# The Nonequivariant Coherent-Constructible Correspondence for Toric Surfaces

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## Motivation -Homological Mirror Symmetry-

Mirror symmetry: A mysterious relationship between symplectic geometry and complex geometry from string theory. (e.g. period integrals and Gromov-Witten invariants)
Homological mirror symmetry: An explanation of mirror symmetry from categorical viewpoint.

derived category of coherent sheaves on $X$
$\cong$ Fukaya-type category of the mirror of $X$

Ex) HMS for $\mathbb{P}^1 \leftrightarrow (\mathbb{C}^\times, W := z + \frac{1}{z})$

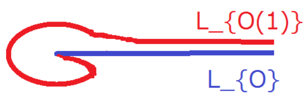
1. Beillinson's theorem

$$D^b(\mathrm{coh}\mathbb{P}^1) \cong \langle \mathcal{O}, \mathcal{O}(1) \rangle$$

2. HMS for $\mathbb{P}^1$

$$D^b(\mathrm{coh}\mathbb{P}^1) \cong DFS(\mathbb{C}^\times, W)$$

Lefshetz thimbles in $\mathbb{C}^\times$




## Nadler-Zaslow microlocalization

$X$: real analytic manifold

$$D^b_c(X) \cong DFuk(T^*X).$$

LHS: the bdd derived cat of constructible sheaves on $X$

How to see this equivalence?

## Constructible sheaf

"Locally constant sheaf on each stratum."

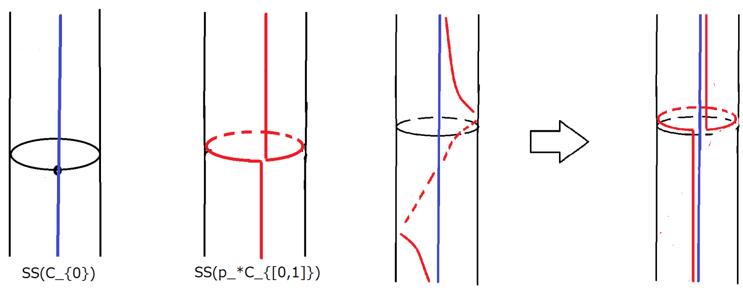
Ex) constant sheaf $\mathbb{C}_{S^1}$, skyscraper sheaf $\mathbb{C}_x$, $p_*\mathbb{C}_{[0,1]}$ where $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$.

## Microsupport $\subset T^*X$

The set of codirections of which "the cohomology of the sheaf does not extend isomorphically".
$(\mathrm{Sh}(S^1) \ni \mathcal{E} \mapsto \mathrm{SS}(\mathcal{E}) \subset T^*S^1)$.

Ex) $T^*S^1 \cong \mathbb{C}^\times$
NZ-microlocalization:

$$L_{\mathcal{O}} \mapsto \mathbb{C}_0$$
$$L_{\mathcal{O}(1)} \mapsto \mathbb{C}_{S^1}$$




$$D^b(\mathrm{coh}\mathbb{P}^1) \xrightarrow{\cong} \langle \mathbb{C}_0, p_!\mathbb{C}_{(0,1)} \rangle \subset D^b_c(S^1) \cong DFuk(T^*S^1).$$

After the works of Bondal and Nadler-Zaslow, Fang-Liu-Treumann-Zaslow proposed the (nonequivariant) coherent-constructible correspondence.

Notations

- $M, N$: free abelian groups of finite rank which are dual each other.
- $\Sigma$: a smooth complete fan defined in $N_\mathbb{R}$.
- $X_\Sigma$: the toric variety associated with $\Sigma$.
- $\overline{\Lambda_\Sigma} := \bigcup_{\sigma \in \Sigma} p(\sigma^\perp) \times (-\sigma)$:the subset of $T \times N_\mathbb{R}$ where $p : M_\mathbb{R} \to M_\mathbb{R}/M := T$ is the quotient map.
- $D^b_c(T, \overline{\Lambda_\Sigma})$: the full subcategory of $D^b_c(T)$ spanned by objects whose microsupports contained in $\overline{\Lambda_\Sigma}$.

## Theorem[Fang-Liu-Treumann-Zaslow, Treumann]

$$\exists \kappa_\Sigma : D^b(\mathrm{coh}X_\Sigma) \hookrightarrow D^b_c(T, \overline{\Lambda_\Sigma}).$$

## Conjecture (the nonequivariant coherent-constructible correspondence) [Fang-Liu-Treumann-Zaslow, Treumann]

$$\kappa_\Sigma : D^b(\mathrm{coh}X_\Sigma) \xrightarrow{\cong} D^b_c(T, \overline{\Lambda_\Sigma}).$$

Known results[Treumann, Scherotzke-Sibilla]

- Zonotopally unimodular fans (e.g. $(\mathbb{P}^1)^n$)
- Cragged fans (e.g. toric Fano surfaces)

## Main Theorem[K]

If $\dim \Sigma = 2$, the above conjecture holds.

Key of the proof: Blow-up formula
$\hat{\Sigma}$: a toric point blow-up of $\Sigma$

## Theorem[Orlov]

$\dim \Sigma = n + 1 \Rightarrow$

$$D^b(\mathrm{coh}X_{\hat{\Sigma}}) \cong \langle \mathcal{O}_E(nE), \ldots, \mathcal{O}_E(E), D^b(\mathrm{coh}X_\Sigma) \rangle$$

## Theorem[K]

$\dim \Sigma = n + 1 \Rightarrow$

$$D^b_c(T, \Lambda_{\hat{\Sigma}}) \cong \langle \kappa_{\hat{\Sigma}}(\mathcal{O}_E(nE)), \ldots, \kappa_{\hat{\Sigma}}(\mathcal{O}_E(E)), D^b_c(T, \overline{\Lambda_\Sigma}) \rangle$$

This formula, Toric MMP and the functorialitiy of $\kappa_\Sigma$ together reduces the proof to the known case if $\dim \Sigma = 2$.
The proof of the formula is relied on microlocal sheaf theory of Kashiwara-Schapira.

Thank you for your attention!