TITLE:
Birational rigidity of complete intersections

AUTHOR(S):
鈴木, 文顕

CITATION:

ISSUE DATE:
2015

URL:
http://hdl.handle.net/2433/218250

RIGHT:
### Birational rigidity of complete intersections

**Fumiaki Suzuki**

Graduate School of Mathematical Sciences, The University of Tokyo

---

#### Background

Recall the following definition:

**Definition**

A Mori fiber space \( X/S \) is called *birationally superrigid* if any birational map to the source of another Mori fiber space is isomorphism.

It implies that \( X \) is *non-rational* and \( \text{Bir}(X) = \text{Aut}(X) \).

Consider a complete intersection \( X \) of type \( X_{d_1, \ldots, d_s} \subset P^{S_1+\ldots+d_s} \) of dimension \( \geq 3 \) with only mild singularities, which is defined by \( s \) hypersurfaces of degree \( d_1, \ldots, d_s \) in a projective space \( P^{S_1+\ldots+d_s} \). It is Fano of index 1 and rationally-connected.

For \( s = 1 \), after the works of Iskovskih-Manin, Pukhlikov and Cheltsov, de Fernex proved:

**Theorem (de Fernex ’13)**

For \( N \geq 4 \), every smooth hypersurface \( X = X_N \subset P^N \) of degree \( N \) is birationally superrigid.

For \( s \geq 2 \), its birational superrigidity is known only when \( X \) is one of the following:

**Known cases when \( s \geq 2 \)**

- a smooth complete intersection \( X = X_{d_1, \ldots, d_s} \subset P^{S_1+\ldots+d_s} \) of dimension \( \geq 12 \) which satisfies so-called regularity conditions, except three infinite series \( X_{2,2}, X_{2,2,2} \) and \( X_{2,2,2,2} \) by Pukhlikov,
- a smooth complete intersection \( X = X_{2,1} \subset P^6 \) not containing planes by Cheltsov.

No explicit examples which satisfy these conditions have been obtained so far.

#### Birational superrigidity complete intersections

We prove birational superrrigidity of smooth and singular complete intersections. For \( s \) positive integers \( d_1, \ldots, d_s \), set

\[
\epsilon_s(d_1, \ldots, d_s) = \frac{2(S_1+d_1+1)}{\sqrt[3]{P_{i=1}^s d_i}} - 5s
\]

in what follows.

**Smooth complete intersections**

**Theorem A**

Let \( d_1, \ldots, d_s \geq 2 \) be integers which satisfy

\[
1 \leq \epsilon_s(d_1, \ldots, d_s).
\]

Then every smooth complete intersection \( X = X_{d_1, \ldots, d_s} \subset P^{S_1+\ldots+d_s} \) is birationally superrigid.

**Corollary**

Every smooth complete intersection

\[
X = X_{d_1, \ldots, d_s} \subset P^{S_1+\ldots+d_s}, X_{i,j,d} \subset P^{i+j}, X_{2,2,d} \subset P^{d+4}
\]

is birationally superrigid for \( d \geq 55, 83, 111, 246 \) respectively.

**Singular complete intersections : Case 1**

Recall that an isolated singularity is called a *semi-homogeneous hypersurface singularity* if its tangent cone is a hypersurface which is smooth away from the vertex.

**Theorem B**

For positive integers \( d_1, \ldots, d_s \geq 2 \), every complete intersection

\[
X = X_{d_1, \ldots, d_s} \subset P^{S_1+\ldots+d_s}, \text{ with only semi-homogeneous hypersurface singularities of multiplicity at most } 
\epsilon_s(d_1, \ldots, d_s) = 2
\]

is birationally superrigid.

#### Singular complete intersections : Case 2

For a complete intersection \( X \subset P^V \), denote by \( X^V \subset (P^V)^* \) the dual variety of \( X \). Let \( d = \deg X \) and \( \pi = \Sigma_{i=1}^s \cdot k \cdot X_i^V \). It is known that \( X^V \) is a hypersurface and \( \deg X^V = dx \) if \( X \) is smooth.

**Theorem C**

Let \( d_1, \ldots, d_s \geq 2 \) be positive integers and \( X = X_{d_1, \ldots, d_s} \subset P^{S_1+\ldots+d_s} \) be a singular complete intersection with \( t \) isolated hypersurface singularities. If \( X^V \) is a hypersurface and

\[
dx - \deg X^V \leq \epsilon_s(d_1, \ldots, d_s) + 2t - 5,
\]

then \( X \) is birationally superrigid.

**Pukhlikov’s multiplicity bounds**

The following is a key proposition, which was known when \( s = 1 \) by Pukhlikov and \( k = 1 \) by Cheltsov.

**Proposition**

Let \( X \) be a complete intersection in \( P^N \) defined by \( s \) hypersurfaces and \( \alpha \) be an effective cycle on \( X \) of pure codimension \( k \) such that

\[
\alpha \sim m \cdot c_i(OX(X)^k \cap [X]).
\]

Assume either that \( X \) is smooth or \( ks + \dim \text{Sing}(X) + 1 < N \). Then \( e_5(\alpha) \leq m \) for every closed subvariety \( S \subset X \) of dimension \( ks \) not meeting the singular locus of \( X \).

**Proof:** We may assume that \( \dim S = ks \) and \( S \subset [\alpha] \). Then we use the method of multiple residual intersection, to construct a cycle \( R \) of pure-dimensional \( k \) on \( X \) such that

1. \( |R| \subset X^{sm} \),
2. \( \alpha \) and \( R \) intersect properly on \( X \), i.e. \( \dim [\alpha] \cap |R| = 0 \),
3. \( S \cap |R| \) contains at least \( \deg R \) points.

Then

\[
m \cdot \deg R = \alpha \cdot R \geq \sum_{i \in |S|} (t \cdot \alpha \cdot \lambda \cdot X) \geq \sum_{i \in |S|} \epsilon_5(\alpha) \cdot e_i(R) \geq \epsilon_5(\alpha) \cdot \deg R.
\]

The proof is done.