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Generalized quark–antiquark potentials from a q-deformed $AdS_5 \times S^5$ background

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We study minimal surfaces with a single cusp in a q-deformed $AdS_5 \times S^5$ background. The cusp is composed of two half-lines with an arbitrary angle and is realized on a surface specified in the deformed AdS_5. The classical string solutions attached to this cusp are regarded as a generalization of configurations studied by Drukker and Forini [J. High Energy Phys. **1106**, 131 (2011) [arXiv:1105.5144 [hep-th]]] in the undeformed case. By taking an antiparallel-lines limit, a quark–antiquark potential for the q-deformed case is derived with a certain subtraction scheme. The resulting potential becomes linear at short distances with a finite deformation parameter. In particular, the linear behavior for the gravity dual of noncommutative gauge theories can be reproduced as a special scaling limit. Finally we study the near straight-line limit of the potential.

Subject Index A10, B21, B81, B82

1. Introduction

One of the most profound subjects in string theory is the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1]. A prototypical example is the conjectured equivalence between type IIB string theory on the $AdS_5 \times S^5$ background and 4D $\mathcal{N} = 4$ $SU(N)$ super Yang–Mills (SYM) theory in the large-N limit. A great discovery is that an integrable structure underlying this duality has been unveiled. This integrability enables us to compute physical quantities at arbitrary coupling constant even in non-BPS sectors; this has led to an enormous amount of support for this duality [2].

The classical action of the $AdS_5 \times S^5$ superstring can be constructed by following the Green–Schwarz formulation with a supercoset [3]:

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}, \tag{1.1}$$

which ensures the classical integrability in the sense of kinematical integrability [4]. As a next step, it would be intriguing to consider integrable deformations of the AdS/CFT and reveal the fundamental mechanism underlying gauge/gravity dualities without relying on the conformal symmetry.

On the string-theory side, in order to study integrable deformations, it is nice to follow the Yang–Baxter sigma-model approach [5–7]. This is a systematic way to consider integrable deformations of 2D nonlinear sigma models. Following this approach, one can specify an integrable deformation by taking a (skew-symmetric) classical r-matrix satisfying the modified classical Yang–Baxter equation

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(mCYBE). The deformed sigma-model action is classically integrable in the sense of the kinematical integrability (i.e., a Lax pair exists).

The original argument was restricted to principal chiral models, but it has been generalized to the symmetric coset case [8][1]. With this success, a q-deformation of the $AdS_5 \times S^5$ superstring action has been studied in Refs. [19,20] by adopting a classical r-matrix of Drinfel'd–Jimbo type [21–23]. This deformed system is often called the η-model. The metric and Neveu–Schwarz–Neveu–Schwarz (NS–NS) two-form are computed in Ref. [24]. Some special limits of deformed $AdS_n \times S^n$ were studied in Refs. [25,26]. The supercoset construction was recently performed in Ref. [27] and the full background was derived (for the associated solution, see Ref. [28]). The resulting background does not satisfy the equations of motion of type IIB supergravity, but it is conjectured that it should satisfy the modified type IIB supergravity equations [29].

For the η-model, a great deal of work has been done so far. A mirror description is proposed in Refs. [30–32]. The fast-moving string limits are considered in Refs. [33,34]. Giant magnon solutions are studied in Refs. [30,35–37]. The deformed Neumann–Rosochatius systems are derived in Refs. [38,39]. A possible holographic setup has been proposed in Ref. [39] and minimal surfaces are studied in Refs. [39–42]. Three-point functions [43–45] and the D1-brane [46] are also discussed. For two-parameter deformations, see Refs. [25,47–49]. Another integrable deformation called the λ-deformation [50–57] is closely linked to the η-model by a Poisson–Lie duality [50–53,58–60].

A possible generalization of the Yang–Baxter sigma model is based on the homogeneous classical Yang–Baxter equation (CYBE) [61]. A strong advantage is that partial deformations of $AdS_5 \times S^5$ can be studied. In a series of works [62–70], a lot of classical r-matrices have been identified with the well-known type IIB supergravity solutions including the γ-deformations of S^5 [71], gravity duals for noncommutative (NC) gauge theories [72,73], and Schrödinger spacetimes [74–76]. The relationship between the gravity solutions and the classical r-matrices is referred to as the gravity/CYBE correspondence (for a short summary, see Refs. [77,78]). This correspondence indicates that the moduli space of a certain class of type IIB supergravity solutions may be identified with the CYBE.

Recently, this identification has been generalized to integrable deformations of 4D Minkowski spacetime in Refs. [79–81]. A q-deformation of the flat space string has also been studied from a scaling limit of the η-deformed $AdS_5 \times S^5$ [82]. Furthermore, as an application, the Yang–Baxter invariance of the Nappi–Witten model [83] has been shown in Ref. [84]. Another remarkable feature is that Yang–Baxter deformations can be applied to nonintegrable backgrounds beyond integrability. An example of nonintegrable backgrounds is $AdS_5 \times T^{1,1}$ [85] and the nonintegrability is supported by the existence of chaotic string solutions [86–88]. Then TsT transformations of $T^{1,1}$ can be reproduced as Yang–Baxter deformations as well [89,90]. This result indicates that the gravity/CYBE correspondence is not limited to the integrable backgrounds.

Here we will focus upon minimal surfaces in the q-deformed $AdS_5 \times S^5$ superstring [19,20,24] (the η-model). It is interesting to argue a holographic relation in the q-deformed background. We have proposed that the singularity surface in the deformed AdS may be treated as the holographic screen [39,40]. For this purpose, it is convenient to introduce a coordinate system that describes only the spacetime enclosed by the singularity surface [39]. Applying this coordinate system, minimal surfaces whose boundaries are straight lines and circles have been considered in Refs. [39,40].

[1] For earlier developments on sigma-model realizations of q-deformed $su(2)$ and $sl(2)$, see Refs. [9–18].

In the $q \to 1$ limit, the solutions correspond to a 1/2-BPS straight Wilson loop [91–93] and a 1/2-BPS circular Wilson loop [94–97], respectively.

In this paper, we continue to study minimal surfaces in the q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ and consider a generalization with a single cusp. The cusp is composed of two half-lines with an arbitrary angle and it is realized on a surface specified in the deformed AdS_5. The classical string solutions attached to this cusp are regarded as a generalization of configurations studied by Drukker and Forini [99] in the undeformed case. By taking an antiparallel-lines limit, we derive a quark–antiquark potential for the q-deformed case with a certain subtraction scheme. The resulting potential becomes linear at short distances with a finite deformation parameter. In particular, the linear behavior for the gravity dual for noncommutative gauge theories can be reproduced by taking a special scaling limit.

This paper is organized as follows. Section 2 gives a short review of string theory on the q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. In Sect. 3, we study a minimal surface whose boundary is given by two half-lines with an angle. The classical action is evaluated with a certain subtraction scheme. In Sect. 4, a quark–antiquark potential is derived by taking an antiparallel-lines limit. The resulting potential exhibits a linear behavior at short distances. In Sect. 5, we study the near straight-line expansion in detail. Section 6 is devoted to the conclusion and discussion.

Appendix A summarizes the definition of elliptic integrals and some properties. Appendix B presents a classical solution of the Wilson loop with a cusp in the global coordinates. Appendix C gives a review of the derivation of a linear potential at short distances from the gravity dual of noncommutative gauge theories.

2. String theory on a q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$

The classical superstring action on the q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background has been constructed by Delduc et al. [19,20]. The metric (in the string frame) and NS–NS two-form have been computed in Ref. [24]. Here, for simplicity, we will focus upon the bosonic part of the classical action with the conformal gauge. The resulting action is composed of the metric part and the Wess–Zumino (WZ) term, as we will show later.

2.1. *The convention of the string action*

Let us here introduce the bosonic part of the string action (with the conformal gauge), which can be divided into the metric part S_G and the Wess–Zumino (WZ) term S_{WZ}:

$$S = S_G + S_{\mathrm{WZ}}.$$

Here S_{WZ} describes the coupling of string to an NS–NS two-form.

The metric part S_G is further divided into the deformed AdS_5 and S^5 parts like

$$\begin{aligned} S_G &= \int d\tau\, d\sigma\, \left[\mathcal{L}_G^{(\mathrm{AdS})} + \mathcal{L}_G^{(\mathrm{S})}\right] \\ &= -\frac{1}{4\pi\alpha'}\int d\tau\, d\sigma\, \eta^{\mu\nu}\left[G^{(\mathrm{AdS})}_{MN}\,\partial_\mu X^M \partial_\nu X^N + G^{(\mathrm{S})}_{PQ}\,\partial_\mu Y^P \partial_\nu Y^Q\right], \end{aligned}$$

where the string world-sheet coordinates are $\sigma^\mu = \left(\sigma^0, \sigma^1\right) = (\tau, \sigma)$ with $\eta_{\mu\nu} = (-1, +1)$.

The WZ term S_{WZ} is also divided into two parts:

$$S_{\mathrm{WZ}} = -\frac{1}{4\pi\alpha'}\int d\tau\, d\sigma\, \epsilon^{\mu\nu}\left[B^{(\mathrm{AdS})}_{MN}\,\partial_\mu X^M \partial_\nu X^N + B^{(\mathrm{S})}_{PQ}\,\partial_\mu Y^P \partial_\nu Y^Q\right].$$

Here the totally antisymmetric tensor $\epsilon^{\mu\nu}$ is normalized as $\epsilon^{01} = +1$.

2.2. *A q-deformed* $AdS_5 \times S^5$

Next, let us introduce the metric and NS–NS two-form of a q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ [24][2].

The metric is divided into deformed AdS_5 and S^5 parts like

$$ds^2_{(\mathrm{AdS}_5)_q} = R^2\sqrt{1+C^2}\left[-\frac{\cosh^2\rho\, dt^2}{1-C^2\sinh^2\rho} + \frac{d\rho^2}{1-C^2\sinh^2\rho} + \frac{\sinh^2\rho\, d\zeta^2}{1+C^2\sinh^4\rho\sin^2\zeta} + \frac{\sinh^2\rho\cos^2\zeta\, d\varphi^2}{1+C^2\sinh^4\rho\sin^2\zeta} + \sinh^2\rho\sin^2\zeta\, d\psi^2\right], \tag{2.1}$$

$$ds^2_{(\mathrm{S}^5)_q} = R^2\sqrt{1+C^2}\left[\frac{\cos^2\gamma\, d\vartheta^2}{1+C^2\sin^2\gamma} + \frac{d\gamma^2}{1+C^2\sin^2\gamma} + \frac{\sin^2\gamma\, d\xi^2}{1+C^2\sin^4\gamma\sin^2\xi} + \frac{\sin^2\gamma\cos^2\xi\, d\phi_1^2}{1+C^2\sin^4\gamma\sin^2\xi} + \sin^2\gamma\sin^2\xi\, d\phi_2^2\right]. \tag{2.2}$$

Here $(t, \varphi, \psi, \zeta, \rho)$ parameterize the deformed AdS_5; $(\vartheta, \gamma, \phi_1, \phi_2, \xi)$ the deformed S^5. The deformation is characterized by a real parameter $C \in [0, \infty)$. When $C = 0$, the geometry is reduced to the usual $\mathrm{AdS}_5 \times \mathrm{S}^5$ with the curvature radius R. Note that a curvature singularity exists at $\rho = \mathrm{arcsinh}(1/C)$ in (2.1).

Let us comment on the causal structure near the singularity surface [40]. For massless particles, it takes infinite affine time to reach the singularity surface, while it does not in the coordinate time. Massive particles cannot reach the surface as well. Thus this property is essentially the same as the conformal boundary of the usual global AdS_5.

The NS–NS two-form $B_2 = B_{(\mathrm{AdS}_5)_q} + B_{(\mathrm{S}^5)_q}$ is given by

$$B_{(\mathrm{AdS}_5)_q} = R^2 C\sqrt{1+C^2}\,\frac{\sinh^4\rho\sin 2\zeta}{1+C^2\sinh^4\rho\sin^2\zeta}\, d\varphi\wedge d\zeta, \tag{2.3}$$

$$B_{(\mathrm{S}^5)_q} = -R^2 C\sqrt{1+C^2}\,\frac{\sin^4\gamma\sin 2\xi}{1+C^2\sin^4\gamma\sin^2\xi}\, d\phi_1\wedge d\xi. \tag{2.4}$$

Note that B_2 vanishes when $C = 0$.

2.3. *Poincaré coordinates*

In Sect. 2.2, we have discussed the bosonic background in the global coordinates [24]. In order to study minimal surfaces, however, it is helpful to adopt Poincaré-like coordinates for the metric of the deformed AdS_5 (2.1) in the Euclidean signature.

After performing the Wick rotation $t \to i\tau$ and the coordinate transformation:

$$\sinh\rho = \frac{r}{\sqrt{z^2 + C^2\left(z^2+r^2\right)}}, \qquad e^{\tau} = \sqrt{z^2+r^2}, \tag{2.5}$$

[2] By performing a supercoset construction, the full component has been obtained recently [27].

the resulting metric describes a deformed Euclidean AdS_5 [40]:

$$ds^2_{(\mathrm{AdS}_5)_q} = R^2\sqrt{1+C^2}\left[\frac{dz^2+dr^2}{z^2+C^2\big(z^2+r^2\big)}+\frac{C^2(z\,dz+r\,dr)^2}{z^2\big(z^2+C^2\big(z^2+r^2\big)\big)}\right.$$
$$\left.+\frac{\big(z^2+C^2\big(z^2+r^2\big)\big)\,r^2}{\big(z^2+C^2\big(z^2+r^2\big)\big)^2+C^2r^4\sin^2\zeta}\left(d\zeta^2+\cos^2\zeta\,d\varphi^2\right)+\frac{r^2\sin^2\zeta\,d\psi^2}{z^2+C^2\big(z^2+r^2\big)}\right]. \tag{2.6}$$

Note that the singularity surface is now located at $z=0$ in the above coordinates as well. It is shown in Ref. [40] that space-like proper distances to the singularity surface are finite, in comparison to the undeformed case. This property might be important in the next section.

When $C=0$, the deformed metric (2.6) is reduced to the Euclidean AdS_5 with Poincaré coordinates as a matter of course. Inversely speaking, one may think of the metric (2.6) giving rise to an integrable deformation of Euclidean Poincaré AdS_5 from the beginning.

3. Cusped minimal surfaces

In this section, we will explicitly derive a classical string solution ending on two half-lines with an arbitrary angle (i.e., a cusp) on the boundary of the Euclidean q-deformed AdS_5. After solving the equations of motion, the Nambu–Goto action is evaluated with a boundary term coming from a Legendre transformation. Our argument basically follows seminal papers [98,99] for the undeformed case.

3.1. *Classical string solutions*

Suppose that the boundary conditions are lines separated by $\pi-\phi$ on the boundary of the deformed AdS_5 and θ on the deformed S^5. Hence it is sufficient to consider an $AdS_3\times S^1$ subspace of the deformed background by requiring that the solution is located at[3]

$$\psi=\zeta=0(\mathrm{AdS}_5),\quad \gamma=\phi_1=\phi_2=\xi=0\Big(\mathrm{S}^5\Big). \tag{3.1}$$

Then the resulting metric of the $AdS_3\times S^1$ subspace is given by

$$ds^2_{(\mathrm{AdS}_3\times\mathrm{S}^1)_q} = R^2\sqrt{1+C^2}\left[\frac{dz^2+dr^2+r^2d\varphi^2}{z^2+C^2\big(z^2+r^2\big)}+\frac{C^2(z\,dz+r\,dr)^2}{z^2\big(z^2+C^2\big(z^2+r^2\big)\big)}+d\vartheta^2\right]. \tag{3.2}$$

Note that the NS–NS two-form (2.4) vanishes under the condition (3.1).

It should be remarked that the metric (2.6) (or (3.2)) is invariant under the rescaling

$$z\to c_0\,z,\quad r\to c_0\,r\quad (c_0>0). \tag{3.3}$$

Then it is helpful to suppose that the tip of the cusp is located at $r=0$ so that the cusp is invariant under the rescaling (3.3).

Let us take r and φ as the string world-sheet coordinates. Then it is natural to suppose that the z coordinate is linear to r so as to respect the rescaling (3.3) like

$$z=r\,v(\varphi),\quad \vartheta=\vartheta(\varphi). \tag{3.4}$$

The coordinate φ extends from $\phi/2$ to $\pi-\phi/2$. We suppose that $v(\varphi)=0$ at the two boundaries where $\varphi=\phi/2$ and $\pi-\phi/2$, while it takes its maximal value v_m at $\varphi=\pi/2$. For the sphere part, the coordinate ϑ ranges from $-\theta/2$ to $\theta/2$.

[3] It seems quite difficult to study a cusped minimal surface solution with a nonvanishing ζ on the q-deformed background; hence, we choose $\zeta=0$ for simplicity.

Then the Nambu–Goto action is given by

$$S_{\mathrm{NG}} = \sqrt{1+C^2}\,\frac{\sqrt{\lambda}}{2\pi}\int\frac{dr}{r}\int_{\phi/2}^{\pi-\phi/2} d\varphi\,\mathcal{L}(\varphi),$$
$$\mathcal{L}(\varphi) = \frac{1}{v\,v_C}\sqrt{v'^2+\left(1+v^2\right)\left(1+v_C^2\,\vartheta'^2\right)}. \tag{3.5}$$

Here, the prime is the derivative with respect to φ, and λ is defined as

$$\sqrt{\lambda} \equiv \frac{R^2}{\alpha'}. \tag{3.6}$$

We have introduced the C-dependent function v_C as

$$v_C(\varphi) \equiv \sqrt{v^2 + C^2\left(1+v^2\right)}, \tag{3.7}$$

which is reduced to v in the $C \to 0$ limit.

Note that the r-dependence is factored out; hence two conserved charges can readily be obtained. The Hamiltonian E that corresponds to ∂_φ translations and the canonical momentum J conjugate to ϑ are given by, respectively,

$$E = \frac{1+v^2}{v\,v_C\sqrt{v'^2+\left(1+v^2\right)\left(1+v_C^2\,\vartheta'^2\right)}}, \qquad J = \frac{v_C\left(1+v^2\right)\,\vartheta'}{v\sqrt{v'^2+\left(1+v^2\right)\left(1+v_C^2\,\vartheta'^2\right)}}. \tag{3.8}$$

It is helpful to introduce the following conserved quantities:

$$p \equiv \frac{1}{E} > 0, \qquad q \equiv \frac{J}{E} = v_C^2\,\vartheta'. \tag{3.9}$$

Then, by using the relation (3.9), the differential equation for $v(\varphi)$ is given by

$$v'^2 = \frac{1+v^2}{v^2\,v_C^2}\left[p^2+\left(p^2-q^2\right)v^2 - v^2 v_C^2\right]$$
$$= \frac{1+v^2}{v^2\,v_C^2}\,\frac{p^2}{v_m^2}\left(1+\frac{b^2}{p^2}\,v^2\right)\left(v_m^2 - v^2\right). \tag{3.10}$$

Here a new parameter b has been introduced as

$$b \equiv \sqrt{\frac{1}{2}\left(p^2-q^2-C^2+\sqrt{\left(p^2-q^2-C^2\right)^2+4\left(1+C^2\right)p^2}\right)} > 0, \tag{3.11}$$

and v_m is the turning point in v where $v'(\varphi=\pi/2)=0$, which is given by

$$v_m^2 = \frac{b^2}{1+C^2}. \tag{3.12}$$

To make Eq. (3.10) more tractable, let us define the following function:

$$x(\varphi) \equiv \sqrt{\frac{v_C^2\left(b^4+\left(1+C^2\right)p^2\right)}{\left(1+C^2\right)\left(b^2+C^2\right)\left(p^2+b^2 v^2\right)}}. \tag{3.13}$$

Then Eq. (3.10) can be expressed as an elliptic equation,

$$x'^2 = \frac{b^2\left[b^4+\left(1+C^2\right)p^2\right]}{\left[p^2+C^2\left(p^2-b^2\right)\right]^2}\left(\frac{b^4+\left(1+C^2\right)p^2}{b^2\left(b^2+C^2\right)x^2}-1\right)^2\left(1-x^2\right)\left(1-k^2x^2\right), \tag{3.14}$$

where k is defined as

$$k \equiv \sqrt{\frac{\left(1+C^2\right)\left(b^2+C^2\right)\left(b^2-p^2\right)}{b^4+\left(1+C^2\right)p^2}}, \tag{3.15}$$

and it satisfies $0 \le k < 1$. Note that k becomes zero when $E = |J|$, i.e., $q = \pm 1$, with the condition $p > 0$, $C \ge 0$. This corresponds to a BPS case in the undeformed limit.

For later purposes, it may be helpful to rewrite p and q in terms of b and k as

$$p^2 = \frac{b^2\left[\left(1+C^2\right)\left(b^2+C^2\right)-b^2k^2\right]}{\left(1+C^2\right)\left(b^2+C^2+k^2\right)}, \tag{3.16}$$

$$q^2 = \frac{\left(1+C^2\right)\left(b^2+C^2\right)-k^2\left[b^4+\left(1+C^2\right)\left(2b^2+C^2\right)\right]}{\left(1+C^2\right)\left(b^2+C^2+k^2\right)}. \tag{3.17}$$

In the $C \to 0$ limit, b and k are reduced to the undeformed ones b_0 and k_0 [99], respectively:

$$b^2 \longrightarrow b_0^2 \equiv \frac{1}{2}\left(p^2-q^2+\sqrt{\left(p^2-q^2\right)^2+4p^2}\right), \quad k^2 \longrightarrow k_0^2 \equiv \frac{b_0^2\left(b_0^2-p^2\right)}{b_0^4+p^2}. \tag{3.18}$$

Note that p and q above are given by (3.8) and (3.9) with $C = 0$.

Classical solution

Let us solve the equation of motion for $x(\varphi)$.

In the first place, one needs to fix a boundary condition. Here let us impose the following boundary condition. For the world-sheet segment i) $\phi/2 \le \varphi \le \pi/2$, $v(\varphi)$ increases monotonically as

$$v(\phi/2) = 0(\text{boundary}) \longrightarrow v(\pi/2) = v_m \quad (\text{midpoint}), \tag{3.19}$$

while for the other segment ii) $\pi/2 < \varphi \le \pi - \phi/2$, $v(\varphi)$ decreases monotonically as

$$v(\pi/2) = v_m(\text{midpoint}) \longrightarrow v(\pi-\phi/2) = 0 \quad (\text{boundary}). \tag{3.20}$$

In terms of $x(\varphi)$ (3.13), the above conditions can be rewritten as

$$\text{i)}\quad x(\phi/2) = x_0(\text{boundary}) \longrightarrow x(\pi/2) = 1 \quad (\text{midpoint}),$$

$$\text{ii)}\quad x(\pi/2) = 1(\text{midpoint}) \longrightarrow x(\pi-\phi/2) = x_0 \quad (\text{boundary}).$$

At the boundary, $x(\varphi)$ takes the minimum value x_0 given by

$$x_0 = \frac{C}{\sqrt{1+C^2}}\sqrt{\frac{b^4+\left(1+C^2\right)p^2}{p^2\left(b^2+C^2\right)}}. \tag{3.21}$$

Note that x_0 satisfies $0 < x_0 < 1$ and vanishes in the $C \to 0$ limit. At the turning point $\varphi = \pi/2$, $x(\varphi)$ take the maximum, i.e., $x(\pi/2) = 1$.

Then let us solve the first-order differential equation (3.14)[4]:

$$x' = \frac{b\sqrt{b^4 + \left(1 + C^2\right) p^2}}{p^2 + C^2\left(p^2 - b^2\right)} \left(\frac{b^4 + \left(1 + C^2\right) p^2}{b^2\left(b^2 + C^2\right) x^2} - 1 \right) \sqrt{\left(1 - x^2\right)\left(1 - k^2 x^2\right)}. \tag{3.22}$$

By integrating (3.22) for $\varphi \geq \phi/2$ with the boundary condition $x(\phi/2) = x_0$, one can obtain the following expression:

$$\int_{\phi/2}^{\varphi} d\tilde{\varphi} = \frac{p^2 + C^2\left(p^2 - b^2\right)}{b\sqrt{b^4 + \left(1 + C^2\right) p^2}} \int_{x_0}^{x} \frac{d\tilde{x}}{\left(\frac{b^4 + \left(1 + C^2\right) p^2}{b^2\left(b^2 + C^2\right) \tilde{x}^2} - 1 \right) \sqrt{\left(1 - \tilde{x}^2\right)\left(1 - k^2 \tilde{x}^2\right)}}. \tag{3.23}$$

Then the world-sheet coordinate $\varphi(\phi/2 \leq \varphi \leq \pi/2)$ can be written in terms of incomplete elliptic integrals of the first and third kinds like

$$\begin{aligned} \varphi = \frac{\phi}{2} + \frac{p^2 + C^2\left(p^2 - b^2\right)}{b\sqrt{b^4 + \left(1 + C^2\right) p^2}} &\left[\Pi\left(\frac{b^2\left(b^2 + C^2\right)}{b^4 + \left(1 + C^2\right) p^2}, \arcsin x | k^2 \right) - F\left(\arcsin x | k^2 \right) \right. \\ &\left. - \Pi\left(\frac{b^2\left(b^2 + C^2\right)}{b^4 + \left(1 + C^2\right) p^2}, \arcsin x_0 | k^2 \right) + F\left(\arcsin x_0 | k^2 \right) \right]. \end{aligned} \tag{3.24}$$

By taking $x(\pi/2) = 1$, the cusp angle $\pi - \phi$ is represented by

$$\begin{aligned} \phi = \pi - 2\,\frac{p^2 + C^2\left(p^2 - b^2\right)}{b\sqrt{b^4 + \left(1 + C^2\right) p^2}} &\left[\Pi\left(\frac{b^2\left(b^2 + C^2\right)}{b^4 + \left(1 + C^2\right) p^2} \,| k^2 \right) - K\left(k^2 \right) \right. \\ &\left. - \Pi\left(\frac{b^2\left(b^2 + C^2\right)}{b^4 + \left(1 + C^2\right) p^2}, \arcsin x_0 | k^2 \right) + F\left(\arcsin x_0 | k^2 \right) \right]. \end{aligned} \tag{3.25}$$

Note here that the incomplete elliptic integrals have been replaced by complete ones.

For the other segment $\pi/2 < \varphi \leq \pi - \phi/2$, an analytical continuation of the solution is necessary. The resulting expression is given by

$$\begin{aligned} \varphi = \pi - \frac{\phi}{2} - \frac{p^2 + C^2\left(p^2 - b^2\right)}{b\sqrt{b^4 + \left(1 + C^2\right) p^2}} &\left[\Pi\left(\frac{b^2\left(b^2 + C^2\right)}{b^4 + \left(1 + C^2\right) p^2}, \arcsin x | k^2 \right) - F\left(\arcsin x | k^2 \right) \right. \\ &\left. - \Pi\left(\frac{b^2\left(b^2 + C^2\right)}{b^4 + \left(1 + C^2\right) p^2}, \arcsin x_0 | k^2 \right) + F\left(\arcsin x_0 | k^2 \right) \right]. \end{aligned} \tag{3.26}$$

By taking the $C \to 0$ limit with p and q fixed, the solutions (3.25) and (3.26) are reduced to undeformed ones [99].

[4] Here the $(+)$-signature is taken, because we consider a classical solution stretching from a boundary at $\varphi = \phi/2$ to the turning point at $\varphi = \pi/2$.

The sphere part

The remaining thing is to get the expression of $\vartheta(\varphi)$. First of all, let us fix a boundary condition. For the world-sheet segment i) $\phi/2 \leq \varphi \leq \pi/2$, $\vartheta(\varphi)$ increases monotonically as

$$\vartheta(\phi/2) = -\theta/2\,(\text{boundary}) \longrightarrow \vartheta(\pi/2) = 0\,(\text{midpoint}), \tag{3.27}$$

while for the other segment ii) $\pi/2 < \varphi \leq \pi - \phi/2$, $\vartheta(\varphi)$ decreases monotonically as

$$\vartheta(\pi/2) = 0\,(\text{midpoint}) \longrightarrow \vartheta(\pi - \phi/2) = \theta/2\ (\text{boundary}). \tag{3.28}$$

By integrating ϑ in (3.9) for $\varphi \geq \phi/2$, one can obtain the following expression:

$$\int_{-\theta/2}^{\vartheta} d\tilde{\vartheta} = \frac{b\,q}{\sqrt{b^4 + \left(1 + C^2\right) p^2}} \int_{x_0}^{x} \frac{d\tilde{x}}{\sqrt{\left(1 - \tilde{x}^2\right)\left(1 - k^2\tilde{x}^2\right)}}. \tag{3.29}$$

This equation can be understood as the relation between x and ϑ:

$$\vartheta = -\frac{\theta}{2} + \frac{b\,q}{\sqrt{b^4 + \left(1 + C^2\right) p^2}} \left[F\left(\arcsin x | k^2\right) - F\left(\arcsin x_0 | k^2\right)\right]. \tag{3.30}$$

When $\varphi = \pi/2$, i.e., $x(\pi/2) = 1$, it reaches the midpoint $\vartheta = 0$; hence the angle θ in the sphere part is determined as

$$\theta = \frac{2b\,q}{\sqrt{b^4 + \left(1 + C^2\right) p^2}} \left[K\left(k^2\right) - F\left(\arcsin x_0 | k^2\right)\right]. \tag{3.31}$$

As a result, the expression (3.30) can be rewritten as

$$\vartheta = \frac{bq}{\sqrt{b^4 + \left(1 + C^2\right) p^2}} \left[F\left(\arcsin x | k^2\right) - K\left(k^2\right)\right]. \tag{3.32}$$

This solution can also reproduce the result in Ref. [99] in the undeformed limit.

3.2. *The classical action*

The next step is to evaluate the value of the classical string action.

First of all, it is helpful to rewrite the Euclidean classical action (3.5) by using the equation of motion (3.14). The resulting action is given by

$$S_{\text{NG}} = \frac{\sqrt{\lambda}}{2\pi} \frac{\sqrt{1 + C^2}}{C} \frac{x_0\sqrt{1 - x_0^2}}{\sqrt{1 - k^2 x_0^2}} \int_{\epsilon_{\text{UV}}}^{R_{\text{IR}}} \frac{dr}{r}\, 2\int_{x_0}^{1} dx\, \frac{\sqrt{1 - k^2 x^2}}{\left(x^2 - x_0^2\right)\sqrt{1 - x^2}}. \tag{3.33}$$

Here R_{IR} and ϵ_{UV} are IR and UV cut-offs for the r-direction, respectively. Then the r-integral is evaluated as

$$\int_{\epsilon_{\text{UV}}}^{R_{\text{IR}}} \frac{dr}{r} = \log \frac{R_{\text{IR}}}{\epsilon_{\text{UV}}} \equiv T. \tag{3.34}$$

Note that this quantity T can be identified with an interval of the global time coordinate[5]. Then the classical action can be calculated as

$$S_{\mathrm{NG}} = -\frac{T\sqrt{\lambda}}{\pi}\frac{\sqrt{1+C^2}}{C}\frac{x_0\sqrt{1-x_0^2}}{\sqrt{1-k^2x_0^2}}\left[k^2\int_{x_0}^{1}\frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\right.$$
$$\left.+\left(x_0^{-2}-k^2\right)\int_{x_0}^{1}\frac{dx}{\left(1-x_0^{-2}x^2\right)\sqrt{1-x^2}\sqrt{1-k^2x^2}}\right]$$
$$= -\frac{T\sqrt{\lambda}}{\pi}\frac{\sqrt{1+C^2}}{C}\frac{x_0\sqrt{1-x_0^2}}{\sqrt{1-k^2x_0^2}}\left(k^2\left[K\left(k^2\right)-F\left(\arcsin x_0|k^2\right)\right]\right.$$
$$\left.+\left(x_0^{-2}-k^2\right)\left[\Pi\left(x_0^{-2}|k^2\right)-\lim_{\epsilon\to 0}\Pi\left(x_0^{-2},\arcsin(x_0+\epsilon)\,|k^2\right)\right]\right), \tag{3.35}$$

where $\Pi\left(\alpha^2,\psi|k^2\right)$ is an incomplete elliptic integral of the third kind (for details, see Appendix A). In general, $\Pi\left(\alpha^2,\psi|k^2\right)$ are interpreted as Cauchy principal values when $\alpha > 1$ [100], and there are singularities on the real axis at $\psi = \arcsin\alpha^{-1}$. Thus, a cut-off ϵ has been introduced as $\epsilon \equiv x - x_0$ for the limit $x \to x_0$, because $\Pi\left(x_0^{-2},\arcsin x|k^2\right)$ diverges logarithmically as $x \to x_0$. Through the relation (3.13), the cut-off ϵ can be converted into v_0, which is a cut-off for small v:

$$\epsilon \equiv x - x_0 = \frac{x_0(1-k^2x_0^2)}{2C^2}\,v_0^2 + \mathcal{O}\left(v_0^4\right). \tag{3.36}$$

The elliptic integrals in (3.35) can be rewritten as follows:

$$\Pi\left(x_0^{-2}|k^2\right) - \lim_{\epsilon\to 0}\Pi\left(x_0^{-2},\arcsin(x_0+\epsilon)\,|k^2\right)$$
$$= \lim_{\epsilon\to 0}\left(\int_0^{x_0-\epsilon}dx + \int_{x_0+\epsilon}^{1}dx\right)\frac{1}{\left(1-x_0^{-2}x^2\right)\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$
$$-\lim_{\epsilon\to 0}\int_0^{x_0-\epsilon}dx\,\frac{1}{\left(1-x_0^{-2}x^2\right)\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$
$$= PV\,\Pi\left(x_0^{-2}|k^2\right) - \lim_{\epsilon\to 0}\Pi\left(x_0^{-2},\arcsin(x_0-\epsilon)\,|k^2\right).$$

Then the principal value can be evaluated as[6]

$$PV\,\Pi\left(x_0^{-2}|k^2\right) = -\frac{x_0\,K\left(k^2\right)\;Z\left(\arcsin x_0|k^2\right)}{\sqrt{\left(1-x_0^2\right)\left(1-k^2x_0^2\right)}}, \tag{3.37}$$

where $Z(\psi|k^2)$ is a Jacobi zeta function given by

$$Z\left(\psi|k^2\right) \equiv E\left(\psi|k^2\right) - \frac{E\left(k^2\right)}{K\left(k^2\right)}\,F\left(\psi|k^2\right). \tag{3.38}$$

[5] The r coordinate is identified with the global time t through $r = \exp t$; hence $\int dr/r = \int dt \equiv T$.
[6] Use the formula 415.01 in Ref. [100].

The incomplete elliptic integral of the third kind can be rewritten as[7]

$$\Pi\left(x_0^{-2}, \arcsin(x_0-\epsilon)\,|k^2\right) = \frac{x_0}{\sqrt{\left(1-x_0^2\right)\left(1-k^2x_0^2\right)}}\left(\frac{1}{2}\log\left[\frac{\vartheta_1(\omega+\nu, q_k)}{\vartheta_1(\omega-\nu, q_k)}\right]\right.$$
$$\left. - F\left(\arcsin\,(x_0-\epsilon)\,|k^2\right)\, Z\left(\arcsin x_0|k^2\right)\right), \tag{3.39}$$

where $\vartheta_1(z, q_k)$ is the Jacobi theta function. The parameters ν, ω, and q_k are defined as

$$\nu \equiv \frac{\pi F\left(\arcsin(x_0-\epsilon)\,|k^2\right)}{2K\left(k^2\right)}, \qquad \omega \equiv \frac{\pi F\left(\arcsin x_0|k^2\right)}{2K\left(k^2\right)}, \qquad q_k \equiv \mathrm{e}^{-\frac{\pi K\left(1-k^2\right)}{K\left(k^2\right)}}. \tag{3.40}$$

3.3. *Separation of the divergence*

Let us decompose $S_{\rm NG}$ (3.35) into the finite part $S_{\rm ren}$ and the divergent part S_0 like

$$S_{\rm NG} = S_{\rm ren} + S_0. \tag{3.41}$$

Here $S_{\rm ren}$ and S_0 are given by, respectively,

$$S_{\rm ren} = \frac{T\sqrt{\lambda}}{\pi}\frac{\sqrt{1+C^2}}{C}\left(K\left(k^2\right)\, Z\left(\arcsin x_0|k^2\right)\right.$$
$$\left. - \frac{k^2x_0\sqrt{1-x_0^2}}{\sqrt{1-k^2x_0^2}}\left[K\left(k^2\right) - F\left(\arcsin x_0|k^2\right)\right]\right), \tag{3.42}$$

$$S_0 = \lim_{\epsilon\to 0}\frac{T\sqrt{\lambda}}{2\pi}\frac{\sqrt{1+C^2}}{C}\log\left[\frac{\vartheta_1(\omega+\nu, q_k)}{\vartheta_1(\omega-\nu, q_k)}\, e^{-2\,F\left(\arcsin(x_0-\epsilon)|k^2\right)\, Z\left(\arcsin x_0|k^2\right)}\right]. \tag{3.43}$$

It is useful to look the origin of the logarithmic divergence in (3.43) in more detail. Let us convert the ϵ-dependence to the v_0 one through the relation (3.36). Then the divergent part S_0 can be expanded around $v_0 = 0$ as

$$S_0 = \frac{T\sqrt{\lambda}}{\pi}\frac{\sqrt{1+C^2}}{C}\left(\log\left[\frac{2C}{\sqrt{1+C^2}\,v_0}\right]\right.$$
$$\left. + \log\left[\left(\frac{\left(1+C^2\right)\sqrt{1-x_0^2}\,K\left(k^2\right)\,\vartheta_1(2\omega, q_k)}{\pi x_0\sqrt{1-k^2x_0^2}\,\vartheta_1'(0, q_k)}\right)^{\frac{1}{2}} e^{-\,F\left(\arcsin x_0|k^2\right)\, Z\left(\arcsin x_0|k^2\right)}\right]\right)$$
$$+\,\mathcal{O}\left(v_0^2\right), \tag{3.44}$$

where $\vartheta_1'(\omega, q_k) = \partial_\omega\vartheta_1(\omega, q_k)$. Now one can see that the first log-term in (3.44) diverges logarithmically as $v_0 \to 0$, while the second log is finite. It is worth noticing that the second log-term vanishes in the undeformed limit $C \to 0$.

Here we should remark that there is an ambiguity in that the second log-term in (3.44) may be included in $S_{\rm ren}$. In particular, the second log-term vanishes in the undeformed limit $C \to 0$, and

[7] Use the formula 436.01 in Ref. [100].

hence one cannot remove this ambiguity by relying on the undeformed limit. Therefore, consistency with the undeformed limit is not enough, and it is necessary to adopt an extra criterion.

Fortunately, there is a definite answer to this issue, i.e., a scaling limit of the q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ to the gravity dual for NC gauge theories [27]. Consistency with this limit gives rise to a sufficiently strong constraint for the regularization. In summary, we will adopt the following criteria to regularize the Nambu–Goto action:

a) S_{ren} is reduced to the usual regularized action in the $C \to 0$ limit.
b) The antiparallel-lines limit of S_{ren} reproduces a quark–antiquark potential derived from the gravity dual for NC gauge theories by taking the scaling limit [27].

According to these criteria, the second log-term in (3.44) should NOT be included in S_{ren} so as to satisfy condition b), as we will see later. Thus, what is the physical interpretation of the second log-term? In the next subsection, we will consider the physical interpretation of the regularization.

3.4. *Interpretation of the regularization*

We will consider here the physical interpretation of the regularization adopted in the previous subsection. Our aim is to compute the quark–antiquark potential and hence it is necessary to subtract the contribution of the static quark mass from the Nambu–Goto action. In addition, we have to take account of the Legendre transformation as usual. In the following, we will see the contributions of the quark mass and the Legendre transformation. Finally, we will check the consistency with the undeformed limit.

The quark mass

Let us first evaluate the static quark mass in the present case. The total mass of a quark and an antiquark $\widetilde{S}_0$ is given by two strings that stretch between the boundary ($v = 0$) and the origin of the deformed $\mathrm{AdS}_5 (v = \infty)$, with a constant of φ:

$$\begin{aligned}\widetilde{S}_0 &= 2\,\frac{T\sqrt{\lambda}}{2\pi}\sqrt{1+C^2}\int_{\widetilde{v}_0}^{\infty}\frac{dv}{v v_C}\\ &= \frac{T\sqrt{\lambda}}{\pi}\frac{\sqrt{1+C^2}}{C}\,\mathrm{arcsinh}\left[\frac{C}{\sqrt{1+C^2}\,\widetilde{v}_0}\right].\end{aligned} \tag{3.45}$$

Here a cut-off $\widetilde{v}_0 (\ll 1)$ has been introduced for small v. When C is fixed, $\widetilde{S}_0$ can be expanded with respect to $\widetilde{v}_0$ like

$$\widetilde{S}_0 = \frac{T\sqrt{\lambda}}{\pi}\frac{\sqrt{1+C^2}}{C}\log\left[\frac{2C}{\sqrt{1+C^2}\,\widetilde{v}_0}\right] + \mathcal{O}\Big(v_0^2\Big). \tag{3.46}$$

Now one can identify $\widetilde{S}_0$ with S_0 in (3.44) through the following relation:

$$\widetilde{v}_0 \equiv v_0\left(\frac{\pi x_0\sqrt{1-k^2x_0^2}\,\vartheta_1'(0,q_k)}{\left(1+C^2\right)\sqrt{1-x_0^2}\,K\left(k^2\right)\,\vartheta_1(2\omega,q_k)}\right)^{\frac{1}{2}} e^{F\left(\arcsin x_0|k^2\right)\,Z\left(\arcsin x_0|k^2\right)}. \tag{3.47}$$

This relation (3.47) can also be expanded around $C = 0$ like

$$\widetilde{v}_0 = v_0 + \mathcal{O}\Big(C^2\Big), \tag{3.48}$$

and hence $\widetilde{v}_0$ is equal to v_0 in the undeformed limit. In other words, there is a slight difference between $\widetilde{v}_0$ and v_0 in the deformed case, and it should be interpreted as a renormalization effect.

Legendre transformation

The next step is to examine an additional contribution that comes from the boundary condition [95]. The total derivative term S_L is given by

$$S_L = \int dr\, 2\int_{\phi/2}^{\pi/2} d\varphi\, \partial_\varphi \left(z \frac{\partial L}{\partial(\partial_\varphi z)} \right). \tag{3.49}$$

By using (3.22), S_L is evaluated as

$$\begin{aligned} S_L &= \sqrt{1+C^2}\,\frac{\sqrt{\lambda}}{2\pi} \int \frac{dr}{r}\, \frac{2\sqrt{b^4 + (1+C^2)\, p^2}}{bp} \int_{x_0}^{1} dx\, \partial_x \left(\frac{1}{x} \sqrt{\frac{1-x^2}{1-k^2x^2}} \right) \\ &= -\sqrt{1+C^2}\,\frac{T\sqrt{\lambda}}{2\pi}\, \frac{2\sqrt{b^4 + (1+C^2)\, p^2}}{bp\, x_0} \sqrt{\frac{1-x_0^2}{1-k^2x_0^2}}. \end{aligned} \tag{3.50}$$

Here the r-integral has led to $\int dr/r = T$ as in (3.34). Note that the expression of (3.50) is just a constant term,

$$S_L = -\frac{T\sqrt{\lambda}}{\pi}\, \frac{\sqrt{1+C^2}}{C}, \tag{3.51}$$

while it diverges as $C \to 0$.

This constant term is necessary to add to the Nambu–Goto action so as to ensure the undeformed limit, as we will see below.

The undeformed limit

Finally, we shall consider the undeformed limit.

By expanding S_{ren} in (3.42) around $C = 0$, the result in Ref. [99] can be reproduced as

$$S_{\text{ren}} = -\frac{T\sqrt{\lambda}}{\pi}\, \frac{\sqrt{1+b_0^2}}{b_0}\, \frac{E(k_0^2) - (1-k_0^2)\, K(k_0^2)}{\sqrt{1-k_0^2}}. \tag{3.52}$$

Here b_0 and k_0 are defined as the $C \to 0$ limit of b and k in (3.18), respectively.

The remaining step is to consider the undeformed limit of $\widetilde{S}_0 (= S_0)$. In the $C \to 0$ limit, $\widetilde{S}_0$ in (3.46) has to be canceled out with S_L in (3.51). The sum of $\widetilde{S} + S_L$ is evaluated as

$$\widetilde{S}_0 + S_L = \frac{T\sqrt{\lambda}}{\pi}\, \frac{\sqrt{1+C^2}}{C} \left(\log\left[\frac{2C}{\sqrt{1+C^2}\, \widetilde{v}_0} \right] - 1 \right). \tag{3.53}$$

Thus this expression tells us that, for the consistency, the undeformed limit should be taken as the following double scaling limit:

$$C \longrightarrow 0, \quad \widetilde{v}_0 \longrightarrow 0 \quad \text{with} \quad \log\left[\frac{2C}{\widetilde{v}_0} \right] = 1 : \text{fixed}. \tag{3.54}$$

In the next section, we will derive a quark–antiquark potential from the regularized action S_{ren} by taking the antiparallel-lines limit of the cusped configuration.

4. A quark–antiquark potential

In this section, we will derive a quark–antiquark potential for the q-deformed case.

It seems quite difficult to realize a rectangle as the boundary of a string solution because the boundary geometry is deformed and hence the rectangular shape is not respected. On the other hand, a quark–antiquark potential can be evaluated from the cusped minimal surface solution by taking an antiparallel-lines limit as in the undeformed case [99], though the potential is valid only at short distances by construction. The antiparallel-lines limit is realized by taking $\phi \to \pi$, and then a quark–antiquark potential is obtained as a function of $\pi - \phi \to L$.

In the undeformed case, the antiparallel-lines limit leads to a Coulomb potential of $-1/L$ as expected from the conformal symmetry of the $\mathcal{N} = 4$ SYM. However, the conformal symmetry is broken in the q-deformed case (though the scaling invariance survives the deformation). Hence the resulting potential may be more complicated at short distances, while it should still have a Coulomb form at large distances because the IR region of the deformed geometry is the same as the usual AdS_5.

In the following, we first examine the antiparallel-lines limit with the finite-C case and derive the the potential. Then we shall consider the limit after expanding around $C = 0$ and see how the deformation modifies the Coulomb potential at short distances $L \ll 1$. Our argument here basically follows the analysis in the undeformed case [99].

4.1. A linear potential at short distances

Let us first express the classical solution in terms of b and k instead of p and q, by using the algebraic relations in (3.17). In the following, the deformation parameter C is kept finite.

Then let us consider the following limit:

$$b \longrightarrow 0, \qquad k: \text{ fixed}, \tag{4.1}$$

and we will ignore $\mathcal{O}\left(b^2\right)$ terms. This limit is nothing but the antiparallel-lines limit [99].

By taking the limit (4.1), ϕ in (3.25) approaches π as

$$\pi - \phi = 2b\, \frac{\sqrt{k^2 + C^2}}{1 + C^2}. \tag{4.2}$$

For the sphere part, θ in (3.31) approaches zero as

$$\theta = 2b\, \frac{\sqrt{1 - k^2}}{C\left(1 + C^2\right)}. \tag{4.3}$$

The regularized action (3.42) is reduced to

$$S_{\text{ren}} = \frac{T\sqrt{\lambda}}{\pi}\, \frac{b}{C^2} \left[\mathrm{E}\left(k^2\right) - \left(1 - k^2\right)\mathrm{K}\left(k^2\right)\right]. \tag{4.4}$$

Substituting $\pi - \phi$ for b, the regularized action leads to the following potential:

$$S_{\text{ren}} = \frac{T\sqrt{\lambda}}{2\pi}\, \frac{1 + C^2}{C^2}\, \frac{\mathrm{E}\left(k^2\right) - \left(1 - k^2\right)\mathrm{K}\left(k^2\right)}{\sqrt{k^2 + C^2}}\, (\pi - \phi). \tag{4.5}$$

This potential is linear, unlike the Coulomb potential in the undeformed case. In particular, the coefficient of $\pi - \phi$ is positive definite; hence it can be regarded as a string tension of the potential. This result is quite similar to the potential obtained from the gravity dual of NC gauge theories [101]. The linear behavior in (4.5) is consistent with the potential for NC gauge theories, as we will see below.

A consistent limit to an NC background

It is worth presenting a connection between the potential (4.5) and another potential for a gravity dual of NC gauge theories. The latter result was originally obtained in Ref. [101]. To be comprehensive, the derivation of the potential is given in Appendix C.

First of all, we will introduce a scaling limit [27] from the q-deformed AdS_5 to a gravity dual of NC gauge theories. This limit is realized by rescaling the coordinates like

$$z = \exp\left[\sqrt{C}\,t\right]\frac{\sqrt{C}}{u}, \quad r = \exp\left[\sqrt{C}\,t\right], \quad \varphi = \frac{\sqrt{C}\,x_2}{\sqrt{1-\mu^2}},$$
$$\psi = \frac{\sqrt{C}\,x_1}{\mu}, \qquad \zeta = \arcsin\mu + \sqrt{C}x_3, \tag{4.6}$$

and taking a $C \to 0$ limit. Then the metric of the (Euclidean) q-deformed AdS_5 with Poincaré coordinates is reduced to that of a gravity dual of NC gauge theories [72,73][8]:

$$ds^2_{\rm NC} = R^2\left[\frac{du^2}{u^2} + u^2\left(dt^2 + dx_1^2 + \frac{dx_2^2 + dx_3^2}{1+\mu^2 u^4}\right)\right], \tag{4.7}$$

$$B_{\rm NC} = R^2\,\mu\,\frac{dx_2\wedge dx_3}{1+\mu^2 u^4}. \tag{4.8}$$

This result indicates that the q-deformed geometry contains a noncommutative space as a special limit.

The next issue is to apply the scaling limit to our classical solution. Let us see the ζ-dependence of the reduction ansatz (3.1) and the scaling limit (4.6). In the ansatz (3.1), ζ is set to be zero, while ζ is expanded around a nonzero constant $\arcsin\mu$. To take account of this gap, it is necessary to take a $\mu \to 0$ limit with a scaling limit (4.6).

Then, after taking the rescaling (4.6), the cusped ansatz (3.4) can be rewritten as

$$v(\varphi) \equiv \frac{z}{r} = \frac{\sqrt{C}}{u(\sigma)}, \quad r = \exp\left[\sqrt{C}\,\tau\right], \qquad \varphi = \frac{\sqrt{C}\,\sigma}{\sqrt{1-\mu^2}}. \tag{4.9}$$

Now $\partial_\varphi v$ is converted to $\partial_\sigma u$ through

$$\partial_\varphi v(\varphi) = \frac{d\sigma}{d\varphi}\,\partial_\sigma\left(\frac{\sqrt{C}}{u(\sigma)}\right) = -\sqrt{1-\mu^2}\,\frac{\partial_\sigma u(\sigma)}{u(\sigma)^2}. \tag{4.10}$$

With the rescaled variables in (4.9), T and $\pi-\phi$ are redefined as new parameters $\widetilde{T}$ and $\widetilde{L}$, respectively:

$$T = \int\frac{dr}{r} = \int_{-\widetilde{T}/2}^{\widetilde{T}/2}\sqrt{C}\,d\tau = \sqrt{C}\,\widetilde{T},$$
$$\pi - \phi = \int d\varphi = \int_{-\widetilde{L}/2}^{\widetilde{L}/2}\frac{\sqrt{C}\,d\sigma}{\sqrt{1-\mu^2}} = \frac{\sqrt{C}\,\widetilde{L}}{\sqrt{1-\mu^2}}. \tag{4.11}$$

By the use of the rescaling (4.9) and the relations in (4.11), an antiparallel-lines limit of the regularized action for the q-deformed case can be rewritten as

$$S_{\rm NG} = \frac{\widetilde{T}\sqrt{\lambda}}{2\pi}\,\frac{1+C^2}{C\sqrt{1-\mu^2}}\,\frac{\mathrm{E}(k^2) - (1-k^2)\,\mathrm{K}(k^2)}{\sqrt{k^2+C^2}}\,\widetilde{L}. \tag{4.12}$$

[8] Now the deformed S^5 part is reduced to the round S^5, though the scaling limit is not described here (for details, see Ref. [27]).

Then we take a double scaling limit:

$$C \longrightarrow 0 \quad \& \quad \mu \longrightarrow 0 \quad \text{with} \quad \frac{C}{\mu} = \frac{\sqrt{2}}{k}\left[\mathrm{E}\left(k^2\right) - \left(1-k^2\right)\mathrm{K}\left(k^2\right)\right] \quad \text{fixed.} \tag{4.13}$$

As a result, the potential (4.12) is reduced to that for the gravity dual of NC gauge theories (with $\mu \to 0$):

$$S_{\rm NG} = \frac{\widetilde{T}\sqrt{\lambda}}{2\pi}\,\frac{\widetilde{L}}{\sqrt{2}\,\mu}. \tag{4.14}$$

Thus we have checked that the scaling limit (4.6) is consistent with our subtraction scheme.

4.2. *Expansion around $C = 0$*

The next step is to study the potential behavior when C is very small. The classical action $S_{\rm ren}$ is first around $C = 0$ while b and k are fixed. Then it is expanded around $b = 0$ with k fixed[9].

As a result, ϕ in (3.25) approaches π like

$$\pi - \phi = \frac{2b}{k}\left[E\left(k^2\right) - \left(1-k^2\right)K\left(k^2\right)\right] + \mathcal{O}\left((C,b)^2\right). \tag{4.15}$$

For the sphere part, θ (3.31) is expanded as

$$\theta = 2\sqrt{1-2k^2}\,K\left(k^2\right) - \frac{2C\sqrt{1-2k^2}}{b\sqrt{1-k^2}} + \mathcal{O}\left((C,b)^2\right). \tag{4.16}$$

Thus the regularized action $S_{\rm ren}$ (3.42) results in

$$S_{\rm ren} = \frac{T\sqrt{\lambda}}{\pi}\left[\frac{E\left(k^2\right) - \left(1-k^2\right)K\left(k^2\right)}{\sqrt{1-k^2}}\left(-\frac{1}{b} - \frac{b}{2}\right) + \frac{C\,k^2}{1-k^2}\left(\frac{1}{b^2}+1\right)\right] + \mathcal{O}\left((C,b)^2\right).$$

Substituting $\pi - \phi$ for b, the leading terms of $S_{\rm ren}$ are evaluated as

$$S_{\rm ren} = \frac{T\sqrt{\lambda}}{4\pi}\,\frac{\left(E\left(k^2\right) - \left(1-k^2\right)K\left(k^2\right)\right)^2}{k\sqrt{1-k^2}}\left[-\frac{8}{\pi-\phi} + \frac{16\,C\,k}{(\pi-\phi)^2\,\sqrt{1-k^2}}\right]. \tag{4.17}$$

Note that the first term is a Coulomb-form potential that agrees with the undeformed result obtained in Ref. [99], while the second term gives rise to a repulsive force with nonvanishing C.

It is worth noting that the sphere-part contribution vanishes when $k^2 = 1/2$, i.e., $\theta = 0$. Then one can obtain the following expression:

$$S_{\rm ren} = \frac{T\sqrt{\lambda}}{4\pi}\,\frac{16\pi^3}{\Gamma\left(\frac{1}{4}\right)^4}\left[-\frac{1}{(\pi-\phi)} + \frac{2C}{(\pi-\phi)^2}\right]. \tag{4.18}$$

The first term of (4.18) precisely agrees with the results of Refs. [91,92] by replacing $\pi - \phi \to L$, and the second term produces a repulsive force, in comparison to the undeformed case.

[9] Note that there is an ambiguity in the order of limits and the order is sensitive to the potential behavior. The opposite order leads to the expansion of (4.5) around $C = 0$; hence, the linear behavior remains.

5. Near straight-line expansion

In the undeformed case, the near straight-line limit is realized as $\phi \to 0$. In this limit, the cusp disappears and the Wilson loop becomes an infinite straight line in R^4, or a pair of antipodal lines on $\mathrm{R}\times\mathrm{S}^3$.

Let us study here an analogue of the near straight-line limit in the deformed case by expanding the classical action around $\phi = \theta = 0$. Now ϕ and θ are expressed in terms of the parameters p and q, and the relevant limit indicates that p becomes large. Note that the modulus k of the elliptic integrals vanishes as $p \to \infty$. Hence we should first expand the elliptic integrals with small k, and then expand around $p \gg 1$.

The classical solution is expanded as

$$\phi = \frac{2}{p}(\operatorname{arccot} C + C) + \mathcal{O}\left(p^{-3}\right), \quad \theta = \frac{2q}{p}\operatorname{arccot} C + \mathcal{O}\left(p^{-3}\right). \tag{5.1}$$

Note here that the $C \to 0$ limit of (5.1) can reproduce the undeformed result [99]:

$$\phi = \frac{\pi}{p} + \mathcal{O}\left(p^{-3}\right), \quad \theta = \frac{\pi q}{p} + \mathcal{O}\left(p^{-3}\right). \tag{5.2}$$

Then the regularized action S_{ren} can be expanded as

$$S_{\mathrm{ren}} = \frac{T\sqrt{\left(1+C^2\right)\lambda}}{4\pi}\,\frac{q^2-1}{p^2}(4\operatorname{arccot} C - \pi) + \mathcal{O}\left(p^{-4}\right). \tag{5.3}$$

Although the action (5.3) is expressed in terms of p and q, it can be rewritten in terms of ϕ and θ through the relations in (5.1). The resulting formula is given by

$$\begin{aligned} S_{\mathrm{ren}} = {} & \frac{T\sqrt{\left(1+C^2\right)\lambda}}{4\pi}\left(\operatorname{arccot} C - \frac{\pi}{4}\right) \\ & \times\left[\frac{\theta^2}{(\operatorname{arccot} C)^2} - \frac{\phi^2}{(\operatorname{arccot} C + C)^2}\right] + \mathcal{O}\left(\left(\phi^2,\theta^2\right)^2\right). \end{aligned} \tag{5.4}$$

In the $C \to 0$ limit, the undeformed result [99] is reproduced like

$$S_{\mathrm{ren}} = \frac{T\sqrt{\lambda}}{4\pi}\,\frac{\theta^2-\phi^2}{\pi} + \mathcal{O}\left(\left(\phi^2,\theta^2\right)^2\right). \tag{5.5}$$

It would be interesting to try to reproduce the result (5.4) from a q-deformed Bethe ansatz by generalizing the methods for the undeformed case [102,103].

6. Conclusion and discussion

In this paper, we have studied minimal surfaces with a single cusp in the q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. By taking an antiparallel-lines limit, a quark–antiquark potential has been computed by adopting a certain regularization. The UV geometry is modified due to the deformation and hence the short-distance behavior may be modified from the Coulomb potential, while the potential should still be of Coulomb type. In fact, the resulting potential exhibits linear behavior at short distances with finite C. In particular, the linear behavior for the gravity dual for noncommutative gauge theories can be reproduced by taking a scaling limit [27]. Finally we have studied the near straight-line limit of the potential.

The most intriguing problem is to unveil the gauge-theory dual for the q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$. In the undeformed case, the potential behaviors at strong and weak coupling can be reproduced from the Bethe ansatz [102,103]. Namely, the gravity and gauge-theory sides are bridged by the Bethe ansatz. In particular, an all-loop expression for the near-BPS expansion of the quark–antiquark potential on an S^3, which was obtained in Refs. [104,105], has been reproduced by an analytic solution of the thermodynamic Bethe ansatz (TBA) [106,107]. Furthermore, for arbitrary ϕ and θ, it is shown in Ref. [108] that the weak-coupling expansion of the TBA reproduces the gauge-theory result up to two loops. Recently, the quantum spectral curve technique [109,110] has been applied to study the behavior of quark–antiquark potentials [111,112]. It would be possible to adopt these methods for the deformed case as well, possibly via a quantum-deformed Bethe ansatz.

We hope that the gauge-theory dual can be revealed in light of our linear potential and the quantum-deformed Bethe ansatz.

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Appendix A. Elliptic integrals

The incomplete elliptic integrals of the first, second, and third kinds are given by

$$
\begin{aligned}
F\left(\psi|k^2\right) &= \int_0^{\sin\psi} \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2\sin^2\psi\right)}}, \\
E\left(\psi|k^2\right) &= \int_0^{\sin\psi} \frac{dx\sqrt{1-k^2\sin^2\psi}}{\sqrt{1-x^2}}, \\
\Pi\left(\alpha^2,\psi|k^2\right) &= \int_0^{\sin\psi} \frac{dx}{\left(1-\alpha^2x^2\right)\sqrt{1-x^2}\sqrt{1-k^2x^2}}.
\end{aligned} \tag{A1}
$$

When $\psi = \pi/2$, these expressions become the complete elliptic integrals:

$$
K\left(k^2\right) = F\left(\pi/2|k^2\right), \quad E\left(k^2\right) = E\left(\pi/2|k^2\right), \quad \Pi\left(\alpha^2|k^2\right) = \Pi\left(\alpha^2,\pi/2|k^2\right).
$$

Note that $\Pi(\alpha^2,\psi|k^2)$ has a pole at $\psi = \arcsin\alpha^{-1}$. Thus it should be interpreted as the Cauchy principal value when $\alpha^2\sin^2\psi > 1$ [100].

Appendix B. Classical solutions in global coordinates

Let us consider here cusped solutions in the $AdS_3 \times S^1$ geometry with global coordinates (in the Lorentzian signature). The global coordinate system for the deformed geometry was originally introduced in Ref. [24]. For our aim of studying minimal surfaces, it would be rather helpful to adopt another coordinate system in which the singularity surface is located at the boundary [39]. Then the deformed $\mathrm{AdS}_3 \times \mathrm{S}^1$ geometry is written as

$$ds^2_{\mathrm{AdS}_3\times\mathrm{S}^1} = R^2\sqrt{1+C^2}\left[-\cosh^2\chi\, dt^2 + \frac{d\chi^2 + \sinh^2\chi\, d\varphi^2}{1+C^2\cosh^2\chi} + d\vartheta^2\right]. \tag{B1}$$

The world-sheet coordinates τ and σ are identified with t and φ, respectively. The other coordinates are supposed to take the following form:

$$\chi = \chi(\varphi), \quad \vartheta = \vartheta(\varphi), \quad \varphi \in \left[\frac{\phi}{2}, \pi - \frac{\phi}{2}\right]. \tag{B2}$$

Note here that χ diverges at $\varphi = \phi/2$ and $\varphi = \pi - \phi/2$. The minimum of χ is realized at the middle point $\varphi = \pi/2$. The range of ϑ is bounded from $-\vartheta/2$ to $\vartheta/2$.

Then the Nambu–Goto action is given by

$$S_{\mathrm{NG}} = \sqrt{1+C^2}\,\frac{\sqrt{\lambda}}{2\pi}\int dt\, d\varphi\, \cosh\chi\, \sqrt{\frac{\left(\partial_\varphi\chi\right)^2 + \sinh^2\chi}{1+C^2\cosh^2\chi} + \left(\partial_\varphi\vartheta\right)^2}. \tag{B3}$$

The energy and the canonical momentum conjugate to φ are given by, respectively,

$$\begin{aligned} E &= \frac{\sinh^2\chi\, \cosh\chi}{\left(1+C^2\cosh^2\chi\right)\sqrt{\frac{(\partial_\varphi\chi)^2+\sinh^2\chi}{1+C^2\cosh^2\chi} + \left(\partial_\varphi\vartheta\right)^2}}, \\ J &= \frac{\partial_\varphi\vartheta\, \cosh\chi}{\sqrt{\frac{(\partial_\varphi\chi)^2+\sinh^2\chi}{1+C^2\cosh^2\chi} + \left(\partial_\varphi\vartheta\right)^2}}. \end{aligned} \tag{B4}$$

It is convenient to introduce the following quantities, like in the Poincaré case:

$$p \equiv \frac{1}{E}, \quad q \equiv \frac{J}{E} = \frac{1+C^2\cosh^2\chi}{\sinh^2\chi}\,\partial_\varphi\vartheta. \tag{B5}$$

By removing $\partial_\varphi\vartheta$, one can obtain the following relation:

$$p = \frac{\left(1+C^2\cosh^2\chi\right)\left(\left(\partial_\varphi\chi\right)^2 + \sinh^2\chi\right) + q^2\,\sinh^4\chi}{\sinh^2\chi\,\cosh\chi}. \tag{B6}$$

Then this expression can be rewritten as

$$\left(\partial_\varphi\chi\right)^2 = \frac{\left(p^2\cosh^2\chi - q^2\right)\sin^4\chi}{1+C^2\cosh^2\chi} - \sinh^2\chi\,. \tag{B7}$$

Note here that the resulting expression (B7) is the same as (3.10) in the Poincaré case through the identification

$$\sinh\chi(\varphi) \quad \longleftrightarrow \quad \frac{1}{v(\varphi)}. \tag{B8}$$

Then, the t-dependence of (B3) is translated to the r-dependence of (3.5) via $t \leftrightarrow \log r$.

Appendix C. A linear potential at short distances in NC gauge theories

Let us consider a minimal surface solution in the gravity dual of NC gauge theories [72,73]. This solution is dual to a rectangular Wilson loop on the gauge-theory side. From the classical action of this solution, a quark–antiquark potential can be evaluated. Then the resulting potential exhibits a linear behavior at short distances [101]. The following argument is just a short review of Ref. [101].

The (Euclidean) metric and B-field for the gravity dual [72,73] are given by

$$ds^2_{\text{NC}} = R^2\left[\frac{du^2}{u^2} + u^2\left(dt^2 + \left(dx^1\right)^2 + \frac{\left(dx^2\right)^2 + \left(dx^3\right)^2}{1+\mu^2u^4}\right)\right], \tag{C1}$$

$$B_{\text{NC}} = R^2\,\mu\,\frac{dx^2\wedge dx^3}{1+\mu^2u^4}. \tag{C2}$$

Here a constant parameter μ measures the noncommutative deformation[10].

To derive a quark–antiquark potential from this background, we study the following configuration of a static string described by

$$u = u(\sigma), \quad t=\tau, \quad x^2=\sigma, \quad x^1 = x^3 = 0. \tag{C3}$$

Then the classical string describes a rectangular loop on the boundary.

Hereafter, we will consider a 4D slice of the metric (C2) at $u=\Lambda$ by following Ref. [101]. Suppose that $\widetilde{L}$ is the distance between two antiparallel lines at $u=\Lambda$, and $\widetilde{T}$ is an interval for the τ-direction. As a result, the classical action is rewritten as

$$S_{\text{NG}} = \frac{\sqrt{\lambda}}{2\pi}\int_{-\widetilde{T}/2}^{\widetilde{T}/2}d\tau\int_{-\widetilde{L}/2}^{\widetilde{L}/2}d\sigma\sqrt{(\partial_\sigma u)^2 + \frac{u^2}{1+\mu^2u^4}}. \tag{C4}$$

From this action, the classical solution can be obtained as

$$\frac{\widetilde{L}}{2} = \frac{1}{u_m}\int_1^{\Lambda/u_m}dy\,\frac{1+\mu^2u_m^4\,y^4}{y^2\sqrt{y^4-1}}, \tag{C5}$$

and then the value of the classical action is evaluated as

$$S_{\text{NG}} = \frac{\widetilde{T}\sqrt{\lambda}}{\pi}\sqrt{1+\mu^2u_m^4}\,u_m\int_1^{\Lambda/u_m}dy\frac{y^2}{\sqrt{y^4-1}}. \tag{C6}$$

Here u_m is the turning point along the u-direction where $\partial_\sigma u = 0$.

---

[10] On the gauge-theory side, the x^2–x^3 plane is deformed to a noncommutative plane with the noncommutativity θ_0. The parameter μ is related to θ_0 through the following relation [101]:

$$\mu = \sqrt{\lambda}\,\theta_0.$$

Now that λ is assumed to be large for the validity of the gravity dual, $\mu \gg \theta_0$.

In the following, we will focus upon a special case, $u_m \sim \Lambda$, and consider the behavior of the classical action (C6) with a double scaling limit:

$$\mu \longrightarrow 0, \quad \Lambda \longrightarrow \infty \quad \text{with } \sqrt{\mu}\,\Lambda \equiv 1. \tag{C7}$$

Then the integrands in (C5) and (C6) can be expanded in terms of y. Hence (C5) and (C6) are evaluated as

$$\frac{\widetilde{L}}{2} = \frac{1+\mu^2 u_m^4}{u_m}\sqrt{\frac{\Lambda}{u_m}-1} + \mathcal{O}\left(\frac{\Lambda}{u_m}-1\right)^{3/2}, \tag{C8}$$

$$S_{\rm NG} = \frac{\widetilde{T}\sqrt{\lambda}}{\pi}\sqrt{1+\mu^2 u_m^4}\,u_m\sqrt{\frac{\Lambda}{u_m}-1} + \mathcal{O}\left(\frac{\Lambda}{u_m}-1\right)^{3/2}. \tag{C9}$$

Note here that the distance $\widetilde{L}$ is very small (in comparison to $\sqrt{\mu}$), because

$$\frac{\widetilde{L}^2}{\mu} \sim \left(\frac{\Lambda}{u_m}-1\right) \ll 1. \tag{C10}$$

As a result, $S_{\rm NG}$ can be expressed as a function of $\widetilde{L}$ like

$$\begin{aligned} S_{\rm NG} &= \frac{\widetilde{T}\sqrt{\lambda}}{2\pi}\,\widetilde{L}\,\frac{u_m^2}{\sqrt{1+\mu^2 u_m^4}} + \mathcal{O}\left(\left(\widetilde{L}/\sqrt{\mu}\right)^3\right) \\ &\simeq \frac{\widetilde{T}\sqrt{\lambda}}{2\pi}\,\frac{\widetilde{L}}{\sqrt{2}\,\mu} + \mathcal{O}\left(\left(\widetilde{L}/\sqrt{\mu}\right)^3\right). \end{aligned} \tag{C11}$$

Here the relation $\sqrt{\mu}\,u_m \sim 1$ has been utilized in the last line. This expression (C11) leads to a linear potential at short distances [101][11].

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