Weak positivity theorem and Frobenius stable canonical rings of geometric generic fibers

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Notation

- Let k be an algebraically closed field.
- Let $f\colon X \to Y$ be a fibration (separable surjective morphism satisfying $f_*\mathcal{O}_X \cong \mathcal{O}_Y$) between smooth projective varieties over k.
- Let $\overline{\eta}$ be the geometric generic point of Y.
- Let $Z := X_{\overline{\eta}}$ be the geometric generic fiber.
- Let $\omega_{X/Y} := \omega_X \otimes f^*\omega_Y^{-1}$ be the relative canonical bundle.

The positivity of $f_*\omega_{X/Y}^m$

Questions

Let m be a positive integer.
- Is $f_*\omega_{X/Y}^m$ a nef vector bundle?
- Is $f_*\omega_{X/Y}^m$ a weakly positive sheaf?

Definition (weak positivity)

A coherent sheaf $\mathcal{F}$ on Y is said to be *weakly positive* if $\forall$ ample divisor H, $\forall a \in \mathbb{Z}_{>0}$, $\exists b \in \mathbb{Z}_{>0}$ s.t. $(\mathrm{Sym}^{ab}\mathcal{F})^{**}(bH)$ is generically globally generated. Here $(\_)^{**} := \mathcal{H}om(\mathcal{H}om(\_, \mathcal{O}_Y))$.

Remarks:
- Every nef vector bundle is weakly positive.
- Every weakly positive vector bundle on projective curve is nef.

Known results (char.$k = 0$)
- $f_*\omega_{X/Y}^m$ is nef vector bundle if
  - f is smooth [P. Griffiths ($m = 1$), O. Fujino ($m \geq 2$)].
  - $\dim Y = 1$ [T. Fujita ($m = 1$), Y. Kawamata ($m \geq 2$)].
  - $m = 1$ and some conditions [Y. Kawamata].
  - Z has a good minimal model (up to a birational modification of f) [O. Fujino].
- $f_*\omega_{X/Y}^m$ is always weakly positive [E. Viehweg].

Remarks: In positive characteristic, $\exists$ counter-examples.
- $\exists$ fibration $g_1 : S_1 \to C_1$ from a smooth projective surface to a smooth projective curve s.t. $g_{1*}\omega_{S_1/C_1}$ is NOT nef [L. Moret-Bailly].
- $\exists$ fibration $g_2 : S_2 \to C_2$ from a smooth projective surface to a smooth projective curve s.t. $\forall m \in \mathbb{Z}_{>0}$, $g_{2*}\omega_{S_2/C_2}^m$ is NOT nef [M. Raynaud, Q. Xie].

The positivity of $f_*\omega_{X/Y}^m$ in char.$k > 0$

From now on, we assume that the characteristic of k is $p > 0$.

Known results (char.$k > 0$)
- $f_*\omega_{X/Y}^m$ is nef vector bundle if
  - $\dim X = 2$, $\dim Y = 1$, Z is a nodal curve, and $m \geq 2$ [J. Kollár].
  - $\dim Y = 1$, Z is normal F-pure, $\omega_{X/Y}$ is f-ample, and $m \gg 0$ [Z. Patakfalvi].
- $f_*\omega_{X/Y}^m$ is weakly positive if $S^0(Z, \omega_Z) = H^0(Z, \omega_Z)$ [J. Jang ($\dim X = 2$), Z. Patakfalvi (general case)].

Definition (F-purity, S^0, and R_S)

Let V be a Gorenstein variety over k.
- $\mathrm{Tr}^{(1)} : F_*\omega_V \cong F_*\mathcal{H}om(\mathcal{O}_V, \omega_V) \cong \mathcal{H}om(F_*\mathcal{O}_V, \omega_V) \to \omega_V$.
- V is said to be *F-pure* if the map $\mathrm{Tr}^{(1)}$ is surjective.
- $\mathrm{Tr}^{(e)} : F^e{}_*\omega_V \xrightarrow{F^{e-1}{}_*\mathrm{Tr}^{(1)}} F^{e-1}{}_*\omega_V \xrightarrow{F^{e-2}{}_*\mathrm{Tr}^{(1)}} \cdots \xrightarrow{\mathrm{Tr}^{(1)}} \omega_V$.

Assume that V is projective.
- For every line bundle $\mathcal{L}$ on V, $S^0(V, \mathcal{L})$ is defined as
$$\bigcap_{e>0} \mathrm{Im}\left(H^0(V, F^e{}_*(\omega_V^{(1-p^e)} \otimes \mathcal{L}^{p^e})) \xrightarrow{H^0(V,\mathrm{Tr}^{(e)}\otimes\omega_V^{-1}\otimes\mathcal{L})} H^0(V, \mathcal{L})\right).$$
- *Frobenius stable canonical ring* $R_S(V, \omega_V)$ is defined as
$$R_S(V, \omega_V) := \bigoplus_{m\geq 0} S^0(V, \omega_V^m) \subseteq \bigoplus_{m\geq 0} H^0(V, \omega_V^m) =: R(V, \omega_V).$$

It is easy to check that $R_S(V, \omega_V)$ is an ideal of $R(V, \omega_V)$.

Examples:
- Let C be a curve. If C is a nodal curve, then C is F-pure. If C has a cusp, then C is NOT F-pure.
- Let V be a Gorenstein projective variety s.t. ω_V is ample. Then
$$\dim_k R(V, \omega_V)/R_S(V, \omega_V) < \infty \Leftrightarrow V \text{ is } F\text{-pure}.$$
- Let C be a F-pure Gorenstein projective curve of arithmetic genus ≥ 2. Then $\forall m \geq 2$, $S^0(C, \omega_C^m) = H^0(C, \omega_C^m)$.

Theorem

In the situation of Notation, assume that
- (i) $R(Z, \omega_Z)$ is finitely generated $k(\overline{\eta})$-algebra, and
- (ii) $\exists m_0 \in \mathbb{Z}_{>0}$, $\forall m \geq m_0$, $S^0(Z, \omega_Z^m) = H^0(Z, \omega_Z^m)$.

Then $f_*\omega_{X/Y}^m$ is weakly positive for $\forall m \geq m_0$.

Corollary

In the situation of Notation, $f_*\omega_{X/Y}^m$ is weakly positive if
- Z is F-pure curve of arithmetic genus ≥ 2 and $m \geq 2$.
- Z is F-pure, ω_Z is ample, and $m \gg 0$.
- Z is smooth surface of general type, $p > 5$, and $m \gg 0$.

Remark: Geometric generic fiber of the fibration $g_2 : S_2 \to C_2$ (see the bottom of the first column) is NOT F-pure (it has a cusp).

An application

Iitaka's conjecture

In the situation of Notation, the inequality
$$\kappa(X) \geq \kappa(Y) + \kappa(Z, \omega_Z)$$
holds. Here $\kappa(Z, \omega_Z)$ is the Iitaka-Kodaira dimension of the dualizing sheaf ω_Z of Z.

In characteristic zero, it is known that this conjecture is true in many cases. In positive characteristic, this conjecture is true if
- $\dim Z = 1$ [Y. Chen, L. Zhang].
- $\dim X = 3$, $k = \overline{\mathbb{F}_p}$, and $p > 5$ [C. Birkar, Y. Chen, L. Zhang].
- Y is of general type and $S^0(Z, \omega_Z) \neq 0$ [Z. Patakfalvi].

Theorem

In the situation of Notation, assume that
- (i) $R(Z, \omega_Z)$ is finitely generated $k(\overline{\eta})$-algebra,
- (ii) $\dim_{k(\overline{\eta})} R(Z, \omega_Z)/R_S(Z, \omega_Z) < \infty$, and
- (iii) Y is of general type or Y is an elliptic curve.

Then
$$\kappa(X) \geq \kappa(Y) + \kappa(Z, \omega_Z).$$

Corollary

Assume that $\dim Y = 1$, $p > 5$, and Z is a smooth surface of general type. Then
$$\kappa(X) \geq \kappa(Y) + \kappa(Z).$$