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Frobenius splitting for some Abelian fiber spaces

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Aim of this poster

Investigate F-split elliptic surfaces, more generally Abelian fiber spaces over curves. We can classify F-split elliptic fibrations (Abelian fiber spaces under some assumptions) by some invariants.

X : smooth projective variety / $k = \bar{k}$, $\mathrm{char}(k) = p > 0$

Definition 1

X is called Frobenius split (F-split) if the absolute Frobenius morphism F_X splits as an O_X-module. (i.e., $\exists \varphi : (F_X)_* O_X \to O_X$ s.t. $\phi \circ (F_X)^\sharp = \mathrm{id}_{O_X}$)

Let $\pi : X \to Y$ be a morphism.

- The absolute Frobenius morphism is NOT k-morphism. Then it is not compatible with restricting to the fibers of π for Frobenius pushforward sheaves $F_*(-)$.

$\longrightarrow$ The relative Frobenius morphism $F_{X/Y}$ is compatible! Moreover we have a characterizing of F-splitting of X via $F_{X/Y}$ as follows.

Proposition 2 (S)

Let $\pi : X \to C$ be a surjective morphism with $\pi_* O_X = O_C$ from n-dimensional smooth projective variety to smooth projective curve C. Then X is F-split if and only if a composition of the following morphisms is nonzero.

$$H^1(C, R^{n-1}\pi_*\omega_X) \xrightarrow{F_C^*} H^1\left(C, F_C^*(R^{n-1}\pi_*\omega_X)\right) \xrightarrow{F_{X/C}^*} H^1(C, R^{n-1}\pi_*\omega_X^{\otimes p}).$$

(1) F_C^* is just the p-th map. Especially we have Watanabe's classification of the F-splitting for the pair $\left(C, \sum_i \frac{b_i - 1}{b_i} P_i\right)$

(2) $F_{X/C}^*$ is closely related to the ordinarity of general fibers if π is elliptic fibration (more generally Abelian fiber spaces).

Theorem 3 (S)

Let $\pi : X \to C$ be a relatively minimal elliptic fibration with only tame fibers.

1. If C is an ordinary elliptic curve, then X is F-split if and only if $\mathrm{ord}(K_X) \mid (p-1)$ and the general fibers of π are ordinary elliptic curves.
2. If C is a projective line $\mathbb{P}^1$, then X is F-split if and only if the general fibers of π are ordinary elliptic curves and one of the following holds :
   (a) $\chi(O_X) = 2$
   NO multiple fibers and $\forall p > 0$.
   (b) $\chi(O_X) = 1$
      i. NO multiple fibers and $\forall p > 0$,
      ii. one multiple fiber and $\forall p > 0$,
      iii. $(2, 2)$ and $p > 2$.
3. $\chi(O_X) = 0$
   (a) NO multiple fibers and $\forall p > 0$,
   (b) two multiple fibers and $\forall p > 0$,
   (c) $(2, 2, 2)$ and $p > 2$,
   (d) $(3, 3, 3)$ and $p \equiv 1 \pmod 3$,

Theorem 3(continued)

(e) $(2, 3, 6)$ and $p \equiv 1 \pmod 3$,
(f) $(2, 4, 4)$ and $p \equiv 1 \pmod 4$,
(g) $(\infty, 0, 1, \lambda; 2, 2, 2, 2)$ and $p = 2n + 1$ such that the coefficient of x^n in the expansions of $(x-1)^n(x-\lambda)^n$ is not zero,

Remark. (1) We have a similar result for Abelian fiber spaces with only tame fibers (under some conditions).

(2) In the case of wild fibers, we have a similar result if $\dim X = 2$.

- We have two applications by using the proof of Proposition 2 and Theorem 3 as follows,

Theorem 4 (S)

Let X be an Enriques surface$/k$, $\mathrm{char}(k) = 2$. Then X is F-split if and only if X is Bombieri-Mumford singular Enriques surface.

This theorem is also obtained by using Crew's theorem which admits etale double K3-cover and some arithmetic facts. But the above theorem is obtained by using elliptic fibration structure and the technique of F-splitting.

Theorem 5 (S)

Let $\pi : X \to C$ be a relatively minimal elliptic fibration with $\kappa(X) = -\infty$. If X is F-split, then the type of multiple fibers are only ordinary elliptic curve or multiplicative type $\mathbb{G}_m$.