On the singularities of varieties admitting an endomorphism

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1 Introduction

We will consider normal projective varieties over $\mathbb{C}$.

During the last thirty years, there has been a lot of interest in the study of varieties admitting an endomorphism :

1.1 Definition. *An endomorphism is a finite surjective morphism $f\colon X \to X$ of degree* $\deg(f) > 1$.

The existence of an endomorphism on a variety X impose strong conditions on the global structure of X. For instance, the variety X cannot be of general type. Under the additional condition that the endomorphism f is polarized (see definition 3.2), then the Kodaira dimension of X is at most 0. For an overview of the classification of varieties admitting an endomorphism, see [FN08].

The following problem concerning the endomorphisms of $\mathbf{P}^n$ comes from complex dynamics. It has originally been studied by Fornaess and Sibony who solved the case $n = 2$ in [FS94].

1.2 Definition. *Let $f\colon X \to X$ be an endomorphism. A subset Z is said totally invariant if*

$$f^{-1}(Z) = Z$$

1.3 Conjecture. *Let $f\colon \mathbf{P}^n \to \mathbf{P}^n$ an endomorphism and $S = f^{-1}(S)$ a totally invariant irreducible hypersurface.*

Then S is linear.

The case of smooth hypersurfaces was solved by Cerveau and Lins Neto in [CLN00]. More generally, Paranjape and Srinivas [PS89] for the quadrics and Beauville [Bea01] for higher degree hypersurfaces proved the following theorem.

1.4 Theorem. *A smooth complex projective hypersurface S of dimension* $\dim(S) \geqslant 2$ *and degree $d \geqslant 2$ admits no endomorphism.*

1.5 Corollary. *Let $f\colon \mathbf{P}^n \to \mathbf{P}^n$ an endomorphism and $S = f^{-1}(S)$ a totally invariant smooth hypersurface.*

Then S is linear.

Thus the conjecture 1.3 is equivalent to the smoothness of hypersurfaces totally invariant by an endomorphism $f\colon \mathbf{P}^n \to \mathbf{P}^n$.

A natural question is to ask if the existence of an endomorphism on a variety X impose conditions on the singularities of X. We cannot expect X to be smooth, see for instance [BdFF12, sections 6.2, 6.3]. However in the article just cited, Boucksom, de Fernex and Favre have shown that in the case of isolated singularities, the singularities of X are controlled in term of the singularities appearing in the minimal model program.

2 Singularities of pairs

We recall briefly the usual terminology for the singularities of pairs. We refer to [KM98] for more details.

Let $\Delta = \sum_i d_i \Delta_i$ be a $\mathbb{Q}$-Weil divisor on X with $d_i \leq 1$ for all i.

We say that the pair (X, Δ) is lc (resp. klt) if

- $K_X + \Delta$ is $\mathbb{Q}$-Cartier and
- for every proper birational morphism $\mu\colon X' \to X$ from a normal variety X' we can write
$$K_{X'} + \mu_*^{-1}(\Delta) = \mu^*(K_X + \Delta) + \sum_j a(E_j, X, \Delta) E_j,$$
where the divisor E_j are μ-exceptional and $a(E_j, X, \Delta) \geq -1$ (resp. $a(E_j, X, \Delta) > -1$) for all j.

The numbers $a(E_j, X, \Delta)$ are called the discrepancies of the divisors E_j.

For a pair (X, Δ), the non-lc locus $\mathrm{Nlc}(X, \Delta)$ is the smallest closed set $W \subset X$ such that $(X \setminus W, \Delta|_{X \setminus W})$ is lc.

If (X, Δ) is lc, we say that a subvariety $Z \subset X$ is an lc centre if there exists a proper birational morphism $\mu\colon X' \to X$ and a μ-exceptional divisor E such that $E \twoheadrightarrow Z$ and $a(E, X, \Delta) = -1$.

3 Singularities of varieties admitting an endomorphism

Unfortunately, for non-isolated singularities the direct generalization of the results of [BdFF12] is not true.

3.1 Example. Let Y be a variety being not log-canonical.

Let $h\colon \mathbf{P^1} \to \mathbf{P^1}$ the endomorphism defined by $f(x_0 : x_1) = (x_0^2 : x_1^2)$.

Let $X = \mathbf{P^1} \times Y$ and $f\colon X \to X$ defined by $f(x, y) = (h(x), y)$.

Then X is not log-canonical and f is an endomorphism of X.

To avoid this 'product' case, we can require the endomorphism to be polarized.

3.2 Definition. *An endomorphism $f\colon X \to X$ is said polarized,*

if there exists an ample divisor A and an integer n such that

$$f^*A \sim nA.$$

We thus obtain the following result on the singularities of varieties admitting a polarized endomorphism.

3.3 Theorem.*[[BH14], corollary 1.3] Let X be a normal projective variety such that K_X is $\mathbb{Q}$-Cartier, and let $f\colon X \to X$ be a polarised endomorphism.*

Then X has at most log-canonical singularities. Moreover X is klt near the ramification divisor R.

If the endomorphism is not polarized, we cannot avoid having non-log-canonical singularities, but the locus of this singularities is totally invariant and satisfy a condition on the degree of the restriction of the endomorphism to it.

3.4 Theorem.*[[BH14], theorem 1.2] Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier, and let $f\colon X \to X$ be an endomorphism.*

Let Z be an irreducible component of the non-lc locus $\mathrm{Nlc}(X)$. *Then (up to replacing f by some iterate) Z is totally invariant. In this case Z is not contained in the ramification divisor R, and the induced endomorphism $f|_Z\colon Z \to Z$ satisfies*

$$\deg(f|_Z) = \deg(f).$$

4 Lifting endomorphisms on a resolution

It is convenient to be able to lift the endomorphism on a resolution of the singularities of X. Although not always possible in general, in the étale in codimension 1 case we can lift the endomorphism to a special partial resolution : the log-canonical modification.

4.1 Definition. *Let (X, Δ) be a log-pair such that $\Delta \geq 0$. A log-canonical model over the pair (X, Δ) is a proper birational morphism*

$$\mu\colon Y \to X$$

such that if we set

$$\Delta_Y := \mu_*^{-1}(\Delta) + E_\mu^{lc},$$

where E_μ^{lc} is the sum of all the μ-exceptional prime divisors taken with coefficient one, the pair (Y, Δ_Y) is log-canonical and $K_Y + \Delta_Y$ is μ-ample.

The existence of the log-canonical models is a consequence of the full minimal model program, including abundance conjecture. In the log-$\mathbb{Q}$-Gorenstein case, the existence of log-canonical models has been proved by Odaka and Xu in [OX12].

4.2 Theorem.

- *If there exists a log-canonical model over a pair (X, Δ), it is unique up to isomorphism.*
- *Suppose now that $\Delta \geq 0$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then there exists a log-canonical model over (X, Δ). Moreover the μ-exceptional locus has pure codimension one. If we write*
$$K_Y + \Delta_Y = \mu^*(K_X + \Delta) + \Delta_Y^{>1},$$
then $\Delta_Y^{>1}$ is anti-effective and $\operatorname{supp} \Delta_Y^{>1} = \operatorname{Exc}(\mu)$.

So we have the following lemma which is the main tool in the étale in codimension 1 case.

4.3 Lemma.*[[BH14], lemma 2.11] Let $f\colon X_1 \to X_2$ be a finite morphism. Let Δ_1 and Δ_2 be reduced effective Weil divisors on X_1 and X_2 such that $\Delta_1 = \operatorname{supp} f^*\Delta_2$ and we have*

$$K_{X_1} + \Delta_1 = f^*(K_{X_2} + \Delta_2).$$

Suppose that there exists a log-canonical model $\mu_2\colon (Y_2, \Delta_{Y,2}) \to (X_2, \Delta_2)$ over the pair (X_2, Δ_2).

Then there exists a log-canonical model $\mu_1\colon (Y_1, \Delta_{Y,1}) \to (X_1, \Delta_1)$ over the pair (X_1, Δ_1), moreover f lifts to a finite morphism $g\colon Y_1 \to Y_2$ such that

$$K_{Y_1} + \Delta_{Y,1} = g^*(K_{Y_2} + \Delta_{Y,2})$$

and $\mu_2 \circ g = f \circ \mu_1$.

5 Proof of the main results

We only sketch the proof and refer to [BH14] for the full details.

Invariance. The following lemma shows the crucial invariance of the non-log-canonical locus.

5.1 Lemma. *Let $f\colon X \to X$ be an endomorphism. Assume that K_X is $\mathbb{Q}$-Cartier.*

Let $Z \subset X$ be an irreducible component of $\mathrm{Nlc}(X, \Delta)$.

Then (up to replacing f by some power) we have

$$f^{-1}(Z) = Z.$$

The ramified case. The following result shows that we can reduce the situation to the non-ramified case.

5.2 Proposition.*[[BH14], proposition 3.1] Let $f\colon X \to X$ be an endomorphism. Denote by R the ramification divisor. Assume that K_X is $\mathbb{Q}$-Cartier.*

Let Z be an irreducible component of $\mathrm{Nlc}(X, 0)$ *that is totally invariant.*

Then $Z \not\subset R$.

The étale case. We now consider an irreducible component Z of $\mathrm{Nlc}(X, 0)$.

Assume that :

- Z is totally invariant (up to replacing f by an iterate).
- The ramification divisor R is null, so that $K_X = f^*K_X$ (using the previous proposition and working locally around Z).

Taking a log-canonical model $\mu : (Y, \Delta_Y) \to (X, 0)$, we can lift $f\colon X \to X$ to an endomorphism $g : Y \to Y$.

We have

$$K_Y + \Delta_Y = g^*(K_Y + \Delta_Y)$$

Letting

$$K_Y + \Delta_Y = \mu^*K_X + \Delta_Y^{>1},$$

We have

$$\mu^*K_X + \Delta_Y^{>1} = g^*(\mu^*K_X + \Delta_Y^{>1})$$

That is

$$\Delta_Y^{>1} = g^*\Delta_Y^{>1}$$

Thus

$$g^*E_i = E_i$$

for every exceptional divisor.

g^{-1} acts by permutation on the μ-exceptional divisors E_i dominating Z. Up to taking an iterate we can assume that $g^{-1}(E_1) = E_1$.

We have
$$\deg(g|_{E_1}) = \deg(f|_Z).$$

Arguing by contradiction, we assume that $\deg(f|_Z) < \deg(f)$.

This means that
$$\deg(g|_{E_1}) < \deg(g).$$

Thus g ramify over E_1 :
$$g^*E_1 = rE_1$$
with $r > 1$. This is a contradiction.

The case of a polarized endomorphism. If $Z \subset X$ is a totally invariant subvariety, the endomorphism $f|_Z \colon Z \to Z$ is polarised by $H|_Z$ (for H such that $f^*H \sim mH$).

We have
$$\deg(f|_Z) = m^{\dim Z} < m^{\dim X} = \deg(f).$$

Thus Z is not a component of $Nlc(X, 0)$.

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