbf{P}_m\, e^n_b k_n{}^p\, \mathbf{P}_p + i e^m_a \mathbf{P}_m\, \bar{\theta}_I \gamma^n D^{IJ}_b \theta_J\, k_n{}^p \mathbf{P}_p\big] \\
&= -\frac{i\sqrt{\lambda_c}}{4}(\gamma^{ab} - \epsilon^{ab})\, \bar{\theta}_I \left(e^n_b k_{nm} \gamma^m D^{IJ}_a + e^m_a k_{nm} \gamma^n D^{IJ}_b\right)\theta_J.
\end{aligned} \tag{3.29}$$

By the same reasoning, it is helpful to evaluate the combination $\mathcal{L}_{(1,1,0)} + \mathcal{L}_{(0,1,1)}$. The resulting expression is given by

$$\begin{aligned}
&\mathcal{L}_{(1,1,0)} + \mathcal{L}_{(0,1,1)} \\
&= -\frac{\sqrt{\lambda_c}}{4}(\gamma^{ab} - \epsilon^{ab})\, \mathrm{STr}\big[\sigma_3^{IJ}\mathbf{Q}^J D^{IK}_a \theta_K \mathcal{O}^{\text{inv}}_{(1)}(e^m_b \mathbf{P}_m) + 2e^m_a \mathbf{P}_m \mathcal{O}^{\text{inv}}_{(1)}(-\mathbf{Q}^I D^{IJ}_b \theta_J)\big] \\
&= -\frac{\sqrt{\lambda_c}}{2}(\gamma^{ab} - \epsilon^{ab})\, \bar{\theta}_I\left[i\eta\lambda_n{}^p \gamma_p \sigma_3^{IJ} - \frac{1}{2}\eta\lambda_n{}^{pq}\gamma_{pq}\sigma_1^{IJ}\right]\left(e^m_b k_m{}^n D^{JK}_a + e^m_a k^n{}_m D^{JK}_b\right)\theta_K.
\end{aligned} \tag{3.30}$$

Finally, $\mathcal{L}_{(0,2,0)}$ and $\mathcal{L}_{(1,0,1)}$ are evaluated as, respectively,

$$\begin{aligned}\mathcal{L}_{(0,2,0)} &= -\frac{\sqrt{\lambda_c}}{4}\,(\gamma^{ab}-\epsilon^{ab})\,\mathrm{STr}\left[2e_a^m\mathbf{P}_m\mathcal{O}^{\mathrm{inv}}_{(2)}(e_b^n\mathbf{P}_n)\right]\\ &= -\frac{\sqrt{\lambda_c}}{2}\,(\gamma^{ab}-\epsilon^{ab})\,e_a^m e_b^n\,\bar{\theta}_I\Big[\epsilon^{IJ}(\mathcal{M}^{P\epsilon}_{(2)})_{nm}+\delta^{IJ}(\mathcal{M}^{P\delta}_{(2)})_{nm}\\ &\qquad\qquad +\sigma_1^{IJ}(\mathcal{M}^{P\sigma_1}_{(2)})_{nm}+\sigma_3^{IJ}(\mathcal{M}^{P\sigma_3}_{(2)})_{nm}\Big]\theta_J,\end{aligned} \tag{3.31}$$

$$\begin{aligned}\mathcal{L}_{(1,0,1)} &= -\frac{\sqrt{\lambda_c}}{4}\,(\gamma^{ab}-\epsilon^{ab})\,\mathrm{STr}\left[\sigma_3^{IJ}\mathbf{Q}^J D_a^{IK}\theta_K(-\mathbf{Q}^L D_b^{LM})\theta_M\right]\\ &= -\frac{i\sqrt{\lambda_c}}{2}\,\epsilon^{ab}\sigma_3^{IJ}\,\bar{\theta}_I e_a^m\gamma_m D_b^{JK}\theta_K.\end{aligned} \tag{3.32}$$

So far, we have derived the deformed Lagrangian at the quadratic level of θ. However, the resulting sum of the components evaluated above is quite intricate and we still need to recast it into the canonical form via coordinate transformations.

3.5. *The canonical form of the Lagrangian*

Here, let us perform coordinate transformations in order to realize the canonical form of the Lagrangian. This process is mainly composed of two steps: 1) the shift of X, and 2) the rotation of θ.

3.5.1. *The canonical form*

First of all, let us present the canonical form of the Lagrangian at order θ^2 [51]:

$$\begin{aligned}\mathcal{L}^{(\theta^2)} &= -\frac{\sqrt{\lambda_c}}{2}\,i\bar{\Theta}_I(\gamma^{ab}\delta^{IJ}+\epsilon^{ab}\sigma_3^{IJ})\,\tilde{e}_a^m\Gamma_m\widetilde{D}_b^{JK}\Theta_K,\\ \widetilde{D}_a^{IJ} &= \delta^{IJ}\left(\partial_a-\frac{1}{4}\tilde{\omega}_a^{mn}\Gamma_{mn}\right)+\frac{1}{8}\sigma_3^{IJ}\tilde{e}_a^m H_{mnp}\Gamma^{np}\\ &\quad -\frac{1}{8}\mathrm{e}^{\Phi}\left[\epsilon^{IJ}\Gamma^p F_p+\frac{1}{3!}\sigma_1^{IJ}\Gamma^{pqr}F_{pqr}+\frac{1}{2\cdot 5!}\epsilon^{IJ}\Gamma^{pqrst}F_{pqrst}\right]\tilde{e}_a^m\Gamma_m.\end{aligned} \tag{3.33}$$

Here, the Γ_ms are 32×32 gamma matrices composed of $\gamma_{\check{m}}$ and $\gamma_{\hat{m}}$ as follows:

$$\Gamma_{\check{m}} = \sigma_1\otimes\gamma_{\check{m}},\qquad \Gamma_{\hat{m}} = \sigma_2\otimes\gamma_{\hat{m}}. \tag{3.34}$$

Then $\Gamma_{m_1\ldots m_2}$ is defined as

$$\Gamma_{m_1\ldots m_n} \equiv \frac{1}{n!}\Gamma_{[m_1}\ldots\Gamma_{m_n]}.$$

Now Θ^I ($I=1,2$) are 32-component Majorana spinors defined as

$$\Theta^I \equiv \begin{pmatrix}1\\0\end{pmatrix}\otimes\theta^I,\qquad \bar{\Theta}\equiv\Theta^\dagger\Gamma^0=\Theta^t\mathcal{C}=(\,0\quad 1\,)\otimes\bar{\theta}^I. \tag{3.35}$$

Here, $\mathcal{C}$ is a charge conjugation matrix defined as

$$\mathcal{C}\equiv i\,\sigma_2\otimes K\otimes K. \tag{3.36}$$

The canonical Lagrangian (3.33) contains the dilaton Φ, the three-form field strength $H_3 = dB_2$ (B_2: NS–NS two-form), the one-form field strength $F_1 = d\chi$ (χ: axion or R–R scalar), the three-form field strength $F_3 = dC_2$ (C_2: R–R two-form), and the five-form field strength $F_5 = dC_4$ (C_4: R–R four-form). Thus, after rewriting the quadratic part of the Lagrangian $\mathcal{L}$ in (3.24) into the canonical form, by comparing the resulting form with the canonical form (3.33) one can read off the component fields of type IIB supergravity.

The remaining task is to rewrite the Lagrangian $\mathcal{L}$ expanded above by performing a shift of X and a rotation of θ. We will explain each of the steps below.

3.5.2. *Shift of X*

Let us see the terms with $\gamma^{ab}\partial_b\theta$ in $\mathcal{L}$. The relevant parts are

$$\text{(a)}\quad \mathcal{L}^{\gamma}_{(2,0,0)} + \mathcal{L}^{\gamma}_{(0,0,2)} \qquad \text{and} \qquad \text{(b)}\quad \mathcal{L}^{\gamma}_{(1,1,0)} + \mathcal{L}^{\gamma}_{(0,1,1)}.$$

One can realize that the terms should appear with δ^{IJ} from the expression of the canonical form (3.33). There is no obstacle for (a), however (b) involves terms like

$$\frac{\sqrt{\lambda_c}}{2}\,\bar{\theta}_I\,\gamma^{ab}\,\sigma_1^{IJ}\,e_a^m\,k_{(mn)}\eta\,\lambda^{n,pq}\gamma_{pq}\partial_b\theta_J. \tag{3.37}$$

Such terms do not appear in the canonical form (3.33) and hence must be removed somehow. A possible resolution is to shift X as [26]

$$X^\mu \longrightarrow X^\mu + \bar{\theta}_I\delta X^{\mu\,IJ}\theta_J, \qquad \delta X^{\mu\,IJ} \equiv \frac{1}{4}\sigma_1^{IJ}e^{n\,\mu}\,\eta\,\lambda_n{}^{pq}\gamma_{pq}. \tag{3.38}$$

While this shift removes the problematic terms, it generates additional ones:

$$\begin{aligned} -\frac{\sqrt{\lambda_c}}{2}\,i\bar{\theta}_I\,\gamma^{ab}\delta^{IJ}\Big[&-\frac{i}{2}\,\sigma_1^{JK}\,e_a^m\,e_N^n\,k_{(mn)}\partial_b(e^{n\,N}\,\eta\,\lambda_n{}^{pq}\gamma_{pq}) \\ &-\frac{i}{4}\sigma_1^{JK}\,\partial_P\tilde{G}_{MN}\partial_aX^M\,\partial_bX^N\,e^{n\,P}\,\eta\,\lambda_n{}^{pq}\gamma_{pq}\Big]\theta_K. \end{aligned} \tag{3.39}$$

Note here that these terms do not involve derivatives of θ.

At this stage, the quadratic Lagrangian including γ^{ab} is written down as

$$\begin{aligned} \mathcal{L}^\gamma = -i\frac{\sqrt{\lambda_c}}{2}\,\gamma^{ab}\delta^{IJ}\bar{\theta}_I\Big[&e_a^p\,k_{(pn)}\,(\eta^{nm} - (-1)^J 2\eta\lambda^{nm})\gamma_m D_b^{JK} \\ &+\frac{1}{2}\sigma_3^{JK}\,e_a^m k_{(mn)}\eta\lambda^{n,pq}\gamma_{pq}\,e_b^r\,\gamma_r - \frac{1}{4}\delta^{JK}\,e_a^m\,k_{nm}\left(\eta^{np} - (-1)^J\,2\eta\,\lambda^{np}\right)\gamma_p\,e_b^q k_q{}^r\,\eta\,\lambda_r{}^{st}\,\gamma_{st} \\ &+\frac{1}{4}\delta^{JK}\,e_a^m k_{mn}\,\eta\,\lambda^{n,pq}\,\gamma_{pq}\,e_b^r\,k_{rs}\left(\eta^{st} - (-1)^J\,2\eta\,\lambda^{st}\right)\gamma_t \\ &+\frac{i}{2}\epsilon^{JK}e_a^m\,k_{nm}\left(\eta^{np} - (-1)^J\,2\eta\,\lambda^{np}\right)\gamma_p\,e_b^q\,k_q{}^r\,\eta\,\lambda_r{}^s\,\gamma_s \\ &-\frac{i}{4}\sigma_1^{JK}\,e_a^m k_{(mn)}\,\eta\,\lambda^{n,pq}\,\gamma_{pq}\,\omega_b^{rs}\,\gamma_{rs} - \frac{i}{2}\epsilon^{JK}\,e_a^m\,k_{nm}\,\eta\,\lambda^{np}\,\gamma_p\,e_b^q\,k_q{}^r\,\gamma_r \\ &-\frac{i}{4}\sigma_1^{JK}\,e_a^m\,k_{nm}\,\eta\,\lambda^{n,pq}\,\gamma_{pq}\,e_b^r\,k_r{}^s\,\eta\,\lambda_s{}^{tu}\,\gamma_{tu} - \frac{i}{2}\sigma_1^{JK}\,e_a^m\,e_M^n\,k_{(mn)}\,\partial_b\left(e^{p,M}\,\eta\,\lambda_p{}^{qr}\,\gamma_{qr}\right) \\ &-\frac{i}{4}\sigma_1^{JK}\,\partial_P\widetilde{G}_{MN}\,\partial_a X^M\,\partial_b X^N\,e^{m,P}\,\eta\,\lambda_m{}^{np}\,\gamma_{np}\Big]\theta_K. \end{aligned} \tag{3.40}$$

The next step is to see the terms with $\epsilon^{ab}\partial_b\theta$ in $\mathcal{L}$. This part has the terms involving σ_1^{IJ} as well. Fortunately, the sift of X in (3.38) can eliminate the problematic terms simultaneously, while some additional terms including ϵ^{ab} are again generated. Then the quadratic Lagrangian including ϵ^{ab} is written down as

$$
\begin{aligned}
\mathcal{L}^{\epsilon} = -i\,\frac{\sqrt{\lambda_c}}{2}\,\epsilon^{ab}\,\bar{\theta}_I\Big[&\Big(\delta^{IJ}\,e_a^m\,k_{[mn]}\,\gamma^n + \sigma_3^{IJ}\big[e_a^m\,k_{[mn]}\,2\,\eta\,\lambda^{np}\,\gamma_p + e_a^m\,\gamma_m)\big]\Big)D_b^{JK}\\
&+ i\,\sigma_1^{IJ}\,e_a^m\,k_{[mn]}\,\eta\,\lambda^{n,pq}\,\gamma_{pq}\left(-\frac{1}{4}\delta^{JK}\,\omega_b^{rs}\,\gamma_{rs} + \frac{i}{2}\epsilon^{JK}e_b^r\,\gamma_r\right)\\
&+ \frac{1}{4}\,\delta^{IK}e_a^m\,k_{nm}\left(\eta^{np} - (-1)^I\,2\eta\,\lambda^{np}\right)\gamma_p\,e_b^q k_q{}^r\,\eta\,\lambda_r{}^{st}\,\gamma_{st}\\
&- \frac{1}{4}\delta^{IK}\,e_a^m k_{mn}\,\eta\,\lambda^{n,pq}\,\gamma_{pq}\,e_b^r\,k_{rs}\left(\eta^{st} - (-1)^I\,2\eta\,\lambda^{st}\right)\gamma_t\\
&- \frac{i}{2}\epsilon^{IK}e_a^m\,k_{nm}\left(\eta^{np} - (-1)^I\,2\eta\,\lambda^{np}\right)\gamma_p\,e_b^q\,k_q{}^r\,\eta\,\lambda_r{}^s\,\gamma_s\\
&+ \frac{i}{2}\epsilon^{IK}\,e_a^m\,k_{nm}\,\eta\,\lambda^{np}\,\gamma_p\,e_b^q\,k_q{}^r\,\gamma_r + \frac{i}{4}\sigma_1^{IK}\,e_a^m\,k_{nm}\,\eta\,\lambda^{n,pq}\,\gamma_{pq}\,e_b^r\,k_r{}^s\,\eta\,\lambda_s{}^{tu}\,\gamma_{tu}\\
&+ \frac{i}{2}\sigma_1^{IK}B_{MN}\partial_a X^M\,\partial_b\left(e^{nN}\,\eta\,\lambda_n{}^{pq}\right)\gamma_{pq}\\
&+ \frac{i}{4}\sigma_1^{IK}\partial_P B_{MN}\partial_a X^M\partial_b X^N e^{nP}\eta\lambda_n{}^{pq}\gamma_{pq}\Big]\theta_K.
\end{aligned}
\tag{3.41}
$$

For the next step, it is convenient to switch from the 16×16 gamma matrices γ to the 32×32 ones Γ, and hence we will work in the 32×32 notation in the following. The lift-up rule is summarized in Appendix A, and it is straightforward to rewrite the Lagrangian.

3.5.3. *Rotation of θ*

After shifting X, the resulting derivative terms of θ take the following form:

$$
-\frac{\sqrt{\lambda_c}}{2}\,i\bar{\Theta}_I\,\gamma^{ab}\delta^{IJ}\,\tilde{e}_{(I)a}{}^m\,\Gamma_m\partial_b\,\Theta_J. \tag{3.42}
$$

Here, the vielbeins[3] $\tilde{e}_{(I)a}{}^m$ are defined as

$$
\tilde{e}_{(I)a}{}^m \equiv e_a^p\,k_{(pn)}\left[\eta^{nm} - (-1)^I 2\eta\lambda^{nm}\right] \tag{3.43}
$$

and depend on the index I. Hence we need to perform a Lorentz transformation for the spinor θ to remove the I dependence.

The first step is to determine the I-independent form of the vielbeins as a reference frame. Hereafter, it is fixed by taking $I=1$ in (3.43) as

$$
\tilde{e}_a^m = e_a^p\,k_{(pn)}\left[\eta^{nm} + 2\eta\lambda^{nm}\right]. \tag{3.44}
$$

[3] Note that $\tilde{e}_{(I)a}{}^m$ satisfy the relation

$$
\tilde{e}_{(I)a}{}^m\,\tilde{e}_{(I)bm} = e_a^m e_b^n\,k_{(mn)} = \widetilde{G}_{\mu\nu}\partial_a X^\mu\partial_b X^\nu \qquad (\text{for } I=1,2).
$$

Then, by performing a Lorentz transformation for θ, this term can be rewritten as

$$\begin{aligned} &\bar{\Theta}_I\, \tilde{e}_{(I)a}^{\ \ m}\, \bar{U}_{(I)}\Gamma_m\, U_{(I)}\partial_b\Theta_I + (\text{the derivative term of } U) \\ &= \bar{\Theta}_I\, \tilde{e}_{(I)a}^{\ \ m}\, \Lambda_{(I)m}{}^{n}\Gamma_n\, \partial_b\Theta_I + (\text{the derivative term of } U). \end{aligned} \tag{3.45}$$

Note that the Lorentz transformation performed here depends on the index I.

In order to realize the I-independent form (3.44), the transformation Λ should be taken as

$$\Lambda_{(I)m}{}^{n} = \left[\delta_m{}^p + (-1)^I 2\eta\lambda_m{}^p\right](\delta - 2\eta\lambda)^{-1}{}_p{}^n. \tag{3.46}$$

Then the spinor transformation $U_{(I)}$ and its inverse $\bar{U}_{(I)}$ have to be determined through the following relation:

$$\bar{U}_{(I)}\Gamma_m U_{(I)} = \Lambda_{(I)m}{}^{n}\Gamma_n. \tag{3.47}$$

It seems difficult to present simple formulae of $U_{(I)}$ and $\bar{U}_{(I)}$ that work for an "arbitrary" classical r-matrix. As a matter of course, given an explicit expression of a classical r-matrix, these quantities can be computed concretely.

However, at least for a simple class of classical r-matrices, we can propose the following concise forms:

$$\begin{aligned} U_{(I)} &= \frac{1}{\det(\mathbf{1}_{32} + \frac{1}{2}\left[1 + (-1)^I\right]\eta\lambda^{mn}\Gamma_{mn})^{\frac{1}{32}}}\left(\mathbf{1}_{32} + \frac{1}{2}\left[1 + (-1)^I\right]\eta\lambda^{mn}\Gamma_{mn}\right), \\ \bar{U}_{(I)} &= \frac{1}{\det(\mathbf{1}_{32} + \frac{1}{2}\left[1 + (-1)^I\right]\eta\lambda^{mn}\Gamma_{mn})^{\frac{1}{32}}}\left(\mathbf{1}_{32} - \frac{1}{2}\left[1 + (-1)^I\right]\eta\lambda^{mn}\Gamma_{mn}\right). \end{aligned} \tag{3.48}$$

For example, these are valid for the examples presented in Sect. 4. It has not been definitely clarified yet to what extent the formulae in (3.48) are valid. With our current techniques, if the expressions in (3.48) do not satisfy the relation (3.47) for a given classical r-matrix, then it is necessary to derive the concrete forms of $U_{(I)}$ and $\bar{U}_{(I)}$ on a case-by-case basis.

After all this, we have obtained the canonical form of the Lagrangian. In the actual derivation of R–R fluxes and the dilaton, we *still* need to use a concrete expression of a classical r-matrix and computation software like Mathematica or Maple, at least at the current level of understanding. We will present the resulting backgrounds for some example classical r-matrices in Sect. 4.

3.6. *The master formula for the dilaton*

We propose the master formula for dilaton, in which the dilaton is described in terms of the classical r-matrix directly. That is, just by putting the elements of the classical r-matrix, the associated dilaton is obtained directly, without passing through the supercoset construction. The formula is given by

$$\begin{aligned} e^{\Phi} &= \frac{1}{\det_{32}(\mathbf{1}_{32} + \eta\,\lambda^{mn}\Gamma_{mn})^{\frac{1}{32}}} \\ &= \frac{1}{\det_{10}(\delta_m{}^n + 2\,\eta\,\lambda_m{}^n)^{\frac{1}{2}}}, \end{aligned} \tag{3.49}$$

where $\det_D$ means the determinant of a $D \times D$ matrix. Recall that $\lambda_m{}^n$, which is defined in (3.15), is determined by putting the elements of the classical r-matrix. Although this formula has not been proven and is just a conjectured form, it works well for well-known examples, including the examples discussed in Sect. 4.

Similar master formulae may be derived for other R–R fluxes, though we have not succeeded in deriving them yet. It is important to try to complete the master formulae and directly check the on-shell condition of type IIB supergravity.

4. Examples

Let us consider some examples of classical r-matrices satisfying the homogeneous CYBE. Then it is possible to complete the supercoset construction and derive the resulting backgrounds.

For the following argument, let us introduce the terms "Abelian" and "non-Abelian" classical r-matrices. Suppose that a classical r-matrix is given by $r = a \wedge b$. It is called "Abelian" when a and b commute with each other. If not, it is "non-Abelian."

4.1. *Gravity duals of non-commutative gauge theories*

Let us discuss gravity duals of non-commutative gauge theories as Yang–Baxter deformations with the following classical r-matrix [40]:

$$r = P_2 \wedge P_3. \tag{4.1}$$

Here it is assumed that P_μ are naturally embedded into 8×8 matrices like

$$\begin{pmatrix} P_\mu & \mathbf{0_4} \\ \mathbf{0_4} & \mathbf{0_4} \end{pmatrix}. \tag{4.2}$$

This is an Abelian classical r-matrix and satisfies the homogeneous CYBE.

The bosonic part has already been studied in [40], where the string frame metric and NS–NS two-form are reproduced with the r-matrix (4.1). The R–R sector and dilaton can be determined by performing the supercoset construction.

The supercoset construction can be carried out by following the general argument in Sect. 3. In the present case, the key ingredient λ_{mn} is given by

$$\lambda_{mn} = \begin{pmatrix} \lambda_{\check{m}\check{n}} & \mathbf{0_5} \\ \mathbf{0_5} & \mathbf{0_5} \end{pmatrix}, \quad \lambda_{\check{m}\check{n}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z^{-2} & 0 \\ 0 & 0 & z^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.3}$$

Then $U_{(I)}$ and $\bar{U}_{(I)}$ are obtained by using the formulae in (3.48).

With the general argument in Sect. 3, one can read off the following background:

$$\begin{aligned} ds^2 &= \frac{-(dx^0)^2 + (dx^1)^2}{z^2} + \frac{z^2\left[(dx^2)^2 + (dx^3)^2\right]}{z^4 + 4\eta^2} + \frac{dz^2}{z^2} + ds^2_{\mathrm{S}^5}, \\ B_2 &= \frac{2\eta}{z^4 + 4\eta^2}\, dx^2 \wedge dx^3, \\ F_3 &= \frac{8\,\eta}{z^5}\, dx^0 \wedge dx^1 \wedge dz, \\ F_5 &= 4\left(\mathrm{e}^{2\Phi}\,\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right), \qquad \Phi = \frac{1}{2}\log\left(\frac{z^4}{z^4 + 4\eta^2}\right). \end{aligned} \tag{4.4}$$

This is nothing but the solution found in [32,33] as a gravity dual of non-commutative gauge theories. Note that the dilaton can be reproduced by using the master formula (3.49) with (4.3).

4.2. γ-deformations of S^5

We shall discuss three-parameter γ-deformations of S^5 with the following classical r-matrix [39]:

$$r = \frac{1}{8}\left(\nu_3\, h_1 \wedge h_2 + \nu_1\, h_2 \wedge h_3 + \nu_2\, h_3 \wedge h_1\right). \tag{4.5}$$

Here, ν_i $(i = 1, 2, 3)$ are real constant parameters, and h_a $(a = 1, 2, 3)$ are the Cartan generators of $\mathfrak{su}(4)$ embedded in 8×8 matrices as the lower diagonal block (for their matrix representation, see Appendix A). This is an Abelian classical r-matrix and satisfies the CYBE.

The bosonic part has already been studied in [39]. The remaining task is to perform supercoset construction in order to determine the R–R sector and dilaton.

The quantities $U_{(I)}$ and $\bar{U}_{(I)}$ are determined by

$$\lambda_{mn} = \begin{pmatrix} \mathbf{0_5} & \mathbf{0_5} \\ \mathbf{0_5} & \lambda_{\hat{m}\hat{n}} \end{pmatrix}, \quad \lambda_{\hat{m}\hat{n}} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\nu_3\,\rho_1\,\rho_2 & 0 & \frac{1}{2}\nu_1\,\rho_2\,\rho_3 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\nu_3\,\rho_1\,\rho_2 & 0 & 0 & 0 & -\frac{1}{2}\nu_2\,\rho_3\,\rho_1 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}\nu_1\,\rho_2\,\rho_3 & 0 & \frac{1}{2}\nu_2\,\rho_3\,\rho_1 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

Then, by following the general discussion, the full solution presented in [34,35] can be reproduced as

$$ds^2 = ds^2_{\mathrm{AdS}_5} + \sum_{i=1}^{3}(d\rho_i^2 + G\,\rho_i^2 d\phi_i^2) + G\rho_1^2\rho_2^2\rho_3^2\left(\sum_{i=1}^{3}\hat{\gamma}_i\,d\phi_i\right)^2, \tag{4.7}$$

$$\begin{aligned} B_2 &= G\left(\hat{\gamma}_3\,\rho_1^2\rho_2^2\,d\phi_1 \wedge d\phi_2 + \hat{\gamma}_1\,\rho_2^2\rho_3^2\,d\phi_2 \wedge d\phi_3 + \hat{\gamma}_2\,\rho_3^2\rho_1^2\,d\phi_3 \wedge d\phi_1\right), \\ F_3 &= -4\sin^3\alpha\,\cos\alpha\,\sin\theta\,\cos\theta\left(\sum_{i=1}^{3}\hat{\gamma}_i\,d\phi_i\right)\wedge d\alpha \wedge d\theta, \\ F_5 &= 4\left(\omega_{\mathrm{AdS}_5} + G\,\omega_{\mathrm{S}^5}\right), \qquad \Phi = \frac{1}{2}\log G. \end{aligned} \tag{4.8}$$

Here we have introduced a scalar function G and $\hat{\gamma}_i$ $(i = 1, 2, 3)$ defined as

$$G^{-1} \equiv 1 + \hat{\gamma}_3^2\,\rho_1^2\rho_2^2 + \hat{\gamma}_1^2\,\rho_2^2\rho_3^2 + \hat{\gamma}_2^2\,\rho_3^2\rho_1^2, \qquad \hat{\gamma}_i \equiv \eta\nu_i. \tag{4.9}$$

Three coordinates ρ_i satisfying the constraint $\sum_{i=1}^{3}\rho_i^2 = 1$ are parametrized by two angle variables α and θ through the relation

$$\rho_1 \equiv \sin\alpha\cos\theta, \quad \rho_2 \equiv \sin\alpha\sin\theta, \quad \rho_3 \equiv \cos\alpha. \tag{4.10}$$

It should be remarked that the resulting background is non-supersymmetric, other than for exceptional cases like $\nu_1 = \nu_2 = \nu_3$. But the supercoset construction still works well. Note that the dilaton can be reproduced by using the master formula (3.49) with (4.6).

4.3. Schrödinger spacetimes

Let us consider Schrödinger spacetimes by employing the following classical r-matrix [41]:

$$r = \frac{i}{4} P_- \wedge (h_1 + h_2 + h_3). \tag{4.11}$$

Here, $P_- \equiv (P_0 - P_3)/\sqrt{2}$ is a light-cone generator in $\mathfrak{su}(2,2)$, and h_1, h_2, h_3 are the Cartan generators in $\mathfrak{su}(4)$.

Note here that the classical r-matrix (4.11) contains a tensor product of an $\mathfrak{su}(2,2)$ generator and an $\mathfrak{su}(4)$ one. Hence the rotation of θ should be a ten-dimensional Lorentz transformation, and becomes intricate. The quantities $U_{(I)}$ and $\bar{U}_{(I)}$ are given by, respectively,

$$\lambda_{mn} = \begin{pmatrix} \mathbf{0_5} & \lambda_{\check{m}\hat{n}} \\ \lambda_{\hat{m}\check{n}} & \mathbf{0_5} \end{pmatrix}, \quad \lambda_{\check{m}\hat{n}} = \begin{pmatrix} \frac{\sin r \sin\zeta}{2\sqrt{2}z} & 0 & \frac{\sin r\cos\zeta}{2\sqrt{2}z} & 0 & \frac{\cos r}{2\sqrt{2}z} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\sin r\sin\zeta}{2\sqrt{2}z} & 0 & \frac{\sin r\cos\zeta}{2\sqrt{2}z} & 0 & \frac{\cos r}{2\sqrt{2}z} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.12}$$

Note that $\lambda_{\hat{m}\check{n}}$ can be obtained from $\lambda_{\check{m}\hat{n}}$, because λ_{mn} is anti-symmetric.

After all, the full solution [36–38] has been reproduced as

$$\begin{aligned} ds^2 &= \frac{-2dx^+dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \eta^2\frac{(dx^+)^2}{z^4} + ds^2_{\mathrm{S}^5}, \\ B_2 &= \frac{\eta}{z^2}\, dx^+ \wedge (d\chi + \omega), \\ F_5 &= 4\left(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right), \qquad \Phi = \text{const.}, \end{aligned} \tag{4.13}$$

and the other fields are zero. Here, the light-cone coordinates are defined as

$$x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^3).$$

The S^5 metric is given by

$$\begin{aligned} ds^2_{\mathrm{S}^5} &= (d\chi + \omega)^2 + ds^2_{\mathbb{C}\mathrm{P}^2}, \\ ds^2_{\mathbb{C}\mathrm{P}^2} &= d\mu^2 + \sin^2\mu\left(\Sigma_1^2 + \Sigma_2^2 + \cos^2\mu\,\Sigma_3^2\right). \end{aligned} \tag{4.14}$$

Namely, the round S^5 is expressed as an S^1-fibration over $\mathbb{C}\mathrm{P}^2$, where χ is the fiber coordinate and ω is a one-form potential of the Kähler form on $\mathbb{C}\mathrm{P}^2$. The symbols Σ_i $(i = 1, 2, 3)$ and ω are defined as

$$\begin{aligned} \Sigma_1 &= \tfrac{1}{2}(\cos\psi\, d\theta + \sin\psi\,\sin\theta\, d\phi), \\ \Sigma_2 &= \tfrac{1}{2}(\sin\psi\, d\theta - \cos\psi\,\sin\theta\, d\phi), \\ \Sigma_3 &= \tfrac{1}{2}(d\psi + \cos\theta\, d\phi), \qquad \omega = \sin^2\mu\,\Sigma_3. \end{aligned} \tag{4.15}$$

It should be remarked that the R–R sector has not been deformed and the dilaton remains constant, though the expression of the fermionic sector is very complicated in the middle of the computation. The cancellation of the deformation effect is really non-trivial. Note also that the background

(4.13) does not preserve any supersymmetries [36–38]. It may sound surprising that the supercoset construction still works well without the help of supersymmetries.

The constant dilaton of this background can be reproduced by using the master formula (3.49) with (4.12) as well.

4.4. *A non-Abelian classical r-matrix*

So far, we have considered Abelian classical r-matrices, for which it seems likely that supercoset construction works well even though the resulting background is non-supersymmetric. The next significant issue is to study non-Abelian classical r-matrices.

As for non-Abelian classical r-matrices, there is no well-known example of the associated background. A nice candidate for non-Abelian classical r-matrices is given by

$$
\begin{aligned}
r &= \frac{1}{\sqrt{2}} E_{24} \wedge (c_1 E_{22} - c_2 E_{44}) \qquad \left[(E_{ij})_{kl} \equiv \delta_{ik}\,\delta_{jl}\right] \\
&= -\frac{1}{2} P_- \wedge \left[\frac{c_1 + c_2}{2}\,(D - L_{03}) + i\,\frac{c_1 - c_2}{2}\left(L_{12} - \frac{i}{2}\,\mathbf{1}_4\right)\right].
\end{aligned}
\tag{4.16}
$$

Note here that $\mathbf{1}_4$ is included in the expression and hence the image is extended from $\mathfrak{su}(2,2|4)$ to $\mathfrak{gl}(4|4)$. However, it can be ignored due to the presence of the projection operator in the classical action as pointed out in [44].

To ensure that the resulting metric and NS–NS two-form are real, it is necessary to impose the reality condition [43]

$$
c_2 = c_1^*. \tag{4.17}
$$

It is now convenient to introduce a_1, a_2 as follows:

$$
a_1 \equiv \frac{c_1 + c_2}{2} = \mathrm{Re}(c_1), \quad a_2 \equiv i\frac{c_1 - c_2}{2} = -\mathrm{Im}(c_1). \tag{4.18}
$$

Note here that the classical r-matrix (4.16) is non-Abelian in general. The case that c_1 is pure imaginary (i.e., $a_1 = 0$) is exceptional and it becomes Abelian.

The bosonic part has already been studied well [42–44], and the remaining task is to determine the R–R sector and dilaton by performing supercoset construction. The quantities $U_{(I)}$ and $\bar{U}_{(I)}$ are obtained by using

$$
\begin{aligned}
\lambda_{mn} &= \begin{pmatrix} \lambda_{\check{m}\check{n}} & \mathbf{0_5} \\ \mathbf{0_5} & \mathbf{0_5} \end{pmatrix}, \\
\lambda_{\check{m}\check{n}} &= \frac{1}{2\sqrt{2}} \begin{pmatrix}
0 & -\frac{a_1 x^1 + a_2 x^2}{z^2} & \frac{a_2 x^1 - a_1 x^2}{z^2} & 0 & -\frac{a_1}{z} \\
\frac{a_1 x^1 + a_2 x^2}{z^2} & 0 & 0 & \frac{a_1 x^1 + a_2 x^2}{z^2} & 0 \\
\frac{a_1 x^2 - a_2 x^1}{z^2} & 0 & 0 & \frac{a_1 x^2 - a_2 x^1}{x^2} & 0 \\
0 & -\frac{a_1 x^1 + a_2 x^2}{x^2} & \frac{a_2 x^1 - a_1 x^2}{z^2} & 0 & -\frac{a_1}{z} \\
\frac{a_1}{z} & 0 & 0 & \frac{a_1}{z} & 0
\end{pmatrix}.
\end{aligned}
\tag{4.19}
$$

After all this, one can read off the resulting background:[4]

$$
\begin{aligned}
ds^2 &= \frac{-2dx^+dx^- + d\rho^2 + \rho^2 d\phi^2 + dz^2}{z^2} - \eta^2 \left[(a_1^2 + a_2^2)\frac{\rho^2}{z^6} + \frac{a_1^2}{z^4} \right] (dx^+)^2 + ds^2_{S^5}, \\
B_2 &= \eta \left[\frac{a_1 x^1 + a_2 x^2}{z^4} dx^+ \wedge dx^1 + \frac{a_1 x^2 - a_2 x^1}{z^4} dx^+ \wedge dx^2 + a_1 \frac{1}{z^3} dx^+ \wedge dz \right], \\
F_3 &= 4\eta \left[\frac{a_2 x^1 - a_1 x^2}{z^5} dx^+ \wedge dx^1 \wedge dz + \frac{a_1 x^1 + a_2 x^2}{z^5} dx^+ \wedge dx^2 \wedge dz + \frac{a_1}{z^4} dx^+ \wedge dx^1 \wedge dx^2 \right], \\
F_5 &= 4 \left(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5} \right), \qquad \Phi = \text{const.},
\end{aligned} \tag{4.20}
$$

and the other components are zero. Here the light-cone coordinates are defined in the same way as the previous section. Notice that the background (4.20) does not satisfy the equation of motion of B_2 because the Bianchi identity for F_3 is broken, namely

$$
dF_3 = 16\eta \frac{a_1}{z^5} dx^+ \wedge dx^1 \wedge dx^2 \wedge dz \neq 0.
$$

Thus the classical r-matrix (4.16) does not lead to a solution of type IIB supergravity.

It is worth noting that the pathology vanishes when $a_1 = 0$. This is an exceptional case in which the classical r-matrix becomes Abelian and the background (4.20) is reduced to the Hubeny–Rangamani–Ross solution [52]. This correspondence was originally argued in [43] and elaborated in [44]. Note here that the constant dilaton can be reproduced by using the master formula (3.49) with (4.19) again.

It would be valuable to see that the background (4.20) is different from the one constructed in [42]. The former (4.20) does not contain the R–R fluxes with the S^5 indices, while the latter does. It was conjectured in [42,43] that the classical r-matrix (4.16) should be associated with the latter, but it was not correct. Our supercoset construction has revealed that the classical r-matrix (4.16) should be associated with the background (4.20).

The result that the Bianchi identity is broken is similar to the q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ [26]. In fact, the background (4.20) satisfies the generalized type IIB supergravity equations of motion proposed in [30]. Detailed analysis will be presented in a future paper.[5]

5. Conclusion and discussion

In this paper, we have discussed the supercoset construction in the Yang–Baxter-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring based on the homogeneous CYBE. We have made a general argument without relying on specific expressions of classical r-matrices. In particular, we have presented the master formula to describe the dilaton in terms of classical r-matrices. The ultimate goal is to represent all of the R–R fluxes as well, and this is a really fascinating future problem. If it is carried out, the on-shell condition of type IIB supergravity can be checked directly for general classical r-matrices and one can test the conjecture of the gravity/CYBE correspondence.

Then we have explicitly performed supercoset construction for some classical r-matrices satisfying the homogeneous CYBE. For Abelian classical r-matrices, perfect agreement has been shown for

[4] As another possibility, one may take a non-constant dilaton $\Phi = \log z$ so as to respect the Bianchi identity $dF_3 = 0$. But in this case the dilaton does not satisfy the equations of motion for the dilaton as well as for other components.

[5] H. Kyono, J. Sakamoto, and K. Yoshida, in preparation.

well-known examples including gravity duals of non-commutative gauge theories, γ-deformations of S^5, and Schrödinger spacetimes. Remarkably, the supercoset construction works well, even if the resulting backgrounds are not maximally supersymmetric. In particular, three-parameter γ-deformations of S^5 and Schrödinger spacetimes do not preserve any supersymmetries. For non-Abelian r-matrices, we have concentrated on a specific example. The resulting background does not satisfy the equation of motion of the NS–NS two-form because the Ramond–Ramond three-form is not closed. Thus, at least so far, it seems likely that there would be no problem for Abelian classical r-matrices, while there are some potential problems in the non-Abelian cases. We will report on results on other examples of Abelian and non-Abelian classical r-matrices in the near future.[6]

There are many open problems. The Yang–Baxter deformation has diverse applicability. For example, it can be applied to the $AdS_5 \times T^{1,1}$ background [53,54]. In this case, the Green–Schwarz string action has not yet been constructed. However, at least for the bosonic sector,[7] it has been shown that three-parameter γ-deformations of $T^{1,1}$ [34,59] can be reproduced as Yang–Baxter deformations with Abelian classical r-matrices [53,54]. It is remarkable that the $AdS_5 \times T^{1,1}$ background is not integrable because chaotic string solutions exist [60,61]. It would be of significance to construct the $AdS_5 \times T^{1,1}$ superstring action and then investigate its Yang–Baxter deformations by following the procedure presented here.

It is also interesting to consider a supersymmetric extension of Yang–Baxter deformations of Minkowski spacetime [55–58]. As a toy model along this direction, it is easier to study the Nappi–Witten model [63]. Yang–Baxter invariance of this model has been discussed in [64]. It would be nice to argue its supersymmetric extension by employing [65] and further generalization with general symmetric two-forms [66].

We believe that our supercoset construction could capture the tip of an iceberg, namely the gravity/CYBE correspondence that denotes a non-trivial relation between type IIB supergravity and the classical Yang–Baxter equation.

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Appendix. A matrix representation of $\mathfrak{su}(2,2|4)$

We present here a matrix representation of the superalgebra $\mathfrak{su}(2,2|4)$. Our notation and conventions basically follow those utilized in [26].

[6] See footnote 5.

[7] The coset structure of $T^{1,1}$ is a little intricate, so even the analysis on the undeformed $T^{1,1}$ is not trivial.

A.1. A representation of $\mathfrak{su}(2,2)$

It is convenient to introduce the following basis of $\mathfrak{su}(2,2) \simeq \mathfrak{so}(2,4)$:

$$\mathfrak{su}(2,2) = \mathrm{span}_{\mathbb{R}} \left\{ \gamma_\mu \,, \gamma_5 \,, n_{\mu\nu} = \frac{1}{4}[\gamma_\mu \,, \gamma_\nu] \,, n_{\mu 5} = \frac{1}{4}[\gamma_\mu \,, \gamma_5] \mid \; \mu, \nu = 0, 1, 2, 3 \right\}. \tag{A.1}$$

Here, γ_μ are gamma matrices satisfying the Dirac algebra:

$$\{\gamma_\mu \,, \gamma_\nu\} = 2\eta_{\mu\nu}, \tag{A.2}$$

where $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric with mostly plus. It is convenient to adopt the following matrix realizations of the γ_μ:

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\gamma_0 = -i\gamma_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.3}$$

A.1.1. A conformal basis

It is also helpful to use the conformal basis

$$\mathfrak{so}(2,4) = \mathrm{span}_{\mathbb{R}}\{ P_\mu \,, L_{\mu\nu} \,, D \,, K_\mu \mid \; \mu, \nu = 0, 1, 2, 3 \}. \tag{A.4}$$

Here, the translation generators P_μ, the Lorentz rotation generators $L_{\mu\nu}$, the dilatation D, and the special conformal generators K_μ are represented by, respectively,

$$P_\mu \equiv \frac{1}{2}(\gamma_\mu - 2n_{\mu 5}), \quad L_{\mu\nu} \equiv n_{\mu\nu}, \quad D \equiv \frac{1}{2}\gamma_5, \quad K_\mu \equiv \frac{1}{2}(\gamma_\mu + 2n_{\mu 5}). \tag{A.5}$$

The non-vanishing commutation relations are given by

$$\begin{aligned} [P_\mu, K_\nu] &= 2(L_{\mu\nu} + \eta_{\mu\nu}\, D\,), \quad [D, P_\mu] = P_\mu \,, \quad [D, K_\mu] = -K_\mu, \\ \left[P_\mu, L_{\nu\rho}\right] &= \eta_{\mu\nu}\, P_\rho - \eta_{\mu\rho}\, P_\nu, \quad [K_\mu, L_{\nu\rho}] = \eta_{\mu\nu}\, K_\rho - \eta_{\mu\rho}\, K_\nu, \\ \left[L_{\mu\nu}, L_{\rho\sigma}\right] &= \eta_{\mu\sigma}\, L_{\nu\rho} + \eta_{\nu\rho}\, L_{\mu\sigma} - \eta_{\mu\rho}\, L_{\nu\sigma} - \eta_{\nu\sigma}\, L_{\mu\rho}. \end{aligned} \tag{A.6}$$

A.2. A representation of $\mathfrak{su}(4)$

It is easy to see that

$$n_{ij} = \frac{1}{4}\left[\gamma_i, \gamma_j\right] \qquad (i,j = 1, \ldots, 5) \tag{A.7}$$

generate $\mathfrak{so}(5)$ by using the Clifford algebra

$$\left\{\gamma_i, \gamma_j\right\} = 2\delta_{ij}. \tag{A.8}$$

Note that $\mathfrak{so}(6)$ is spanned by the set of the Hermite generators,

$$\mathfrak{so}(6) = \mathrm{span}_{\mathbb{R}}\left\{ \frac{1}{2}\gamma_i\,,\ i\,n_{ij} \right\}. \tag{A.9}$$

It is convenient to introduce the Cartan generators of $\mathfrak{su}(4)$ as follows:

$$h_1 \equiv 2i\,n_{12}, \qquad h_2 \equiv 2i\,n_{43}, \qquad h_3 \equiv \gamma_5. \tag{A.10}$$

A.3. An 8 × 8 supermatrix representation

By using the gamma matrices introduced above, let us represent the $\mathfrak{su}(2,2|4)$ generators by 8×8 supermatrices.

It is helpful to introduce the following quantities:

$$\boldsymbol{\gamma}_{\check{m}} \equiv \{\gamma_0\,,\gamma_1\,,\gamma_2\,,\gamma_3\,,\gamma_5\,\}\,, \qquad \boldsymbol{\gamma}_{\hat{m}} \equiv \{-\gamma_4\,,-\gamma_1\,,-\gamma_2\,,-\gamma_3\,,-\gamma_5\,\}\,. \tag{A.11}$$

For later convenience, we introduce the following metrics:

$$\eta_{\check{m}\check{n}} = \mathrm{diag}(-1,1,1,1,1), \qquad \eta_{\hat{m}\hat{n}} = \mathrm{diag}(1,1,1,1,1),$$
$$\eta_{mn} = \begin{pmatrix} \eta_{\check{m}\check{n}} & 0 \\ 0 & \eta_{\hat{m}\hat{n}} \end{pmatrix}. \tag{A.12}$$

Then the $\mathfrak{su}(2,2|4)$ generators $(\mathbf{P}_{\check{m}}\,,\mathbf{P}_{\hat{m}}\,,\mathbf{J}_{\check{m}\check{n}}\,,\mathbf{J}_{\hat{m}\hat{n}}\,,\mathbf{Q}^I)$ $(I = 1,2)$ can be represented by the following 8×8 supermatrices:

$$\mathbf{P}_{\check{m}} = \begin{pmatrix} -\frac{1}{2}\boldsymbol{\gamma}_{\check{m}} & \mathbf{0_4} \\ \mathbf{0_4} & \mathbf{0_4} \end{pmatrix}, \qquad \mathbf{J}_{\check{m}\check{n}} = \begin{pmatrix} +\frac{1}{2}\boldsymbol{\gamma}_{\check{m}\check{n}} & \mathbf{0_4} \\ \mathbf{0_4} & \mathbf{0_4} \end{pmatrix} \qquad (\check{m},\ \check{n} = 0,\ldots,4),$$
$$\mathbf{P}_{\hat{m}} = \begin{pmatrix} \mathbf{0_4} & \mathbf{0_4} \\ \mathbf{0_4} & +\frac{i}{2}\boldsymbol{\gamma}_{\hat{m}} \end{pmatrix}, \qquad \mathbf{J}_{\hat{m}\hat{n}} = \begin{pmatrix} \mathbf{0_4} & \mathbf{0_4} \\ \mathbf{0_4} & +\frac{1}{2}\boldsymbol{\gamma}_{\hat{m}\hat{n}} \end{pmatrix} \qquad (\hat{m},\ \hat{n} = 5,\ldots,9),$$
$$\left(\mathbf{Q}^{\check{\alpha}\hat{\alpha}}\right)^I = \begin{pmatrix} \mathbf{0_4} & {m_I}^{\check{\alpha}\hat{\alpha}} \\ -\,{\bar{m}_I}^{\check{\alpha}\hat{\alpha}} & \mathbf{0_4} \end{pmatrix}. \tag{A.13}$$

Here we have used the following notation,

$$\boldsymbol{\gamma}_{\check{m}\check{n}} \equiv \frac{1}{2}[\boldsymbol{\gamma}_{\check{m}},\boldsymbol{\gamma}_{\check{n}}], \qquad \boldsymbol{\gamma}_{\hat{m}\hat{n}} \equiv \frac{1}{2}[\boldsymbol{\gamma}_{\hat{m}},\boldsymbol{\gamma}_{\hat{n}}],$$

and introduced the following quantities,

$$({m_I}^{\check{\alpha}\hat{\alpha}})_i^{\ j} \equiv \mathrm{e}^{(1-(-1)^I)i\,\frac{\pi}{4}}K^{j\check{\alpha}}\delta_i^{\hat{\alpha}},$$
$$({\bar{m}_I}^{\check{\alpha}\hat{\alpha}})_i^{\ j} \equiv \mathrm{e}^{(1+(-1)^I)i\,\frac{\pi}{4}}K^{\hat{\alpha}j}\delta_i^{\check{\alpha}}, \tag{A.14}$$

with the matrix K defined as

$$K \equiv i\,\gamma_2\,\gamma_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A.15}$$

It is helpful to summarize the action of d and $\tilde{d}$ defined in (2.7) and (3.23), respectively:

$$d(\mathbf{P}_m) = \tilde{d}(\mathbf{P}_m) = 2\,\mathbf{P}_m, \quad d(\mathbf{J}_{mn}) = \tilde{d}(\mathbf{J}_{mn}) = 0, \quad d(\mathbf{Q}^I) = -\tilde{d}(\mathbf{Q}^I) = \sigma_3^{IJ}\mathbf{Q}^J. \tag{A.16}$$

A.4. Supertrace formulae

When the Lagrangian is evaluated, the following supertrace formulae are useful:

$$\begin{aligned}
\mathrm{Str}\,[\,\mathbf{P}_m\mathbf{P}_n\,] &= \eta_{mn},\\
\mathrm{Str}\left[\,\mathbf{J}_{\check{m}\check{n}}\mathbf{J}_{\check{p}\check{q}}\,\right] &= -(\eta_{\check{m}\check{p}}\eta_{\check{n}\check{q}} - \eta_{\check{m}\check{q}}\eta_{\check{n}\check{p}}),\\
\mathrm{Str}\left[\,\mathbf{J}_{\hat{m}\hat{n}}\mathbf{J}_{\hat{p}\hat{q}}\,\right] &= +(\eta_{\hat{m}\hat{p}}\eta_{\hat{n}\hat{q}} - \eta_{\hat{m}\hat{q}}\eta_{\hat{n}\hat{p}}),\\
\mathrm{Str}\left[(\mathbf{Q}^{\check{\alpha}\hat{\alpha}})^I(\mathbf{Q}^{\check{\beta}\hat{\beta}})^J\right] &= -2\epsilon^{IJ}K^{\check{\alpha}\check{\beta}}K^{\hat{\alpha}\hat{\beta}}.
\end{aligned} \tag{A.17}$$

A.5. A lift up to the 16×16 matrix representation

Here let us consider a lift up of the 8×8 matrix representation to the 16×16 one that appears in the $\mathfrak{su}(2,2|4)$ superalgebra (3.1).

The 16×16 gamma matrices γ_m in the superalgebra (3.1) can be realized as follows:

$$\gamma_m = (\gamma_{\check{m}}, \gamma_{\hat{m}}) \qquad (m = 0, \dots, 9), \tag{A.18}$$

where $\gamma_{\check{m}}$ and $\gamma_{\hat{m}}$ are constructed as a tensor product with $\mathbf{1_4}$ like

$$\gamma_{\check{m}} \equiv \boldsymbol{\gamma}_{\check{m}} \otimes \mathbf{1_4}, \qquad \gamma_{\hat{m}} \equiv \mathbf{1_4} \otimes i\,\boldsymbol{\gamma}_{\hat{m}}. \tag{A.19}$$

More explicitly, the index structure can be displayed as

$$\begin{aligned}
(\gamma_m)_{\check{\alpha}\hat{\alpha}}{}^{\check{\beta}\hat{\beta}} &= \left((\gamma_{\check{m}})_{\check{\alpha}\hat{\alpha}}{}^{\check{\beta}\hat{\beta}}, (\gamma_{\hat{m}})_{\check{\alpha}\hat{\alpha}}{}^{\check{\beta}\hat{\beta}}\right),\\
(\gamma_{\check{m}})_{\check{\alpha}\hat{\alpha}}{}^{\check{\beta}\hat{\beta}} &= \left(\boldsymbol{\gamma}_{\check{m}}\right)_{\check{\alpha}}{}^{\check{\beta}} \otimes (\mathbf{1_4})_{\hat{\alpha}}{}^{\hat{\beta}}, \qquad (\gamma_{\hat{m}})_{\check{\alpha}\hat{\alpha}}{}^{\check{\beta}\hat{\beta}} = (\mathbf{1_4})_{\check{\alpha}}{}^{\check{\beta}} \otimes i\left(\boldsymbol{\gamma}_{\hat{m}}\right)_{\hat{\alpha}}{}^{\hat{\beta}}.
\end{aligned} \tag{A.20}$$

Then the gamma matrices γ_m act on the spinor $\theta_I = (\theta_{\check{\alpha}\hat{\alpha}})_I$ like

$$\gamma_m\theta_I = (\gamma_m)_{\check{\alpha}\hat{\alpha}}{}^{\check{\beta}\hat{\beta}}\left(\theta_{\check{\beta}\hat{\beta}}\right)_I. \tag{A.21}$$

A.6. A lift up to the 32×32 matrix representation

In Sect. 3, it is necessary to rewrite the deformed Lagrangian in terms of ten-dimensional 32×32 gamma matrices Γ in order to read off the component fields of type IIB supergravity. Hence we introduce a concise rule to switch from the 16×16 notation to the 32×32 one.

Let us first define the following rules:

$$\begin{aligned}
\bar{\Theta}_I\Gamma_m\Theta_J &\equiv \bar{\theta}_I\gamma_m\theta_J,\\
\bar{\Theta}_I\Gamma_m\Gamma_{np}\Theta_J &\equiv \bar{\theta}_I\gamma_m\gamma_{np}\theta_J.
\end{aligned} \tag{A.22}$$

Here we have defined γ_{mn} as

$$\begin{aligned}
\gamma_{\check{m}\check{n}} &\equiv \boldsymbol{\gamma}_{\check{m}\check{n}} \otimes \mathbf{1_4},\\
\gamma_{\check{m}\hat{n}} &\equiv \boldsymbol{\gamma}_{\check{m}} \otimes i\,\boldsymbol{\gamma}_{\hat{n}},\\
\gamma_{\hat{m}\hat{n}} &\equiv \mathbf{1_4} \otimes \boldsymbol{\gamma}_{\hat{m}\hat{n}}.
\end{aligned} \tag{A.23}$$

Then the other combinations of gamma matrices are automatically lifted up as follows:

$$\bar{\theta}_I\,\gamma_m\,\gamma_n\,\theta_J = \bar{\Theta}_I\,\Gamma_m\,\Gamma_{01234}\,\Gamma_n\,\Theta_J,$$

$$\bar{\theta}_I \gamma_{mn} \theta_J = \bar{\Theta}_I \Gamma_{01234} \Gamma_{mn} \Theta,$$
$$\bar{\theta}_I \gamma_{mn} \gamma_p \theta_J = \bar{\Theta}_I \Gamma_{01234} \Gamma_{mn} \Gamma_{01234} \Gamma_p \Theta_J,$$
$$\bar{\theta}_I \gamma_{mn} \gamma_{pq} \theta_J = \bar{\Theta}_I \Gamma_{01234} \Gamma_{mn} \Gamma_{pq} \Theta_J. \tag{A.24}$$

Note here that the right-hand side of (A.24) contains the factor Γ_{01234}, which is evaluated as

$$\Gamma_{01234} = \frac{1}{5!}\Gamma_{[0}\cdots\Gamma_{4]} = \sigma_1 \otimes \mathbf{1_4} \otimes \mathbf{1_4}.$$

The insertion of this factor is necessary for an appropriate lift-up. For the detail of the lift-up, see, for example, [67].

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