Gluing stability conditions on ruled surfaces with base curve of positive genus

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Motivations

Bridgeland stability conditions on ruled surfaces S

1. geometric stability conditions
   - existence? - Yes!!
   - potentially stable objects - all point sheaves
2. gluing stability conditions …… from Orlov's SOD
   - existence? - Unknown......
   - potentially stable objects - Unknown......

Questions (→ Main Result)

1. Declare all gluing stability conditions.
2. Find potentially stable objects of gluing stability conditions.
3. Positional relation between gluing and geometric on Stab S.

Gluing stability conditions

$p : S \to C$, ruled surface S with base curve C

- $\exists s : C \to S$ s.t. $p \circ s = \mathrm{id}_C$ (projection and section)
  $C_0 := s(C)$, $f :=$ fiber of p
- Orlov's semiorthogonal decomposition:
  $D^b(S) = \langle D_1, D_2 \rangle$,
  $D_1 := p^* D^b(C) \otimes \mathcal{O}_S(-C_0)$ and $D_2 := p^* D^b(C)$

$\sigma_i \in \mathrm{Stab}\, D_i$ for $i = 1, 2$

- $D_1, D_2 \simeq D^b(C)$
- $\exists$standard stability condition σ_{st} on Stab C
- $g(C) \geq 1 \Rightarrow \sigma_{st}.\widetilde{\mathrm{GL}^+(2,\mathbb{R})} = \mathrm{Stab}\, C$ ([Bri],[Mac])

Gluing $\sigma_1 = (Z_1, \mathcal{A}_1) \in \mathrm{Stab}\, D_1$ and $\sigma_2 = (Z_2, \mathcal{A}_2) \in \mathrm{Stab}\, D_2$

- $Z := Z_1 \circ \lambda_1 + Z_2 \circ \rho_2$
  - λ_1: left adjoint functor of the inclusion $D_1 \hookrightarrow D^b(S)$
  - ρ_2: right adjoint functor of the inclusion $D_2 \hookrightarrow D^b(S)$
- $\mathcal{A} := \{E \mid \lambda_1(E) \in \mathcal{A}_1 \text{ and } \rho_2(E) \in \mathcal{A}_2\}$ (gluing heart)

→ $\sigma = (Z, \mathcal{A})$, gluing **pre**-stability condition

Difficulties

- Confirm gluing property between σ_1 and σ_2:
  $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2[i]) = 0$ for any $i \leq 0$
  → $\sigma = (Z, \mathcal{A})$, gluing stability condition

Main Result [U]

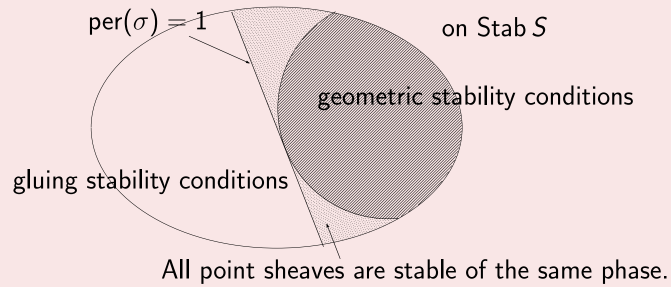
σ: gluing pre-stability condition of σ_1 and σ_2

- $\sigma_1 = (Z_1, \mathcal{P}_1)$: $\mathcal{P}_1(0) = p^*\mathcal{P}_{st}(\phi_1) \otimes \mathcal{O}_S(-C_0)$
- $\sigma_2 = (Z_2, \mathcal{P}_2)$: $\mathcal{P}_2(0) = p^*\mathcal{P}_{st}(\phi_2)$

→ $\mathrm{per}(\sigma) := \phi_1 - \phi_2$, **gluing perversity**.

Theorem

1. σ: gluing pre-stability condition
   σ gluing stability condition $\Leftrightarrow$ $\mathrm{per}(\sigma) \geq 1$
2. Potentially stable objects: $\mathcal{O}_f$ and $\mathcal{O}_f(-C_0)$
3. Positional relation between gluing and geometric on Stab S:
   Please see the right figure!!



Idea of gluing perversity

Theorem: σ gluing pre-stability condition
σ gluing stability condiition $\Leftrightarrow$ $\mathrm{per}(\sigma) \geq 1$
(Proof)
- assume: $\mathcal{A}_1 = p^*\mathcal{P}_{st}(\phi, \phi+1] \otimes \mathcal{O}_S(-C_0)$, $\mathcal{A}_2 = p^*\,\mathrm{Coh}\, C$.

- $\mathrm{per}(\sigma) < 1$
  1. $\forall q \in \frac{1}{\pi}\arctan \frac{1}{\mathbb{Z}}$ $\exists L$ line bundle s.t. $L \in \mathcal{P}_{st}(q)$
     → $\exists q \in (\phi - \lfloor \phi \rfloor, 1)$ s.t. $\exists L$ line bundle $L \in \mathcal{P}_{st}(q)$
  2. $\phi < 1 \Rightarrow \mathrm{Hom}(p^*L \otimes \mathcal{O}_S(-C_0)[\lfloor \phi \rfloor], p^*L[\lfloor \phi \rfloor]) \neq 0$
     → $\phi < 1 \Rightarrow \exists i \leq 0$ s.t. $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2[i]) \neq 0$
- $\mathrm{per}(\sigma) \geq 1$
  1. $Rp_*\mathcal{O}_S(C_0)$ locally free sheaf
  2. $p^*F \otimes \mathcal{O}_S(-C_0) \in \mathcal{A}_1$, $p^*G \in \mathcal{A}_2$
     $\mathrm{Hom}(p^*F \otimes \mathcal{O}_S(-C_0), p^*G[i]) = \mathrm{Hom}(F, G \otimes Rp_*\mathcal{O}_S(C_0)[i])$
     $\phi_{st}(F) \in (\phi, \phi+1]$, $\phi_{st}(G \otimes Rp_*\mathcal{O}_S(C_0)[i]) \in (i, i+1]$
     → $\phi \geq 1 \Rightarrow \forall i \leq 0\, \mathrm{Hom}(F, G \otimes Rp_*\mathcal{O}_S(C_0)[i]) = 0$

Further study

Stability space of ruled surfaces:

- topology
  simply connectedness? - Unknown......
  ex) Stability space of $\mathbb{P}^1$ is simply connected. ([Mac])
- of Hirzeburch surface, especially
  $\{S_k\}_{k \in \mathbb{Z}}$: family of exceptional objects on $D^b(\mathbb{P}^1)$
  $\Theta_k := \{\sigma \in \mathrm{Stab}\, \mathbb{P}^1 \mid S_k \text{ and } S_{k+1} \text{ are stable.}\}$
  - $\mathrm{Stab}\, \mathbb{P}^1 = \bigcup_{k \in \mathbb{Z}} \Theta_k$
  - $i \neq j \Rightarrow \Theta_i \cap \Theta_j = \sigma_{st}.\widetilde{\mathrm{GL}^+(2,\mathbb{R})}$

  → Gluing is more complicated......

My dreams

S, ruled surface

- Understand gluing stability conditions on $D^b(S)$.
- Get a complete description of Stab S!!
  positional relation, topology, ...

References

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