NO SMOOTH JULIA SETS FOR POLYNOMIAL DIFFEOMORPHISMS OF $\mathbb{C}^2$ WITH POSITIVE ENTROPY

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1. INTRODUCTION

Let $f : X \to X$ be a holomorphic mapping of a complex manifold X. To understand the mapping f as a dynamical system, it is essential to understand invariant sets. Two important invariant sets are Fatou sets and Julia sets. The Fatou set of f is the set where the sequence f^n are locally equicontinuous, in other word the Fatou set is the set where the dynamics of f is regular. In case X is not compact, we may consider a one point compactification of X so that a uniform divergence sequence can be considered as equicontinuous. By definition, the Fatou set is an open set. We define the Julia set as the complement of the Fatou set and thus the chaotic dynamics of f occurs on the Julia set.

Having a smooth Julia set is very special and rare. For instance, if f is a polynomial map of $\mathbb{C}$ with the smooth Julia set, f has to be equivalent to either $g(x) = x^d$ where d is an integer with $|d| \geq 2$ or a Chebyshev polynomial. Here we consider the smoothness of Julia sets of polynomial automorphisms of $\mathbb{C}^2$ and conclude that there is no polynomial automorphism of $\mathbb{C}^2$ whose Julia set is a C^1-smooth manifold. Since a polynomial automorphism f has a polynomial inverse, ∞ is an attracting fixed point for both f and f^{-1}. We can define Julia sets $J^{\pm}$ for f and f^{-1} by $J^{\pm} = \partial K^{\pm}$ where $K^{\pm} = \{x : f^{\pm n}(x) \text{ is bounded as } n \to \infty\}$. Fiedland and Milnor [7] and Smillie [10] showed the topological entropy of a polynomial automorphism f is equal to $\log(\text{degree } f)$.

Theorem 1 (Friedland and Milnor [7])**.** *Suppose f is a polynomial automorphism of $\mathbb{C}^2$. If the degree of $f > 1$ then f is conjugate to a composition of generalized Hénon maps. If degree of $f = 1$, then f is conjugate to either an affine map or an elementary map.*

The image of a line $\{ y = \text{ const } \}$ under an elementary map is another line $\{ y = \text{ const}' \}$. In fact elementary maps and affine maps have simple dynamics. (See [7].) We focus on the composition of generalized Hénon maps. A generalized Hénon map $h : \mathbb{C}^2 \to \mathbb{C}^2$ is given by

$$h(x,y) = (y, p(y) - \delta x)$$

where $p(y)$ is a monic polynomial of degree $d \geq 2$ and $\delta \in \mathbb{C} \setminus \{0\}$. The complex Jacobian of h is δ and the degree of h, the largest degree of coordinate polynomials, is d. Suppose

$$f = h_k \circ \cdots \circ h_1 \tag{1}$$

a composition of generalized Hénon maps $h_i(x,y) = (y, p_i(y) - \delta_i x)$. Then the degree of f, d is given by the product of degree of h_i's, $d := d_k \cdots d_1$, the Jacobian is $\delta = \delta_k \cdots \delta_1$, and the entropy is given by $\log d > 0$. Dynamics of these mappings are nontrivial and have been considered by many authors. For instance, it is shown that for any saddle point q, the stable manifold $W^s(q)$ is a Riemann surface dense in J^+. Also since J^+ has no interior and is can not be a Riemann surface, if J^+ is a manifold, then the real dimension of J^+ has to be 3. (See for example, [4], [3], [2], [9], [8], [6] and [7])

In this article we will discuss the following theorem

Main Theorem (Bedford and Kim [1]). *For any polynomial automorphism of $\mathbb{C}^2$ of positive entropy, neither J^+ nor J^- is smooth of class C^1, in the sense of manifold-with-boundary.*

2. Polynomial Dynamics of $\mathbb{C}$

Let us consider a polynomial automorphisms f of $\mathbb{C}$ with $\deg(f) > 1$. Since the ∞ is an attracting fixed point, the Julia set $J = \partial K$ where $K = \{z : f^n(z)$ is bounded as $n \to \infty\}$. It is known that there is a Green function $G(z) := \lim \frac{1}{d^n} \log(|f^n(z)| + 1)$ such that $G \circ f = d \cdot G$, G is harmonic on $\mathbb{C} - K$, and $G \equiv 0$ on K.

If the Julia set J is smooth, then every point $z \in J$, then the Green function is piecewise smooth and thus there exists a normal derivative toward outside K. This follows that $|f'(z)| = d$ and $|(f^n)'(z)| = d^n$ for all periodic points z of period n in J. Since the degree of $f > 1$, the number of periodic points on J grows like d^n. It follows that having a smooth Julia set requires infinitely many conditions for a polynomial automorphism of degree d which only have finitely many parameters. In other words, it is very special to have a smooth Julia set.

Fatou showed that there are only two possibilities for smooth Julia sets.

Case I. $f(z) = z^d, |d| \geq 2$: In this case the Julia set is a unit circle $J = \{z : |z| = 1\}$. There is the unique fixed point $z = 1$ on J with $f'(z) = d$. If z is a periodic points of period n, then $z = e^{2\pi i p/q}$ where p and q are relatively prime and $q = d^n - 1$ and therefore $(f^n)'(z) = d^n$.

Case II. f is equivalent to a Chebyshev polynomial: To simplicity, let us consider one case $f(z) = z^2 - 2$. The Julia set is given by an interval $[-2, 2]$. There are two fixed points on J, $z = -1$ and $z = 2$. Their derivatives are given by $f'(-1) = -2$ and $f'(2) = 4$ respectively. Notice that 2 is on the end point of the Julia set. For all other periodic points of period $n \geq 2$ inside J, one can check that their derivative are given by 2^n. In this case, the critical point $z = 0$ is on the Julia set. But $z = 0$ is a pre-periodic : $f : 0 \mapsto -2 \mapsto 2 \mapsto 2 \mapsto \cdots$

It is natural to ask whether there are special cases with smooth Julia sets for polynomial automorphisms in the higher dimension.

3. Dynamics of Complex Hénon maps

The proof of main theorem consists of two parts. In the first part, we used known theories about generalized Hénon maps to show that

Proposition 2. *Suppose f is a composition of generalized Hénon maps with $\deg(f) = d$. If J^+ is a C^1 smooth manifold, then f satisfies the followings:*

(i) *f is a volume decreasing map, i.e. the constant Jacobian δ satisfies $|\delta| < 1$.*
(ii) *f has d distinct fixed points and the differential Df has multiplier d at least $d-1$ fixed points.*

Friedland and Milnor [7] showed that if $\delta| \geq 1$ then K^+ has no interior outside the bidisk $\Delta_R = \{(x,y) : |x| \leq R, |y| \leq R\}$ and f has exactly d isolated fixed points counted with multiplicity. If J^+ is C^1 then we can conclude that J^+ is orientable 3-manifold. However if $|\delta| \geq 1$, then a small neighborhood of a point in $J^+ \cap \Delta_R^c$ only contains J^+ and an attracting basin of ∞.

If $|\delta| < 1$, assuming that J^+ is C^1 smooth we see the every fixed point in J^+ has to be saddle and has d as its multiplier. To conclude that there are exactly $d-1$ such saddle point in J^+ we used the following theorem:

Theorem 3 (Bedford and Smillie[4]). *The boundary of any basin of attraction is J^+. Thus if f has more than one basin components, then J^+ is not an embedded topological manifold at any point.*

For the detailed proof of Proposition 2, let us refer to the original paper [1]. Here we will focus the second part which we showed that there is no composition of generalized Hénon maps satisfies the second condition above Proposition.

4. Non-smoothness of J^+ for a generalized Hénon map

In this section, we will explain the basic idea using one generalized Hénon map, $h(x,y) = (y, p(y) - \delta x)$ with $p(y)$ is a monic polynomial of degree $d \geq 2$ and $\delta \in \mathbb{C} - \{0\}$. Let us set

$$p(y) = y^d + q(y), \quad \text{and} \quad q(y) \; = \; \sum_{i=0}^{d-1} c_i y^i.$$

To get a contradiction, let us assume that the Julia set of h is C^1 smooth manifold.

4.1. Conditions from Fixed Points.

From the Proposition 2, it follows that h has d distinct fixed points and at least $d-1$ fixed points has multiplier d. Fixed points of h are given by $\{(x,y) | x = y \ , \ p(y) - \delta y = y\}$. Let us set

$$\phi(y) := p(y) - (\delta + 1)y.$$

The complex differential at the fixed point (y,y) is $Dh(y,y) = \begin{pmatrix} 0 & 1 \\ -\delta & p'(y) \end{pmatrix}$. If d is a multiplier, $\det|Dh(y,y) - dI_2|(:= (p'(y) - d)(-d) + \delta) = 0$. Let us also set

$$\Phi(y) := p'(y) - (d^2 + \delta)/d.$$

That is, if (y_*, y_*) is a fixed point for h whose multiplier $= d$, then $\phi(y_*) = 0$ and $\Phi(y_*) = 0$.

If all fixed points for h have the same multiplier d, then Φ vanishes at every root of $\phi(y) = 0$. However the zero locus of ϕ, $Z(\phi)$ consists of d distinct numbers, ϕ can not divide Φ due to the reason of the degree. Now let us suppose h has one fixed point whose multiplier is not equal to d. It follows that there is $a \in \mathbb{C}$ such that $(y-a)\Phi(y)$ vanishes at every fixed point. Since $p'(y) = dy^{d-1} + q'(y)$, we have

$$\begin{aligned} (y-a)\Phi(y) &= (y-a)p'(y) - (y-a)(d + \delta/d) \\ &= dy^d + yq'(y) - ady^{d-1} - aq'(y) - (y-a)(d+\delta/d) \qquad (2) \\ &= d\phi(y) + R \end{aligned}$$

where the remainder is a polynomial of degree $\leq d-1$.

$$R = -dq(y) + d(\delta+1)y + yq'(y) - ady^{d-1} - aq'(y) - (y-a)(d+\delta/d)$$

To make $(y-a)\Phi(y)$ divisible by ϕ, that is $R \equiv 0$, we must have

$$\begin{aligned} c_i &= (-a)^{d-i} \cdot \binom{d}{d-i} \qquad \forall\, i = 2,\dots,d-1, \\ c_1 &= (-a)^{d-1} \cdot d + \delta(d+1)/d, \\ c_0 &= (-a)^d + a(1-1/d). \end{aligned} \tag{3}$$

Thus we have $p(y) = (y-a)^d + \delta y + \delta/d\, y + a(1-1/d)$ and thus

$$\phi(y) = (y-a)^d + (\delta/d - 1)y + a(1-1/d).$$

Since we assume (a,a) is a fixed point for h, $\phi(a) = (\delta-1)a/d$ must be equal to zero. Using the fact that $|\delta| < 1$ and $d \geq 2$, we see that a has to be zero and therefore $p(y) = y^d + \delta(1+1/d)y$.

4.2. **Conditions from Period** 2 **Points.** Taking n-th iteration would not change the Julia set. If the Julia set of h is C^1-smooth then all but one fixed points for h^2 must have the same multiplier d^2 while all but one fixed points for h must have the same multiplier d. From the previous subsection, we have seen that if all but one fixed points for h must have the same multiplier d then the only possibility is $p(y) = y^d + \delta(1+1/d)y$. Now suppose (y_0,y_1) is a point of period 2:

$$h : (y_0,y_1) \mapsto (y_1,y_2) \mapsto (y_0,y_1)$$

It follows that $y_2 = y_0$ and two polynomial conditions on y_0 and y_1.

$$\begin{aligned} \phi_0 &:= y_0^d + \delta(1+1/d)y_0 - (\delta+1)y_1 = 0 \\ \phi_1 &:= y_1^d + \delta(1+1/d)y_1 - (\delta+1)y_0 = 0. \end{aligned} \tag{4}$$

Using the theorem 1 by Friedland and Milnor, we see that there are exactly d^2 fixed points for h^2 and d of them are fixed points for h. The one extra fixed points with different multiplier must be the fixed point $(0,0)$. Using the chain rule, we see that

$$Dh^2(y_0,y_1) = Dh(y_1,y_0) \cdot Dh(y_0,y_1) = \begin{bmatrix} 0 & 1 \\ -\delta & p'(y_0) \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -\delta & p'(y_1) \end{bmatrix}$$

To have a multiplier d^2, (y_0,y_1) must satisfies the polynomial equation $\Phi = 0$ where

$$\Phi = (-1/d^2) \cdot \mathrm{Det}|Dh^2(y_0,y_1) - d^2 I_2| = p'(y_1)p'(y_0) - (1+\delta/d)^2.$$

Since the origin is the extra fixed point, to have a C^1 smooth Julia set the polynomial equation $y_1\Phi$ must vanish at all d^2 roots of the system of equations (4). However

$$\begin{aligned} y_1\Phi =& d^2 y_2^{d-1}\phi_1 + d^2(\delta+1)\phi_0 \\ &- \delta(d^2-1)y_0^{d-1}y_1 - d\delta(\delta+1)(d+1)y_0 + ((\delta d+1)^2 - (\delta/d+1)^2)y_1 \end{aligned}$$

Since all fixed points except the origin are saddle points, those d^2 roots of equations (4) are all distinct. However there are at most d of them that belongs to the zero set of $y_1\Phi$ and thus all not fixed periodic 2 cycles doesn't have a multiplier d^2. It follows that

Lemma 4. *If f is a generalized Hénon map, then the Julia set is not C^1 smooth.*

5. n-Fold Composition of Generalized Hénon Maps

Let us suppose f is a composition of n generalized Hénon maps with $n \geq 3$ and its Julia set is C^1 smooth. As we have seen in the previous section, we can always consider the iteration to increase the number of Hénon maps in the composition. When n is too small, it is necessary to consider periodic points. It turns out that a composition of 3 maps is already sufficient to get the desired contradiction.

$$f = h_n \circ \cdots \circ h_1 \quad \text{and} \quad h_i(x,y) = (y, p_i(y) - \delta_i x)$$

where p_i is a degree d_i monic polynomial, $d = d_n \cdots d_1$, and δ_i is a non-zero complex number satisfying $|\delta| = |\delta_n \cdots \delta_1| < 1$. Let us set $p_i(y) = y_i^{d_i} + \zeta_i(y_i)$ then $\zeta_i(y_i)$ is a polynomial of degree $\geq d_i - 1$.

5.1. An Ideal for Fixed Points.

If $q = (x,y)$ is a fixed point for f, there exist n points in $\mathbb{C}^2$ such that $(x_1,y_1) = (x,y)$, $(x_{i+1},y_{i+1}) = h_i(x_i,y_i)$ for $i = 1,\ldots,n$ and $(x_{n+1},y_{n+1}) = (x_1,y_1)$. Since $h_i(x,y) = (y, p_i(y) - \delta_i x)$, we see that $x_{i+1} = y_i$ for all i. Furthermore $y_{i+1} = p_i(y_i) - \delta_i y_{i-1}$ for all i where the subscription is defined cyclically in mod n. Let us define n polynomials in $\mathbb{C}[y_1,\ldots,y_n]$

$$\begin{aligned}
\phi_1 &:= p_1(y_1) - \delta_1 y_n - y_2 \\
\phi_i &:= p_i(y_i) - \delta_i y_{i-1} - y_{i+1}, \quad i = 2,\ldots,n-1 \\
\phi_n &:= p_n(y_n) - \delta_n y_{n-1} - y_1
\end{aligned} \tag{5}$$

Each fixed point $q = (y_n, y_1)$ can be identified with a point $\tilde{q} = (y_1, y_2, \ldots, y_n)$ in the zero locus of $Z(\phi_1,\ldots,\phi_n)$. From the Proposition 2, we see that $Z(\phi_1,\ldots,\phi_n)$ consists of d distinct points. It follows that every polynomial which vanishes at every fixed point belongs to the ideal $I_\phi = \langle \phi_1, \phi_2, \ldots, \phi_n \rangle$. To check this, we construct the Gröbner basis for I_ϕ. For a zero dimensional ideal, it is relatively easy to find its Gröbner basis. (See for example [5].) In our case, the formula of a generalized Hénon map makes it possible to compute the Gröbner basis of the ideal generated by $\phi_1,\ldots,\phi_n$. Notice the fact that each ϕ_i only depends on three variables and its non linear terms only depends on one variable y_i. Using this, we first get the Gröbner basis and perform multivariate division algorithm to show that Φ can not be in the ideal I_ϕ.

Let us fix the graded lexicographical order in monomials in $\{y_1,\ldots y_n\}$. Notice that in ϕ_i all non-linear terms belong to $p_i(y_i)$. The least common multiple of the leading terms of ϕ_i, ϕ_j is $y_i^{d_i} y_j^{d_j}$. Now because $d_i \geq 2$, $q_i := \phi_i - y_i^{d_i}$ is a polynomial whose leading term is strictly smaller than the leading term of ϕ_i. Thus we have

$$y_j^{d_j}\phi_i - y_i^{d_i}\phi_j = (\phi_j - q_j)\phi_i - (\phi_i - q_i)\phi_j = -q_j\phi_i + q_i\phi_j.$$

Using Buchberger's Algorithm, we see that $\{\phi_1,\ldots,\phi_n\}$ is in fact a Gröbner basis of the ideal I_ϕ.

Lemma 5. *With the graded lexicographical order, $\{\phi_1,\ldots,\phi_n\}$ is a Gröbner basis of I_ϕ.*

5.2. **Multiplier Condition.** The complex differential of f at the fixed point $q \leftrightarrow \tilde{q} = (y_1,\ldots,y_n)$ is given by

$$Df(q) = Dh_n(y_{n-1},y_n)\cdot Dh_{n-1}(y_{n-2},y_{n-1})\cdots Dh_1(y_n,y_1)$$

where $Dh_i(x,y) = \begin{pmatrix} 0 & 1 \\ -\delta_i & p_i'(y) \end{pmatrix}$ for all i. If a fixed point q has d as its multiplier, then $\Phi := \det|Df(q) - dI_2| = 0$. Direct computation shows that Φ is a linear combination of products of $p_i'(y_i)$

$$\Phi = \sum_{J\subset\{1,\ldots,n\}} a_J \left(\prod_{i\in J} p_i'(y_i)\right). \tag{6}$$

Using the fact that Φ is the determinant of 2×2 matrix and the induction on n, one can also show that the index set J satisfies $|J| = n-2k$ for some $k = 0,1,\ldots\lfloor n/2\rfloor$. The leading monomial of Φ is given by the product of leading monomials of $p_i'(y_i), i = 1,\ldots,n$.

$$\text{The Leading Monomial of } \Phi,\ \ LM(\Phi) = y_1^{d_1-1}y_2^{d_2-1}\cdots y_n^{d_n-1}.$$

With the graded lexicographical order, $LM(\Phi)$ is not divisible by ϕ_i for each $i = 1,\ldots,n$ since the leading monomial of ϕ_i, $LM(\phi_i) = y_i^{d_i}$. It follows that $\Phi \notin I_\phi$ and thus f must have one fixed point with different multiplier.

5.3. **Division Algorithm.** To simplicity let us assume that one extra fixed point is $(0,0)$. Otherwise we can always use affine conjugation to move the extra fixed point to the origin. The leading monomial of $y_1\Phi$ is $y_1^{d_1}y_2^{d_2-1}\cdots y_n^{d_n-1}$ which is divisible by the leading monomial of ϕ_1. In fact, we have

$$y_1\Phi = y_1^{d_1}\cdot Q(y_2,\ldots,y_n) + R_1(y_1,\ldots,y_n)$$

where R_1 is a polynomial in $y_1,\ldots,y_n$ and none of $LM(\phi_i)$ divides $LM(R)$ for $i = 1,\ldots,n$ and Q is a polynomial in $y_2,\ldots,y_n$ with $LM(Q) = y_2^{d_2-1}\cdots y_n^{d_n-1}$. Since Φ is written in a special form in (6), we see that Q also has the same form in (6) with variables $y_2,\ldots,y_n$. Recall that a monic polynomial $p_i(y_i) = y_i^{d_i} + \zeta_i(y_i)$ and $y_1^{d_1} = \phi_1 - \zeta_1(y_1) + \delta_1 y_n + y_2$. Thus we have

$$\begin{aligned} y_1\Phi = &Q(y_2,\ldots,y_n)\phi_1 - \zeta_1(y_1)Q(y_2,\ldots,y_n) \\ &+ \delta_1 y_n\cdot Q(y_2,\ldots,y_n) + y_2\cdot Q(y_2,\ldots,y_n) + R_1(y_1,\ldots,y_n). \end{aligned}$$

Since both y_2Q and $\delta_1 y_n Q$ have the essentially same form as $y_1\Phi$ with the reduced number of variables, we can apply the multivariable division algorithm with ϕ_2 and ϕ_n respectively and get

$$\begin{aligned} y_1\Phi = &Q(y_2,\ldots,y_n)\phi_1 + \delta_1 Q_n(y_2,\ldots,y_{n-1})\phi_n + Q_2(y_3,\ldots,y_n)\phi_2 \\ &+ \delta_1\delta_n y_{n-1}Q_n(y_2,\ldots,y_{n-1}) + \delta_1 y_1 Q_n(y_2,\ldots,y_{n-1}) \\ &+ \delta_2 y_1 Q_2(y_3,\ldots,y_n) + y_3 Q_2(y_3,\ldots,y_n) \\ &- \zeta_1(y_1)Q(y_2,\ldots,y_n) - \delta_1\zeta_n(y_n)Q_n(y_2,\ldots,y_{n-1}) \\ &- \zeta_2(y_2)Q_2(y_3,\ldots,y_n) + R_1(y_1,\ldots,y_n). \end{aligned} \tag{7}$$

Since the cardinality of the index sets in the summation in (6) decreases by 2, each time we remove the leading terms, we only create the remainder terms with smaller number of variables. For instance the term $y_1^k y_2^{d_2-1} y_n^{d_n-1}$, $k \leq d_1 - 1$ only appears in either $\zeta_1(y_1)Q(y_2,\ldots,y_n)$ or $R_1(y_1,\ldots,y_n)$. Continuing this division algorithm we get

$$y_1\Phi = A_1\phi_1 + A_2\phi_2 + \cdots + A_n\Phi_n + R$$

where the leading monomial of the remainder R is not divisible by any of the leading monomial of ϕ_i for $i = 1,\ldots,n$. Since $\zeta_1(y_1)Q(y_2,\ldots,y_n)$ has terms of the form $y_1^k \prod_{i\in J} p_i(y_i)$ with $|J| \leq n-2$ and $k \leq d_1 - 1$, the monomial $y_1 y_n^{d_n-1} \prod_{i=3}^{n-1} y_i^{d_i-1}$ only appears in $\delta_2 y_1 Q_2(y_3,\ldots,y_n)$ in (7). Similarly the monomial $y_1 y_2^{d_2-1} \prod_{i=3}^{n-1} y_i^{d_i-1}$ is in only $\delta_1 y_1 Q_n(y_2,\ldots,y_{n-1})$. It follows that the remainder term R can be written as sum of two polynomials $R = R_* + \tilde{R}$ and

$$R_* = (d_1 d_2 \delta_2 y_1 y_n^{d_n-1} + d_1 d_n \delta_1 y_1 y_2^{d_2-1}) \prod_{i=3}^{n-1} y_i^{d_i-1}$$

and $\tilde{R}$ doesn't have monomials in R_*. Since $n \geq 3$, we conclude that R_* is not equal to zero and thus

$$y_1\Phi \notin \langle \phi_1, \phi_2, \ldots, \phi_n \rangle$$

Thus we have

Lemma 6. *If f is a composition of generalized Hénon maps, then the Julia set is not C^1 smooth.*

Since both f and f^{-1} are equivalent to composition of generalized Hénon maps, applying above Lemma for both f and f^{-1} we get the main theorem. In fact the argument in this section works for any multiplier.

Corollary 7. *Suppose f is a n-fold composition of the generalized Hénon maps with $n \geq 3$ and suppose $\delta \neq 0$. Let d be the degree of f. Then for all $\lambda \in \mathbb{C}$, the number of fixed points with the multiplier λ is at most $d-2$.*

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