DFG GRK 1463 Analysis, Geometry and String Theory

Orbifold Jacobian algebras for invertible polynomials

Elisabeth Werner

Leibniz Universität Hannover, Institut für Algebraische Geometrie

joint work with Alexey Basalaev (Mannheim) and Atsushi Takahashi (Osaka)

Leibniz Universität Hannover

Introduction

Mirror symmetry gives an identification between two objects coming from different mathematical origins. It has been studied intensively by many mathematicians for more than twenty years since it yields important, interesting and unexpected geometric information. The use of orbifold constructions is the cornerstone of the original mirror construction. The orbifolds under study in that context are so called Landau-Ginzburg orbifolds. Here we want to find an orbifolded version of a Frobenius algebra to a pair (f, G) of a polynomial f and a certain group of symmetries of f. Certain work was also done previously by R. Kaufmann ([K03],[K06]) and M. Krawitz ([Kr]).

Preliminaries

- $f(x_1,\dots,x_N) \in \mathbb{C}[x_1,\dots,x_N]$
  - *non-degenerate* : $\Leftrightarrow$ isolated singularity at the origin $\Leftrightarrow$ The *Jacobian algebra*
  $$\mathrm{Jac}(f) = \mathbb{C}[x_1,\dots,x_N]\Big/\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_N}\right)$$
  is a finite-dimensional algebra over $\mathbb{C}$.
  - *invertible*: non-degenerate weighted homogeneous polynomial in N variables which contains N monomials, i.e.
  $$f(x_1,\dots,x_N) = \sum_{i=1}^{N} a_i \prod_{j=1}^{N} x_j^{E_{ij}}$$
  $a_i \in \mathbb{C}^*$, E_{ij} non-negative integers, $E = (E_{ij})$ invertible over $\mathbb{Q}$.
- $\Omega_f := \Omega^N(\mathbb{C}^N)/(df \wedge \Omega^{N-1}(\mathbb{C}^N))$
- Ω_f is naturally a free $\mathrm{Jac}(f)$-module of rank one, namely, by choosing a nowhere vanishing N-form we have the following isomorphism
$$\mathrm{Jac}(f) \xrightarrow{\cong} \Omega_f, \quad \phi \mapsto \phi \cdot [dx_1 \wedge \dots \wedge dx_N]. \qquad (1)$$
- *residue pairing:* Ω_f is equipped with a non-degenerate $\mathbb{C}$-bilinear form $J_f : \Omega_f \otimes_{\mathbb{C}} \Omega_f \longrightarrow \mathbb{C}$
$$J_f\left([\phi(\mathbf{x})dx_1 \wedge \dots \wedge dx_N], [\psi(\mathbf{x})dx_1 \wedge \dots \wedge dx_N]\right) := \mathrm{Res}_{\mathbb{C}^N}\begin{bmatrix} \phi(\mathbf{x})\psi(\mathbf{x})dx_1 \wedge \dots \wedge dx_N \\ \frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_N} \end{bmatrix}.$$

Remark 1. Under the isomorphism (1), the residue pairing endows the Jacobian algebra $\mathrm{Jac}(f)$ with a structure of a Frobenius algebra, see also [AGV85],
$$J_f(\phi\cdot\psi, \vartheta) = J_f(\phi, \psi\cdot\vartheta) \quad \phi,\psi,\vartheta \in \mathrm{Jac}(f).$$

The purpose is to generalize these results to pairs (f, G), where $G \subset \mathrm{SL}(N;\mathbb{C})$ is a finite abelian subgroup leaving f invariant. If f is weighted homogeneous, such a pair is also called an orbifold Landau-Ginzburg model because f is the potential of such a model.

Example

Let us take the polynomial $f := x_1^3 + x_2^3 + x_3^2$ and as the group
$$G := \langle g\rangle\,, \; g := \left(e^{2\pi i 1/3}, e^{2\pi i 2/3}, 1\right).$$
$$\mathrm{Jac}(f) \cong \mathbb{C}[x_1,x_2,x_3]\Big/\left(3x_1^2, 3x_2^2, 2x_3\right) \cong \langle [1],[x_1],[x_2],[x_1x_2]\rangle_{\mathbb{C}}$$
$$J_f\left([dx_1\wedge dx_2\wedge dx_3], [x_1x_2 dx_1\wedge dx_2\wedge dx_3]\right) = \frac{1}{18}$$
$$\mathrm{Jac}(f,G) \cong \langle [1],[x_1x_2]\rangle_{\mathbb{C}} \oplus \left\langle e_g, e_{g^{-1}}\right\rangle_{\mathbb{C}}$$
$$J_{f,\mathrm{id}}\left([dx_1\wedge dx_2\wedge dx_3], [x_1x_2 dx_1\wedge dx_2\wedge dx_3]\right) = 3\cdot\frac{1}{18} = \frac{1}{6}$$
$$J_{f,g}\left([dx_3],[dx_3]\right) = -1\cdot\frac{1}{3}\cdot\frac{1}{2} = -\frac{1}{6}$$
$$\Rightarrow \quad e_g \circ e_{g^{-1}} = -[x_1x_2], \quad e_g^2 = 0, \quad e_{g^{-1}}^2 = 0$$
We have that there is an isomorphism of Frobenius algebras
$$\mathrm{Jac}(f,G) \cong \mathrm{Jac}(\overline{f}),$$
for the polynomial $\overline{f} = y_1^2 + y_3y_2^2 + y_2y_3^2$. This $\overline{f}$ we get by describing explicitly the geometry of vanishing cycles for the proper transform of $f^{-1}(0)/G$ in a crepant resolution of $\mathbb{C}^3/G$. The singularity of the proper transform is contained in the zero locus of $\overline{f}$ on one chart isomorphic to $\mathbb{C}^3$, see also [ET].

(f, G)

Definition 1. Let G be a finite subgroup of $\mathrm{SL}(N;\mathbb{C})$ and of the *group of maximal diagonal symmetries* of f
$$G_f := \left\{(\lambda_1,\dots,\lambda_N) \in (\mathbb{C}^*)^N \mid f(\lambda_1x_1,\dots,\lambda_Nx_N) = f(x_1,\dots,x_N)\right\}.$$

The pair (f, G) is often called a *Landau-Ginzburg orbifold.*
For $g \in G$ we define
- $\mathrm{age}(g) := \sum_{i=1}^N a_i$ for $g = (e^{2\pi i a_1},\dots,e^{2\pi i a_N})$, $0 \le a_i < 1$,
- $\mathrm{Fix}(g) := \{\mathbf{x} \in \mathbb{C}^N \mid g\cdot\mathbf{x} = \mathbf{x}\}$ the fixed locus of g,
- $N_g := \dim \mathrm{Fix}(g)$ its dimension,
- $f^g := f|_{\mathrm{Fix}(g)}$ the restriction of f to the fixed locus of g.

Proposition 1. *f^g has an isolated singularity at the origin and there is a natural surjective $\mathbb{C}$-algebra homomorphism $\mathrm{Jac}(f) \to \mathrm{Jac}(f^g)$ by setting variables not fixed by g equal to zero.*

Corollary 1. *For each $g \in G$, Ω_{f^g} is naturally equipped with a structure of $\mathrm{Jac}(f)$-module.*

Definition 2.
$$\mathrm{Aut}(f,G) := \{\varphi \in \mathrm{Aut}(\mathbb{C}[x_1,\dots,x_N]) \mid \varphi(f) = f,\ \varphi\circ g\circ\varphi^{-1} \in G\, \forall g \in G\}.$$

It is obvious that G is naturally identified with a subgroup of $\mathrm{Aut}(f,G)$.

$\Omega_{f,G}$

Definition 3. Define a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-module
$$\Omega'_{f,G} = \bigoplus_{\substack{g\in G\\ N-N_g\equiv 0 \ (\mathrm{mod}\ 2)}} \Omega'_{f,g} \oplus \bigoplus_{\substack{g\in G\\ N-N_g\equiv 1 \ (\mathrm{mod}\ 2)}} \Omega'_{f,g} = \left(\Omega'_{f,G}\right)_{\overline{0}} \oplus \left(\Omega'_{f,G}\right)_{\overline{1}},$$
where $\Omega'_{f,g} := \Omega_{f^g}$ and for each $g \in G$ with $\mathrm{Fix}(g) = \{0\}$ we define Ω_{f^g} to be the $\mathbb{C}$-module of rank one generated by the symbol 1_g. $\mathrm{Aut}(f,G)$ acts on each Ω_{f^g} by the pull-back of forms via its action on $\mathrm{Fix}(g)$.

Let $\Omega_{f,G} = \left(\Omega'_{f,G}\right)^G$ be the G-invariant part.

The *orbifold residue pairing* is a non-degenerate $\mathbb{Z}/2\mathbb{Z}$-graded symmetric $\mathbb{C}$-bilinear form $J_{f,G} := \bigoplus_{g\in G} J_{f,g} : \Omega'_{f,G} \otimes_{\mathbb{C}} \Omega'_{f,G} \longrightarrow \mathbb{C}$, where $J_{f,g}$ is a perfect $\mathbb{C}$-bilinear form $J_{f,g} : \Omega'_{f,g} \otimes_{\mathbb{C}} \Omega'_{f,g^{-1}} \longrightarrow \mathbb{C}$ defined by
$$J_{f,g}\left([\phi(\mathbf{x})dx_{i_1}\wedge\dots\wedge dx_{i_{N_g}}], [\psi(\mathbf{x})dx_{i_1}\wedge\dots\wedge dx_{i_{N_g}}]\right)$$
$$:= (-1)^{N-N_g-\mathrm{age}(g)}|G/K_g|\cdot|K_g|^{-1}\cdot\mathrm{Res}_{\mathbb{C}^{N_g}}\begin{bmatrix} \phi\psi dx_{i_1}\wedge\dots\wedge dx_{i_{N_g}} \\ \frac{\partial f^g}{\partial x_{i_1}} \cdots \frac{\partial f^g}{\partial x_{i_{N_g}}} \end{bmatrix}$$
where K_g is the maximal subgroup of G fixing the space $\mathrm{Fix}(g)$.

$\mathrm{Jac}(f, G)$

Definition 4. A *G-twisted Jacobian algebra* of f is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra $\mathrm{Jac}'(f,G) = \mathrm{Jac}'(f,G)_{\overline{0}} \oplus \mathrm{Jac}'(f,G)_{\overline{1}}$ satisfying the following axioms:

1. For each $g \in G$, there is a $\mathbb{C}$-module $\mathrm{Jac}'(f,g)$ isomorphic to $\Omega'_{f,g}$ as a $\mathbb{C}$-module. In particular, for the identity id of G, $\mathrm{Jac}'(f,\mathrm{id}) = \mathrm{Jac}(f)$.We have
$$\mathrm{Jac}'(f,G)_{\overline{0}} := \bigoplus_{\substack{g\in G\\ N-N_g\equiv 0 \ (\mathrm{mod}\ 2)}} \mathrm{Jac}'(f,g),$$
$$\mathrm{Jac}'(f,G)_{\overline{1}} := \bigoplus_{\substack{g\in G\\ N-N_g\equiv 1 \ (\mathrm{mod}\ 2)}} \mathrm{Jac}'(f,g).$$
2. The $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra structure $\circ$ on $\mathrm{Jac}'(f,G)$ satisfies
$$\mathrm{Jac}'(f,g)\circ\mathrm{Jac}'(f,h) \subset \mathrm{Jac}'(f,gh), \quad g,h \in G,$$
and the $\mathbb{C}$-subalgebra $\mathrm{Jac}'(f,\mathrm{id}) \cong$ of $\mathrm{Jac}'(f,G)$ as $\mathbb{C}$-algebras.
3. The $\mathbb{C}$-module $\Omega'_{f,G}$ has a structure of G-equivariant $\mathrm{Jac}'(f,G)$-module
$$\mathrm{Jac}'(f,G)\otimes\Omega'_{f,G} \longrightarrow \Omega'_{f,G}, \quad X\otimes\omega \mapsto X\cdot\omega.$$
Moreover, by choosing a nowhere vanishing G-invariant N-form we have the following isomorphism
$$\mathrm{Jac}'(f,G) \xrightarrow{\cong} \Omega'_{f,G}, \quad \phi \mapsto \phi\cdot\zeta,$$
where ζ is the residue class in $\Omega_{f,\mathrm{id}} = (\Omega_{f,\mathrm{id}})^G = (\Omega_f)^G$ of the N-form, such that
   (a) the restriction to $\mathrm{Jac}'(f,\mathrm{id}) = \mathrm{Jac}(f)$ coincides with (1),
   (b) for any $g,h \in G$ we have
   $$\mathrm{Jac}'(f,g)\cdot\Omega'_{f,h} \subset \Omega'_{f,gh},$$
   and the $\mathrm{Jac}'(f,\mathrm{id})$-module structure on $\Omega'_{f,g}$ coincides with the $\mathrm{Jac}(f)$-module structure on Ω_{f^g} given by Corollary 1.
   (c) for homogeneous elements $X \in \mathrm{Jac}'(f,G)$, $\omega,\omega' \in \Omega'_{f,G}$, we have
   $$J_{f,G}(X\cdot\omega,\omega') = (-1)^{\overline{X}\overline{\omega}}J_{f,G}(\omega, X\cdot\omega'),$$
   where $\overline{X}$ and $\overline{\omega}$ are the $\mathbb{Z}/2\mathbb{Z}$-grading of X and ω respectively.
4. On $\mathrm{Jac}'(f,G)$ we have the induced action of $\mathrm{Aut}(f,G)$ on $\mathrm{Jac}'(f,G)$ given by
$$\varphi_*(X)\cdot\varphi^*(\zeta) := \varphi^*(X\cdot\zeta), \quad \varphi \in \mathrm{Aut}(f,G),\ X \in \mathrm{Jac}'(f,G).$$
We require that the algebra structure of $\mathrm{Jac}'(f,G)$ is $\mathrm{Aut}(f,G)$-invariant, namely,
$$\varphi_*(X)\circ\varphi_*(Y) = \varphi_*(X\circ Y), \quad \varphi \in \mathrm{Aut}(f,G),\ X,Y \in \mathrm{Jac}'(f,G),$$
and it is G-twisted $\mathbb{Z}/2\mathbb{Z}$-graded commutative, namely, for any $g,h \in G$ and $X \in \mathrm{Jac}'(f,g)$, $Y \in \mathrm{Jac}'(f,h)$ we have
$$X\circ Y = (-1)^{\overline{XY}}g_*(Y)\circ X,$$
where g_* is the induced action of g considered as an element of $\mathrm{Aut}(f,G)$.

Definition 5. The $\mathbb{Z}/2\mathbb{Z}$-graded commutative Frobenius algebra

$\mathrm{Jac}(f,G) := \left(\mathrm{Jac}'(f,G)\right)^G$ is called the *orbifold Jacobian algebra* of (f, G).

Theorem

Let $f = f(x_1,x_2,x_3)$ be an invertible polynomial and G a subgroup of $G_f \cap \mathrm{SL}(3;\mathbb{C})$.

Then the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$ is a Frobenius algebra and is uniquely determined by the axioms in Definition 4.

More precisely, there exists a G-twisted Jacobian algebra $\mathrm{Jac}'(f,G)$ and $\mathrm{Jac}(f,G)$ is independent of the choice of $\mathrm{Jac}'(f,G)$ and uniquely determined by (f,G) up to isomorphism.

References

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