Perverse Schobers and birational geometry

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This is a report on a joint work in progress with A. Bondal and V. Schechtman. *Perverse Schobers*, a term introduced in [KS2], are conjectural categorical analogs of perverse sheaves. While we do not yet have a general definition of a perverse Schober, there are cases when such definition can be given and has geometric sense. In these cases one can use known classification of perverse sheaves in terms of linear algebra data and then category these data.

1. Reminder on perverse sheaves. Let X be a complex manifold and $S = (X_\alpha)$ be a complex analytic stratification of X. Thus each X_α is smooth but the closure $\overline{X}_\alpha$ can be a singular complex analytic space. Thus there is an open dense stratum X_0. Let $\mathbf{k}$ be a field. We denote by $\mathrm{Perv}(X, S)$ the full subcategory in $D^b \mathrm{Sh}_X$, the derived category of sheaves of $\mathbf{k}$-vector spaces on X, formed by complexes $\mathcal{F}$ such that:

(P^-) Each cohomology sheaf $\underline{H}^i(\mathcal{F})$ is locally constant on each X_α and is supported on an analytic subset of complex codimension $\geq i$.

(P^+) The sheaves $\underline{\mathbb{H}}^i_{X_\alpha}(\mathcal{F})$ are zero for $i < \mathrm{codim}_{\mathbb{C}}(X_\alpha)$.

Objects of $\mathrm{Perv}(X, S)$ are called S-smooth perverse sheaves on X. It follows that for any such object $\mathcal{F}$ the restriction $\mathcal{F}|_{X_0}$ is a local system in degree 0. So a perverse sheaf can be seen as a particular way to extend a local system on the open stratum to strata of higher codimension.

2. Classification of perverse sheaves. $\mathrm{Perv}(X, S)$ is an Artinian and Noetherian abelian category. In many cases it can be identified with the category of representations of an explicit quiver with relations. In such cases people often talk about "explicit classification of perverse sheaves".

Example 1. Let $X = D \subset \mathbb{C}$ be the unit disk, and let S consist of $\{0\}$ and $D - \{0\}$. We denote the corresponding category of perverse sheaves by $\mathrm{Perv}(D, 0)$.

Classical Theorem ([5] [13]) The category $\mathrm{Perv}(D, 0)$ is equivalent to the category of diagrams of finite-dimensional $\mathbf{k}$-vector spaces

$$\begin{tikzcd} \Phi \arrow[r, shift left, "a"] & \Psi \arrow[l, shift left, "b"] \end{tikzcd}$$

such that $T_\Psi = \mathrm{Id}_\Psi - ab$ and $T_\Phi = \mathrm{Id}_\Phi - ba$ are invertible. The spaces $\Phi = \Phi(\mathcal{F})$ and $\Psi = \Psi(\mathcal{F})$ corresponding to $\mathcal{F} \in \mathrm{Perv}(D, 0)$ are called the spaces of *vanishing* and *nearby cycles* of $\mathcal{F}$. Explicitly, if we denote by $K = [0, 1)$ the radius of the disk D, and form the sheaf $\mathcal{R} = \mathcal{R}_K(\mathcal{F}) = \underline{\mathbb{H}}^1_K(\mathcal{F})$ on K, then Φ is the stalk of $\mathcal{R}$ at 0 and Ψ is the stalk at any other point. The map a is the generalization map describing the sheaf structure.

Example 2 Alternative ("Dirac")description of the same category [KS1]. It can be considered as a "square root" of the previous description. Namely, $\mathrm{Perv}(D, 0)$ is equivalent to the category of diagrams

$$\begin{tikzcd} E_- \arrow[r, shift right, "\delta_-"'] & E_0 \arrow[l, shift right, "\gamma_-"'] \arrow[r, shift left, "\gamma_+"] & E_+ \arrow[l, shift left, "\delta_+"] \end{tikzcd}$$

satisfying the two following conditions:

(1) $\gamma_-\delta_- = \mathrm{Id}_{E_-}, \ \ \gamma_+\delta_+ = \mathrm{Id}_{E_+}$.

(2) The maps $\gamma_-\delta_+ : E_+ \to E_-, \ \ \gamma_+\delta_- : E_- \to E_+$ are invertible.

Explicitly, we take $k = (-1, 1)$ to be the diameter of the disk, and form $\mathcal{R} = \underline{\mathbb{H}}^1_K(\mathcal{F})$. Then $E_\pm$ are the stalks of $\mathcal{R}$ are ± 1, while E_0 is the stalk at 0.

Example 3. [KS1] Let $X = \mathbb{C}^n$, and $\mathcal{H}$ be an arrangement of hyperplanes in $\mathbb{R}^n$. The complexification $\mathcal{H}_\mathbb{C}$ defines a natural stratification of $\mathbb{C}^n$, and we denote by $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ the corresponding category of perverse sheaves. In [KS1] we have described it in terms of the *chamber structure* on $\mathbb{R}^n$ given by the arrangement $\mathcal{H}$. More precisely, $\mathcal{H}$ subdivides $\mathbb{R}^n$ into locally closed real *cells* (of all dimensions). Open cells are called *chambers*. We have a poset $(\mathcal{C}, \leq)$ formed by cells and inclusions of closures.

Theorem. $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ is equivalent to the category of diagrams formed by finite-dimensional vector spaces $E_C, C \in \mathbb{C}$, maps

$$E_C \underset{\delta_{C'C}}{\overset{\gamma_{CC"}}{\rightleftarrows}} E_{C'}, \qquad C' \le C,$$

satisfying an explicit set of relations, of which we mention:

$$\gamma_{CC'}\delta_{C'C} = \mathrm{Id},$$

as well as transitivity for $C'' \le C \le C'$ and certain invertibility conditions. In particular, for $n = 1$ and the arrangement consisting of the single hyperplane $\{0\} \subset \mathbb{R}$ we get the "Dirac" description above for perverse sheaves on $\mathbb{C}$ (which is topologically the same as D) smooth outside of 0.

3. Idea of perverse Schobers. We want to "upgrade" structures involving $\mathbf{k}$-vector spaces to those involving triangulated categories $\mathcal{V}$, so that if we pass to the Grothendieck groups $K_0(\mathcal{V}) \otimes_{\mathbb{Z}} \mathbf{k}$, we will get the previous theory. In this upgrading, the difference $a - b$ of vectors has to be replaced by the mapping cone $\mathrm{Cone}\{f : A \to B\}$ of a morphism in a triangulated category. In order to be able to speak about canonical cones, we need to work in the pre-triangulated context [10].

Various quiver descriptions of perverse sheaves admit transparent categorifications.

Example 4 (Sperical functors). This concept was introduced in [1, 2]. Suppose we have a diagram of pre-triangulated categories consisting of one functor S and its right adjoint

$$\mathcal{D}_0 \underset{S^*}{\overset{S}{\rightleftarrows}} \mathcal{D}_1 \ .$$

S is called spherical, if the functors

$$T_1 = \mathrm{Cone}\{S \circ S^* \xrightarrow{c} \mathrm{Id}_{\mathcal{D}_1}\}, \quad T_0 = \mathrm{Cone}\{\mathrm{Id}_{\mathcal{D}_0} \xrightarrow{e} S^* \circ S\}$$

are equivalences of categories. So this is completely analogous to Example 1. We can then say that a perverse Schober on $(D, 0)$ is a the same as a spherical functor.

An example of a spherical functor is given by a *spherical object* of dimension d, which is an object $E \in \mathcal{D} = \mathcal{D}_1$ with $\mathrm{Ext}^i(E, E) = \mathbf{k}$ for $i = 0, d$

and 0 for other i (plus some additional mild condition). It gives a spherical functor

$$\mathcal{D}_0 = D^b \operatorname{Vect}_{\mathbf{k}} \longrightarrow \mathcal{D}, \quad V \mapsto V \otimes_{\mathbf{k}} E.$$

A particular example of a spherical object is provided by a (-1)-*curve* on a smooth algebraic 3-fold X, i.e., by a rational curve $C \simeq \mathbb{P}^1$ with normal bundle $N_{C/X} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. In this case the sheaf $\mathcal{O}_C$ is a spherical object in $D^b(X)$, the derived category of coherent sheaves on X.

Example 5 (Spherical pairs: analog of Dirac description of $\operatorname{Perv}(D, 0)$**).** A spherical pair [KS2] is a diagram of triangulated categories

$$\mathcal{E}_- \underset{\delta_-}{\overset{\gamma_-}{\rightleftarrows}} \mathcal{E}_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} \mathcal{E}_+$$

such that $\gamma_\pm$ is the left adjoint of $\delta_\pm$ (and $\delta_\pm$ has also left adjoints) which satisfies the analogs of the conditions of Example 2:

$$\gamma_\pm \delta_\pm = \operatorname{Id} \text{ (this implies that } \mathcal{E}_\pm \text{ are admissible subcategories in } \mathcal{E}_0)$$
$$\gamma_\mp \delta_\pm : \mathcal{E}_\pm \to \mathcal{E}_\mp \text{ is an equivalence,}$$

and similar invertibility conditions imposed for the $\mathcal{E}_\pm^\perp$, the right orthogonals to $\mathcal{E}_\pm$ in $\mathcal{E}_0$, see [KS2]. A spherical pair gives rise to a spherical functor $S : \mathcal{E}_- \to \mathcal{E}_+^\perp$ (projection along $\mathcal{E}_+$).

4. Schobers and flops. The classical (Atiyah) flop is the diagram

$$X_- \xrightarrow{f_-} Z \xleftarrow{f_+} X_+$$

formed by two desingularizations of a 3-fold Z with one quadratic singular point z. In this case each $f_\pm^{-1}(z)$ is a (-1)-curve on $X_\pm$. Forming $X_0 = X_- \times_Z X_+ = \operatorname{Bl}_z(Z)$, we have a diagram

$$X_- \xleftarrow{p_-} X_0 \xrightarrow{p_+} X_+.$$

It was proved by Bondal and Orlov [11] that $D^b(X_+)$ and $D^b(X_-)$ are equivalent. More precisely, each of the functors

$$T_{+,-} = Rp_{-*} \circ p_+^* : D^b(X_+) \longrightarrow D^b(X_-),$$
$$T_{-,+} = Rp_{+*} \circ p_-^* : D^b(X_-) \longrightarrow D^b(X_+),$$

is an equivalence.

Observation. This means that the $\mathcal{E}_\pm = D^b(X_\pm) \subset \mathcal{E}_0 = D^b(X_0)$ (embedded via pullback) form a spherical pair.

This was recently generalized by Bodzenta-Bondal [9] who constructed spherical pairs associated to a wide class of flops. This suggests that various derived equivalences that appear in birational geometry should be included into perverse Schobers on some stratified algebraic varieties.

5. Web of flops. More precisely, the Atiyah flop is but a simplest 3-dimensional flop. A more complicated example can be obtained by taking a reducible curve $C = \bigcup C_i$ in a 3-fold X such that we can make a flop along some component C_i, getting a new 3-fold X_1, then flop X_1 along the strict preimage of some other component C_j, getting X_2 and so on. We get in this way a "web of flops", a system of 3-folds X_ν and flops connecting them. This system of flops can have loops: we may be able to obtain the same X_ν by two or more different sequences of flops.

According to Kawamata [15], the structure of iterated flops is governed by the *chamber structure on the movable cone* M_X. By definition, $M_X \subset \mathrm{Pic}(X) \otimes \mathbb{R}$ is the cone generated by line bundles $\mathcal{L}$ on X such that the image $X_\mathcal{L}$ of X under the rational map given by linear system $|\mathcal{L}|$ has the same dimension as X. The open part of this cone is subdivided into *chambers*; inside each chamber the variety $X_\mathcal{L}$ is the same, and when we cross walls (cells of codimension 1) between neighboring chambers, the $X_\mathcal{L}$ undergoes a flop. Thus cells of codimension ≥ 2 can be considered as relations, syzygies etc. among the flops.

Note that all the derived categories $D^b(X_\nu)$ are equivalent because each individual flop leads to a derived equivalence. The *Homological Minimal Model Program* (HMMP) studies such derived equivalences and relations among them. We have a local system of categories on the graph whose vertices are the X_ν and edges are the flops. In fact, it was proved by Donovan-Wemyss [12] that this extends to a local system of categories on $\mathbb{C}^n - \mathcal{H}_\mathbb{C}$, the complement of a certain hyperplane arrangement on $\mathbb{C}^n$ with real equations (a sub-arrangement of a root arrangement of type ADE). Individual X_ν correspond thereby to chambers of the real arrangement.

The above examples suggest that one can extend HMMP to the following conjectural picture.

Schober HMMP. Local systems of triangulated categories appearing in

HMMP (these are typically defined on open strata of some stratified spaces) admit natural extensions to perverse Schobers on the entire spaces.

6. Grothendieck resolution: "the mother of all flops". Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, with the corresponding algebraic group G of adjoint type, and G/B be its flag variety. It can be seen as parametrizing all the Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$, so there is a tautological bundle $\pi : \underline{\mathfrak{b}} \to G/B$. The *Grothendieck resolution*, see, e.g. [7] [8] is the total space $\widetilde{\mathfrak{g}}$ of this bundle, i.e.,

$$\widetilde{\mathfrak{g}} \;=\; \big\{(x, \mathfrak{b}) \in \mathfrak{g} \times G/B \big| x \in \mathfrak{b}\big\}.$$

It comes with a natural projection to $\mathfrak{g}$ which is, over a generic point of $\mathfrak{g}$, an unramified Galois covering with Galois group W (the Weyl group of $\mathfrak{g}$). In fact, this projection factorizes through a birational map

$$f : \widetilde{\mathfrak{g}} \longrightarrow Z = Z_{\mathfrak{g}} \;=\; \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}.$$

Here $\mathfrak{h}$ is the Cartan subalgebra in $\mathfrak{g}$, and we use the identification $\mathfrak{g}/\operatorname{Ad}(G) = \mathfrak{h}/W$. The variety Z is singular while $\widetilde{\mathfrak{g}}$ is smooth.

Example 6. Let $\mathfrak{g} = \mathfrak{sl}_2$, then $G/B = \mathbb{P}^1$, and $\underline{\mathfrak{b}} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The variety Z is the 3-dimensional quadratic cone (the 2-sheeted covering of $\mathfrak{g} \simeq \mathbb{C}^3$ ramified along the quadratic cone in $\mathbb{C}^3$). So $\mathfrak{g}$ is the flopping contraction for the Atiyah flop (local model).

Remark 7. The contraction f for general $\mathfrak{g}$ can be seen as the group-theoretical source of 3-dimensional flops for the following reasons. First, according to M. Reid's "elephant" picture, the contraction of a flopping curve on a 3-fold can be seen as a 1-parameter deformation of a Kleinian (ADE) singularity. So it can obtained by base change from the universal deformation of the ADE as constructed by Brieskorn.

Further, Brieskorn has interpreted each ADE singularity as the generic 2-dimensional transversal slice to the nilpotent cone of the corresponding Lie algebra $\mathfrak{g}$. Now, the Grothendieck resolution can be seen as the universal deformation of the nilpotent cone itself. So it is the fundamental object underlying 3d flops.

7. The $\mathfrak{g}$-web of flops. The Grothendieck resolution gives rise to a very explicit "$\mathfrak{g}$-web of flops" constructed as follows. Note that the variety Z is acted upon by W, so we define the w-flopped contraction as the base change

$$f_w : X_w := w^* \widetilde{\mathfrak{g}} \longrightarrow Z, \quad w \in W.$$

Although all the X_w are isomorphic as algebraic varieties, they are connected by nontrivial birational isomorphisms coming from the birational identifications of all of them with Z. This is similar to defining a 3d flop by base change along an involution.

The picture with the movable cone becomes particularly transparent in this "local" case. Denote $X = \widetilde{\mathfrak{g}}$. We have $\mathrm{Pic}(X) = \mathrm{Pic}(G/B) = \Lambda$, the lattice of weights, the isomorphism given by pullback along π. For $\lambda \in \Lambda$ we denote by $\mathcal{L}(\lambda) = \pi^*\mathcal{O}(\lambda)$ the corresponding line bundle. Then all $\mathcal{L}(\lambda)$ are movable. If λ lies (strictly) inside the dominant Weyl chamber C_+, then $X_{\mathcal{L}(\lambda)} = X$, since $\mathcal{L}(\lambda)$ is ample. If λ lies inside the chamber $C_w = w(C_+)$, then $X_{\mathcal{L}(\lambda)} = X_w$. When λ and λ' lie in adjacent Weyl chambers (separated by a wall of codimension 1), then $X_{\mathcal{L}(\lambda)}$ and $X_{\mathcal{L}(\lambda')}$ differ by a birational transformation essentially reduced to the Atiyah flop (case $\mathfrak{g} = \mathfrak{sl}_2$).

Further, choosing λ on various cells of codimension ≥ 1 (of the arrangement of coroot hyperplanes in $\mathfrak{h}^*_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$) gives partial contractions of X. In particular, the most degenerate case $\lambda = 0$ corresponds to $X_{\mathcal{L}(\lambda)} = Z$ being the affinization of X.

The results of [12] for 3d flops are analogous to the results of [8] who constructed the action of the braid group (associated to $\mathfrak{g}$) on the derived category $D^b(\widetilde{\mathfrak{g}})$. This means that we have a local system of categories on $\mathfrak{h}^*_{\mathbb{C}} - \mathcal{H}_{\mathbb{C}}$, where $\mathcal{H}$ is the arrangement of the co-root hyperplanes.

8. Fiber products and configuration spaces. Thus it is natural to expect that in this situation we have not just a local system but a perverse Schober on the entire $\mathfrak{h}^*_{\mathbb{C}}$, given by a type of categorical data analogous to those in Example 3. Comparison with the basic example in §4 (which can be seen as corresponding to $\mathfrak{sl}_2$) shows that we need not just the partial contractions $X_{\mathcal{L}(\lambda)}$ but rather their fiber products. In particular, we need to work with X_0, the fiber product of all the X_w over Z. Such fiber products can be reducible and singular, so care is needed in dealing with them. We call the *main component* of X_0 the closure of the image of the generic part of Z under the diagonal embedding.

Example 8 (Variety of simplices). Suppose $\mathfrak{g} = \mathfrak{sl}_n$. The part of $X = \widetilde{\mathfrak{g}}$ lying over 0 is the flag variety $F = F(1, 2, \cdots, n; \mathbb{C}^n) = SL_n/B$. So the preimage of $0 \in Z$ in X_0 will lie in the product of $n!$ copies of F. Embedding each copy of F into the product of Grassmannians, we identity the preimage of 0 in (he main component of X_0 with the main component of the following incidence variety, sometimes called the *variety of simplices* [3, 4]:

$$M \subset \prod_{I \subset \{1,2,\cdots,n\}} \mathrm{Gr}(|I|, \mathbb{C}^n), \quad M = \big\{(V_I)_{I\subset\{1,2,\cdots,n\}} : \ V_I \subset V_J, \ I \subset J\big\}.$$

For $n = 2$ we get $M = \mathbb{P}^1 \times \mathbb{P}^1$ which is the preimage of the singular point $z \in Z$ in its blowup, i.e., in the fiber product of the two desingularizations related by an Atiyah flop.

For $n = 3$ we get the *variety of triangles*

$$M = \big\{p_1, p_2, p_3, l_{12}, l_{13}, l_{23}) \in (\mathbb{P}^2)^3 \times (\check{\mathbb{P}}^2)^3 \big| \ p_i \subset l_{ij}\big\}.$$

This variety is reducible but singular and has classical desingularizations: the Semple space of complete triangles and the Fulton-MacPherson space $\mathbb{P}^2[3]$, see [14]. In the case $n = 4$ the variety M is reducible, and the desingularization of its main component was constructed in [3, 4]. Using these explicit descriptions, we can get partial results about the diagram of triangulated categories formed by the derived categories of the fiber products.

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