CAMPANA-PETERNELL CONJECTURE FOR n-FOLDS WITH $\rho > n - 5$

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1. Introduction

Mori's solution of Hartshorne's conjecture [19] asserts that any projective manifold with ample tangent bundle is a projective space. As a generalization of Mori's result, Campana and Peternell conjectured the following:

Conjecture 1.1 (Campana-Peternell Conjecture [3])**.** Fano manifolds with nef tangent bundles are homogeneous.

In this report, we explain about the following result obtained by the author:

Theorem 1.2 ([9, 10])**.** *Conjecture 1.1 is true for n-folds with Picard number $\rho > n - 5$.*

The same result in the case of $\rho > n - 4$ is independently obtained by Kiwamu Watanabe [28].

We will work over the field of complex numbers and mainly in the category of smooth projective varieties.

I. By a theorem of Borel and Remmert [2], every homogenous manifold is isomorphic to a product $A \times G/P$, where A is an abelian variety and G/P is a rational homogeneous manifold with a semisimple algebraic group G and a parabolic subgroup P of G. Of course an abelian variety has a trivial canonical bundle. On the other hand a rational homogeneous manifold G/P has an ample anticanonical divisor, i.e. it is a Fano manifold. Hence the affirmative answer to Conjecture 1.1 implies that every Fano manifold with nef tangent bundle is isomorphic to a rational homogenous manifold.

II. By a theorem of Demailly, Peternell and Schneider [7], every manifold with nef tangent bundle admits a smooth Fano fibration onto an abelian variety after taking a finite étale base change. Then the fibers have nef tangent bundles. Hence the affirmative answer to Conjecture 1.1 gives a decomposition theorem for manifolds with nef tangent bundles.

Note that there is a manifold with nef tangent bundle which is not an étale quotient of homogenous one [7, Example 3.5].

III. Conjecture 1.1 is known to be true in some cases. For example, as for lower dimensional cases, Conjecture 1.1 is known to be true for

(1) 3-folds or 4-folds with Picard number $\rho > 1$ [3, 4],
(2) 4-folds with Picard number one [8, 18],
(3) 5-folds with Picard number $\rho > 1$ [26].

Modulo the above known results, our proof of Theorem 1.2 is divided into three parts:

(1) $\rho \geq 3$,
(2) 6-folds with Picard number $\rho = 2$,
(3) 5-folds with Picard number $\rho = 1$.

The case $\rho \geq 3$ is discussed in Section 2. In Sections 3 and 4, we explain the case of 5-folds with Picard number one.

IV. Recently, a characterization of complete flag manifolds G/B is obtained by Muñoz, Occhetta, Solá Conde, Watanabe and Wiśniewski [20, 23]:

Theorem 1.3 ([23])**.** *A Fano manifold M is a complete flag manifold if and only if every elementary contraction of M is a smooth $\mathbb{P}^1$-fibration.*

Moreover they proposed the following conjecture which is equivalent to Conjecture 1.1:

Conjecture 1.4 ([20, Conjecture 2], [23, Conjecture 6.3])**.** For any Fano manifold X with nef tangent bundle, there exists a contraction $f : M \to X$ from a complete flag manifold M.

In our proof of Theorem 1.2, Theorem 1.3 plays an important role to study manifolds with two $\mathbb{P}^1$-fibrations in Section 3 and Section 4, and to prove homogeneity of certain Fano manifolds with nef tangent bundles in Proposition 4.8 (1).

For further results or background materials about Conjecture 1.1, we refer the reader to the article [22] and references therein.

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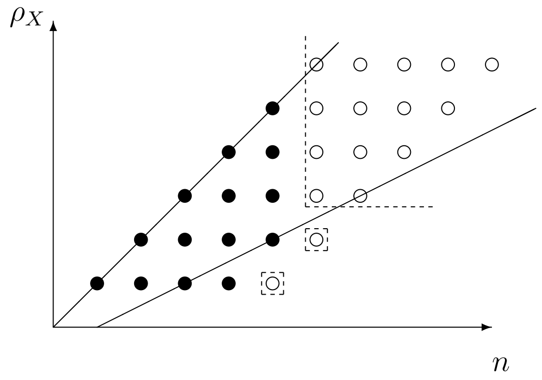
2. CP manifolds with large Picard numbers

In this section, we explain our inductive approach to prove Theorem 1.2. The key proposition is the following Proposition 2.1.

Proposition 2.1 ([10, Theorem 4.1]). *Let X be a CP manifold. If $n \leq 2\rho_X + 1$, then X is one of the following:*

(1) *$Y \times M$, where Y is a CP manifold and M is a complete flag manifold.*
(2) *X is homogeneous.*

In the first case, the homogeneity of X is reduced to that of Y. Furthermore we have $\dim Y - \rho_Y \leq \dim X - \rho_X$. Hence, by induction and modulo known results, the proof of Theorem 2.1 is reduced to the case of 6-folds with $\rho_X = 2$ and 5-folds with $\rho_X = 1$.



For a more precise statement and the detailed proof of Proposition 2.1 we refer the reader to the original article [10, Theorem 4.1]. Here we only mention that the key to the proof of Proposition 2.1 is Theorem 1.3 and the following proposition.

Proposition 2.2 ([20, Proposition 5]). *If a CP manifold X admits a contraction onto a complete flag manifold M, then X is a product $Y \times M$.*

In the rest of this paper, we will explain the proof of Conjecture 1.1 for 5-folds with Picard number one.

3. Manifolds with two smooth $\mathbb{P}^1$-fibrations

The result of this section (=Theorem 3.1) will be used twice in the proof; first to prove the existence of the morphism p and q in Theorem 4.6 (2), and second to prove Proposition 4.8 (1).

Let U be a manifold with two smooth $\mathbb{P}^1$-fibrations π and e. Note that here we do *not* assume neither $\rho_M = 2$ nor M is Fano.

$$\begin{tikzcd} & U \arrow[dl, "\pi"'] \arrow[dr, "e"] & \\ S & & W \end{tikzcd}$$

Then the image of π-fiber on W defines a half line $R_S \subset \overline{\mathrm{NE}}(W)$.

Theorem 3.1 ([9, Theorem 2.2]). *R_S is an extremal ray and the contraction of R_S is smooth.*

Hence there exists the following diagram, where $f = f_{R_S}$ is the contraction of the ray R_S:

$$\begin{array}{ccc} U & \xrightarrow{e} & W \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & Z. \end{array}$$

For the detailed proof, we refer the reader to the paper [9, Section 2]. The key point is the following:

(1) Every S-rationally chain connected equivalent class has the same dimension. Hence S-rationally chain connected quotient map $W \dashrightarrow Z$ is a morphism, which proves that R_S is a ray [12, I. Theorem 3.17 and Theorem 3.21] or [1, Proposition 1]. For accounts of rationally connected quotients, we refer the reader to [6, Chapter.5], [12, Chapter IV] [13, Section 1].
(2) e^*T_W (resp. π^*T_S) is π-nef (resp. e-nef). Hence, by a similar argument of [25, The proof of Theorem 4.4] (cf. [7, Theorem 5.2]), one can show that the contraction of the ray R_S is smooth..

4. CP 5-folds with $\rho_X = 1$

In this section we explain our proof of Conjecture 1.1 for 5-folds based on [9].

Definition 4.1. Let X be a CP manifold with Picard number one. The pseudoindex of X is defined to be the minimum anticanonical degree of rational curves on X:

$$i_X \coloneqq \min\{\, -K_X.C \mid C \text{ is a rational curve on } X \,\}.$$

By Mori's bend and break lemma, the pseudoindex i_X is not greater than $n+1$ and, by theorems of Cho, Miyaoka and Shepherd-Barron or Miyaoka [5, 17], X is a projective space or a hyperquadric if $i_X \geq n$.

On the other hand the pseudoindex is not less than two and X is isomorphic to $\mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{Q}^3$ or $K(G_2)$ if $i_X \leq 3$ [8, 18].

Hence the Conjecture 1.1 for the case of dimension 5 is reduced to the case of CP 5-folds with Picard number one and $i_X = 4$.

4.2. **Family of minimal rational curves.** We fix a CP manifold X with Picard number one and denote by $\mathrm{RatCurves}^n(X)$ the normalization of the scheme parametrizing rational curves on X [12, II, Section 2].

Definition 4.3. A family of minimal rational curves is an irreducible component $V \subset \mathrm{RatCurves}^n(X)$ which parametrizes rational curves of minimum anticanonical degree i_X.

For a family of minimal rational curves V, there exists the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{e} & X \\ {\scriptstyle\pi}\downarrow & & \\ V, & & \end{array}$$

where $\pi\colon U \to V$ is the universal family of the minimal rational curves and $e\colon U \to X$ is the evaluation morphism of the minimum rational curves. In particular, π is a $\mathbb{P}^1$-fibration [12, II. Corollary 2.12].

The pseudoindex determines the dimension of the family of minimal rational curves:

Proposition 4.4 ([12, II. Theorem 1.2, Proposition 2.14, Theorem 2.15 and Corollary 3.5.3], [14, Corollary 1.4])**.** *In the above notation, e is a smooth projective morphism with connected fibers of dimension $i_X - 2$ and V is a smooth projective manifold of dimension $n + i_X - 3$.*

From now on, we assume that the pseudoindex i_X is four. Then we have the following lemma by the isotriviality of smooth projective family of minimal surfaces over $\mathbb{P}^1$ due to Migliorini, Oguiso and Viehweg [16, 24] and a similar argument as in [18, Lemma 1.2.2]:

Lemma 4.5 ([9, Lemma 3.5])**.** *Let X be a CP manifold with Picard number one and pseudoindex four. Then there exists a K_U-negative extremal ray $R \subset \overline{\mathrm{NE}}(U/X)$.*

Hence there exists the contraction of the ray R ([11] or [15]) and the detailed study of the contraction gives the following:

Theorem 4.6 ([9, Theorem 3.3])**.** *One of the following holds:*

(1) *e is a smooth $\mathbb{P}^2$-fibration.*

(2) *we have the following commutative diagram with properties (a) and (b):*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend left, "e"] \arrow[d, "\pi"'] & W \arrow[r, "g"] \arrow[d, "q"] & X \\ V \arrow[r, "p"] & Y & \end{tikzcd}$$

(a) *f and g are smooth $\mathbb{P}^1$-fibrations,*

(b) *p and q are smooth elementary Mori contraction.*

4.7. **Conjecture 1.1 for 5-folds.** The following completes the proof of Conjecture 1.1 for 5-folds.

Proposition 4.8 ([9, Section 4])**.**

(1) *If Theorem 4.6* (1) *occurs, then* X *is homogeneous.*
(2) *If Theorem 4.6* (2) *occurs, then* $\dim X \neq 5$.

Proof. (1) The proof is based on the strategy described in Section 1 IV. Set $M \coloneqq \mathbb{P}(T_e)$ and let $p : M \to U$ be the projection. Then,

(1) $e \circ p$-fiber is isomorphic to $\mathbb{P}(T_{\mathbb{P}^2})$,
(2) $\pi \circ p$-fiber is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ since $T_e|_C \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ for a π-fiber C.

Hence, by [25, Theorem 4.4] (cf. [7, Theorem 5.2]), there exists the following diagram with smooth $\mathbb{P}^1$-fibrations q and r:

$$\begin{array}{ccccc} L & \xleftarrow{\ r\ } & M & \xrightarrow{\ q\ } & N \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ V & \xleftarrow{\ \pi\ } & U & \xrightarrow{\ e\ } & X. \end{array}$$

Then Theorem 3.1 implies that M is Fano and every elementary contraction of M is p, q or r. Hence M is homogeneous by Theorem 1.3 and so is X.

(2) To obtain a contradiction, we assume that $\dim X = 5$.

For a π-fiber C, we have

$$-K_f.C = -K_g.f(C) = -1.$$

The fiber of $h \coloneqq q \circ f = p \circ \pi$ is a complete flag manifold with Picard number $\rho = 2$ by Theorem 1.3 and, since $-K_f.(\pi\text{-fiber}) = -1$, all f-fibers are the same and isomorphic to $\mathbb{P}^2$, $\mathbb{Q}^3$ or $K(G_2)$ by the classification of rational homogeneous manifolds (see also [21, 27]).

First assume that q is a smooth $\mathbb{P}^2$-fibration. Then we have

$$b_4(W) = 1 + b_2(Y) + b_4(Y),$$
$$b_6(W) = b_2(Y) + b_4(Y) + b_6(Y).$$

On the other hand, since g is a smooth $\mathbb{P}^1$-fibration, we have

$$b_4(W) = b_2(X) + b_4(X),$$
$$b_6(W) = b_4(X) + b_6(X).$$

Combined with duality $b_4(X) = b_6(X)$ and $b_2(Y) = b_6(Y)$, we have $b_4(Y) = 0$. This gives a contradiction.

Next assume that the relative dimension of q is not less than 3. Then $\dim Y \leq 3$. Let H be the g-pullback of an ample divisor on X and τ a real number which satisfies that $-K_g + \tau H$ is nef but not ample. Then the non-nefness of $-K_g$ gives $\tau \neq 0$.

On the other hand, a direct computation with numerical relations

$$(-K_g + \tau H)^4 = \cdots = (-K_g + \tau H)^6 = 0.$$

and the Chern-Wu relation ($K_g^2 \equiv_{\mathrm{num}} g^*Z$ for $Z \in N^2(X)$) gives $\tau = 0$. This gives a contradiction. For the detailed calculation, we refer the reader to [9, Proposition 4.4]. $\square$

References

1. Laurent Bonavero, Cinzia Casagrande, and Stéphane Druel, *On covering and quasi-unsplit families of curves*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 1, 45–57.
2. A. Borel and R. Remmert, *Über kompakte homogene Kählersche Mannigfaltigkeiten*, Math. Ann. **145** (1961/1962), 429–439.
3. Frédéric Campana and Thomas Peternell, *Projective manifolds whose tangent bundles are numerically effective*, Math. Ann. **289** (1991), no. 1, 169–187.
4. ______, *4-folds with numerically effective tangent bundles and second Betti numbers greater than one*, Manuscripta Math. **79** (1993), no. 3-4, 225–238.
5. Koji Cho, Yoichi Miyaoka, and N. I. Shepherd-Barron, *Characterizations of projective space and applications to complex symplectic manifolds*, Higher dimensional birational geometry (Kyoto, 1997), Adv. Stud. Pure Math., vol. 35, Math. Soc. Japan, Tokyo, 2002, pp. 1–88.
6. Olivier Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001.
7. Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom. **3** (1994), no. 2, 295–345.
8. Jun-Muk Hwang, *Rigidity of rational homogeneous spaces*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 613–626.
9. Akihiro Kanemitsu, *Fano 5-folds with nef tangent bundles*, arXiv:1503.04579v1, submitted, 2015.
10. ______, *Fano n-folds with nef tangent bundle and picard number greater than $n-5$*, arXiv:1505.02952v1, submitted, 2015.
11. Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki, *Introduction to the minimal model problem*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 283–360.
12. János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996.
13. János Kollár, Yoichi Miyaoka, and Shigefumi Mori, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. **36** (1992), no. 3, 765–779.
14. ______, *Rationally connected varieties*, J. Algebraic Geom. **1** (1992), no. 3, 429–448.
15. János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
16. Luca Migliorini, *A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial*, J. Algebraic Geom. **4** (1995), no. 2, 353–361.
17. Yoichi Miyaoka, *Numerical characterisations of hyperquadrics*, Complex analysis in several variables—Memorial Conference of Kiyoshi Oka's Centennial Birthday, Adv. Stud. Pure Math., vol. 42, Math. Soc. Japan, Tokyo, 2004, pp. 209–235.

18. Ngaiming Mok, *On Fano manifolds with nef tangent bundles admitting 1-dimensional varieties of minimal rational tangents*, Trans. Amer. Math. Soc. **354** (2002), no. 7, 2639–2658 (electronic).
19. Shigefumi Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979), no. 3, 593–606.
20. Roberto Muñoz, Gianluca Occhetta, Luis E. Solá Conde, and Kiwamu Watanabe, *Rational curves, Dynkin diagrams and Fano manifolds with nef tangent bundle*, Math. Ann. **361** (2015), no. 3-4, 583–609.
21. Roberto Muñoz, Gianluca Occhetta, and Luis Eduardo Solá Conde, *On rank 2 vector bundles on Fano manifolds*, Kyoto J. Math. **54** (2014), no. 1, 167–197.
22. Roberto Muñoz, Gianluca Occhetta, Luis Eduardo Solá Conde, Kiwamu Watanabe, and Jarosław A. Wiśniewski, *A survey on the Campana-Peternell Conjecture*, arXiv:1407.6483v3, 2014.
23. Gianluca Occhetta, Luis Eduardo Solá Conde, Kiwamu Watanabe, and Jarosław A. Wiśniewski, *Fano manifolds whose elementary contractions are smooth $\mathbb{P}^1$-fibrations*, arXiv:1407.3658v2, 2014.
24. Keiji Oguiso and Eckart Viehweg, *On the isotriviality of families of elliptic surfaces*, J. Algebraic Geom. **10** (2001), no. 3, 569–598.
25. Luis Eduardo Solá Conde and Jarosław A. Wiśniewski, *On manifolds whose tangent bundle is big and 1-ample*, Proc. London Math. Soc. (3) **89** (2004), no. 2, 273–290.
26. Kiwamu Watanabe, *Fano 5-folds with nef tangent bundles and Picard numbers greater than one*, Math. Z. **276** (2014), no. 1-2, 39–49.
27. ———, *$\mathbb{P}^1$-bundles admitting another smooth morphism of relative dimension one*, J. Algebra **414** (2014), 105–119.
28. ———, *Fano manifolds with nef tangent bundle and large Picard number*, Proc. Japan Acad. Ser. A Math. Sci. **91** (2015), no. 6, 89–94.

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