Geometric description of the moduli space of parabolic connections on $\mathbb{P}^1 \setminus \{t_1, \dots, t_5\}$ and the universal family

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Introduction

(Joint work with M.-H. Saito)

- n is a positive integer.
- $0, 1, t_1, \dots, t_{n-3}, \infty$ are n-distinct points on $\mathbb{P}^1$, and $D := 0 + 1 + t_1 + \cdots + t_{n-3} + \infty$ is an effective divisor.
- $\nu_1, \dots, \nu_n$ are complex numbers satisfying some conditions.

Moduli space of $\mathfrak{sl}_2$-connections ($\dim = 2(n-3)$)

$$M := \{(E, \nabla, \varphi)\}/\cong$$

1. E is a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\nabla \colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$ is a connection,
3. $\varphi \colon \bigwedge^2 E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ is a horizontal isomorphism,
4. the residue $\mathrm{res}_{t_i}(\nabla)$ of the connection ∇ at t_i has eigenvalues $\nu_i^{\pm}$, $1 \le i \le n$ ($\nu_i^{\pm} := \pm\nu_i$, $\nu_n^+ := \nu_n$, and $\nu_n^- := 1 - \nu_n$).

Moduli space of $\mathfrak{sl}_2$-Higgs bundles ($\dim = 2(n-3)$)

$$M_H := \{(E, \Phi, \varphi)\}/\cong$$

1. E: a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\Phi \colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$ is a Higgs field,
3. $\varphi \colon \bigwedge^2 E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ is an isomorphism and $\mathrm{tr}(\Phi) = 0$,
4. the residue $\mathrm{res}_{t_i}(\nabla)$ of the connection ∇ at t_i has eigenvalues $\nu_i^{\pm}$, $1 \le i \le n$. ($\nu_i^{\pm} := \pm\nu_i$).

Stratifications

$$M = M^0 \cup \cdots \cup M^{[(n-3)/2]}, \quad M_H = M_H^0 \cup \cdots \cup M_H^{[(n-3)/2]}$$

where $M^k := \{(E, \nabla, \varphi) \mid E \cong \mathcal{O}(k) \oplus \mathcal{O}(-k-1)\}/\cong$ and $M_H^k := \{(E, \Phi, \varphi) \mid E \cong \mathcal{O}(k) \oplus \mathcal{O}(-k-1)\}/\cong$.

Problem

Give explicit descriptions of the moduli spaces.

Canonical coordinate of the moduli spaces

Apparent singularities and dual parameters for M_H.

$$(E, \Phi, \varphi, \sigma) \longmapsto (G, C_s, \sigma_G)$$

- $\sigma \in H^0(\mathbb{P}^1, E)$,
- C_s is the spectral curve of (E, Φ),
- G is the torsion free sheaf of rank 1 on C_s associated to E (Beauville-Narasimhan-Ramanan correspondence), and
- $\sigma_G \in H^0(C_s, G) \cong H^0(\mathbb{P}^1, E)$ associated to σ.

For (G, C_s, σ_G),

$$0 \longrightarrow \mathcal{O}_{C_s} \xrightarrow{\sigma_G} G \longrightarrow T \longrightarrow 0$$

where T is a torsion sheaf on C_s of length $n-3$.

$(E, \Phi, \varphi, \sigma) \longmapsto \mathrm{Supp}(T) = \{(q_1, p_1), \dots, (q_{n-3}, p_{n-3})\} \in \mathrm{Sym}^{n-3}(\mathbb{L})$

where $\mathbb{L}$ is the total space of $\Omega^1_{\mathbb{P}^1}(D)$.

Apparent singularities and dual parameters

$\{q_1, \dots, q_{n-3}\}$: apparent singularities of $(E, \Phi, \varphi, \sigma)$

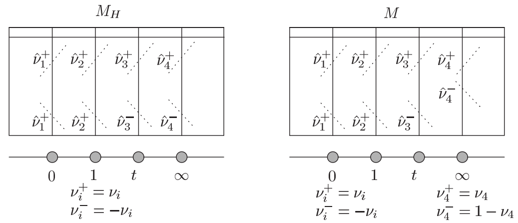
$\{p_1, \dots, p_{n-3}\}$: dual parameters of $(E, \Phi, \varphi, \sigma)$

Apparent singularities and dual parameters for M.

We can define the apparent singularities for M. On the other hand, we can define the dual parameters for only M^0, which is a Zariski open set.

$n = 4$

In this case, $M = M^0$ and $M_H = M_H^0$.



$n = 5$ (Higgs case)

In this case, $M = M^0 \cup M^1$ and $M_H = M_H^0 \cup M_H^1$.

Moduli space of $\mathfrak{sl}_2$-Higgs bundles with a cyclic vector

$$\widehat{M}_H := \{(E, \Phi, \varphi, [\sigma])\}/\cong$$

1. $(E, \Phi, \varphi) \in M_H$,
2. $[\sigma] \subset H^0(\mathbb{P}^1, E)$ is a 1-dimensional subspace generated by a nonzero section $\sigma \in H^0(\mathbb{P}^1, E)$.

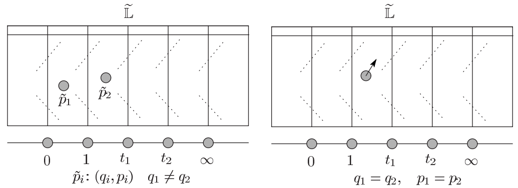
- $\widehat{M}_H$ is the blowing up of M_H along M_H^1.
- By the apparent singularities and the dual parameters, we have $\widehat{M}_H \to \mathrm{Sym}^2(\mathbb{L})$.

Theorem

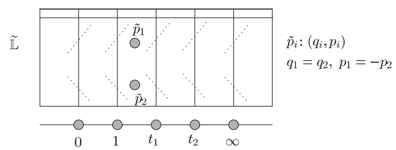
The map $\widehat{M}_H \to \widetilde{\mathrm{Hilb}}^2(\widetilde{\mathbb{L}})$ is injective.

Here, $\widetilde{\mathrm{Hilb}}^2(\widetilde{\mathbb{L}})$ is a blowing up of $\mathrm{Hilb}^{n-3}(\widetilde{\mathbb{L}})$.

The locus of $\mathcal{O} \oplus \mathcal{O}(-1)$.



The locus of $\mathcal{O}(1) \oplus \mathcal{O}(-2)$.



$n = 5$ (Connection case)

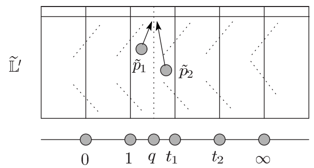
By the apparent singularities and the dual parameters, we obtain that

$$M^0 \longrightarrow \mathrm{Hilb}^2(\widetilde{\mathbb{L}}')$$

is injective. We extend the map to the moduli space $\widehat{M}$ of $\mathfrak{sl}_2$-connection with a cyclic vector.

Strategy

1. Construct a jumping family Φ_{jump} of Higgs bundles, that is, Φ_{jump} is type $\mathcal{O} \oplus \mathcal{O}(-1)$ for $q_1 \neq q_2$ and $\lim_{q_2 \to q_1} \Phi_{\mathrm{jump}}$ is type $\mathcal{O}(1) \oplus \mathcal{O}(-2)$.
2. Construct a connection ∇_0 such that
   - $\lim_{q_2 \to q_1} \nabla_0$ is type $\mathcal{O}(1) \oplus \mathcal{O}(-2)$.
   - $\nabla_0 + \Phi_{\mathrm{jump}}$ satisfies the conditions of the residue matrices.
3. Follow up the apparent singularities and the dual parameters of $\nabla_0 + \Phi_{\mathrm{jump}}$ as $q_2 \to q_1$.



References

[1] A. Komyo, M.-H. Saito, *Geometric description of the moduli space of parabolic connections on $\mathbb{P}^1 \setminus \{t_1, \dots, t_5\}$ and the universal family*. in preparation.

[2] M.-H. Saito, S. Szabo, *Apparent singularities and canonical coordinates of moduli spaces of parabolic connections and parabolic Higgs bundles*. in preparation.