PROJECTIVE SPACES IN FERMAT VARIETIES

ALEX DEGTYAREV

ABSTRACT. We give a brief systematic overview of a few results concerning the Néron–Severi lattices of Fermat varieties and Delsarte surfaces.

CONTENTS

1. INTRODUCTION

The goal of this survey is to give a brief systematic overview of a few results, both recent and old, concerning the generators of the Néron–Severi lattices of Fermat varieties and of closely related to them Delsarte surfaces.

Citing T. Shioda [1], the Néron–Severi group "...is a rather delicate invariant of arithmetic nature. Perhaps for this reason it usually requires some nontrivial work before one can determine the Picard number of a given variety, let alone the

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full structure of its Néron–Severi group." The Picard ranks $\operatorname{rk} NS(X)$ of Fermat varieties and Delsarte surfaces were computed in [17, 16, 18]; these results are outlined in §2.1. Comparing $\operatorname{rk} NS(X)$ with the rank of the subgroup $\mathbf{S} \subset NS(X)$ generated by a certain set $\mathcal{S}$ of "immediately seen" subvarieties of X (projective spaces or images thereof, see §2.2 and §2.3), it was observed that, under some rather general assumptions, the subvarieties constituting $\mathcal{S}$ generate the rational Néron–Severi group: $\mathbf{S} \otimes \mathbb{Q} = NS(X; \mathbb{Q})$. Naturally, the question arose whether one also has $\mathbf{S} = NS(X)$ over the integers; the affirmative answer to this question would give one the complete structure of the Néron–Severi lattice.

The question remained unsettled for almost 30 years, until the first numerical evidence suggesting the positive answer appeared in 2010, see [14, 15]. The original case of Fermat surfaces was finally settled (in the affirmative) in [4]. The situation with Delsarte surfaces turned out more complicated: it was shown in [3] that the answer depends on the structure of the defining equation and typically is in the negative, although the torsion of the quotient $NS(X)/\mathbf{S}$ is bounded; *e.g.*, its length does not exceed 7. The techniques used in the proofs are outlined in §3.1 and §3.2, and a brief account of the results is found in §3.3.

The most recent achievement is an algebraic restatement (similar to that used in [4]) of the original question for Fermat varieties of higher dimension, see [6] and §4.1: the answer is given in terms of the integral torsion of certain modules over polynomial rings. Unfortunately, we failed to prove that this torsion vanishes. So far, only some numerical evidence and a few partial vanishing results are available, see §4.2. Some of these partial results have geometric implications to a wider class of varieties; they are discussed in §4.3.

In §5, I briefly state a few open problems that seem to be of general interest.

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2. The rational Néron–Severi lattice

2.1. **Fermat varieties.** Consider the Fermat variety $\Phi^d_m \subset \mathbb{P}^{d+1}$ given by

$$z_0^m + \ldots + z_{d+1}^m = 0,$$

and let G_m be the group

$$G_m := \bigl\{(\epsilon_0, \ldots, \epsilon_{d+1}) \in (\mathbb{C}^\times)^{d+2} \bigm| \epsilon_0^m = \ldots = \epsilon_{d+1}^m = 1\bigr\}/\text{diagonal}. \tag{2.1}$$

This group acts on Φ^d_m *via*

$$(\epsilon_0, \ldots, \epsilon_{d+1})\colon (z_0 : \ldots : z_{d+1}) \mapsto (\epsilon_0 z_0 : \ldots : \epsilon_{d+1} z_{d+1}),$$

inducing a decomposition $H_2(\Phi^d_m;\mathbb{C}) = \bigoplus_\omega H^\omega_d(\Phi^d_m)$, where ω runs through the dual group $G^\vee_m := \operatorname{Hom}(G_m,\mathbb{C}^\times)$,

$$G^\vee_m = \bigl\{(\omega_0,\ldots,\omega_{d+1}) \in (\mathbb{C}^\times)^{d+2} \bigm| \omega^m_0 = \ldots = \omega^m_{d+1} = \omega_0\ldots\omega_{d+1} = 1\bigr\}.$$

According to [16], one has $H^\omega_d = 0$ unless either $\omega = 1$ or

$$\omega \in \mathfrak{A}^d_m := \bigl\{\omega \in G^\vee_m \bigm| \omega_i \neq 1 \text{ for each } i = 0,\ldots,d+1\bigr\}.$$

Furthermore, the dimension of each nontrivial eigenspace is 1 and the Hodge weight of the eigenspace H^ω_d corresponding to a character $\omega \in \mathfrak{A}^d_m$ equals $\operatorname{Log}\omega - 1$, where $\operatorname{Log}\omega := \sum_i \operatorname{Log}\omega_i \in \{1,\ldots,d\}$ and $\operatorname{Log}\omega_i$ is the argument of the complex number ω_i *specialized to* $[0,2\pi)$ and divided by 2π. (Note a mysterious similarity between these formulas and the signature of a generalized Hopf link, see [5]; I do not know a conceptual explanation of this fact.)

The group of units $(\mathbb{Z}/m)^\times$ acts on the character group $G^\vee_m$ *via* $u\colon (\omega_i) \mapsto (\omega^u_i)$ and it is clear that a sum $\bigoplus_\omega H^\omega_d$, $\omega \in \Omega$, of eigenspaces is defined over $\mathbb{Q}$ if and only if the index set $\Omega \subset G^\vee_m$ is invariant under this action. Therefore, assuming that $d = 2k$ is even, the dimension of the rational Néron–Severi lattice

$$N\!S(\Phi^d_m;\mathbb{Q}) := N\!S(\Phi^d_m) \otimes \mathbb{Q} = H_d(\Phi^d_m;\mathbb{Q}) \cap H^{k,k}(\Phi^d_m)$$

equals $|\mathfrak{B}^d_m| + 1$, where

$$\mathfrak{B}^d_m := \bigl\{\omega \in \mathfrak{A}^d_m \bigm| \operatorname{Log}(\omega^u) = k+1 \text{ for each } u \in (\mathbb{Z}/m)^\times\bigr\},$$

see [16, 17].

In the case of surfaces ($d = 2$), the set $\mathfrak{B}^2_m$ has been studied in [17]. In particular, it has been shown that

$$\begin{aligned}\dim N\!S(\Phi^2_m;\mathbb{Q}) = 3(m-1)(m-2) + \delta_m + 1 \\ + 24(m/3)^* + 48(m/2)^* + 24\epsilon(m),\end{aligned} \tag{2.2}$$

where $\delta_m := 1 - (m \bmod 2) \in \{0,1\}$, the expression $(q)^*$ stands for q if $q \in \mathbb{Z}$ and 0 otherwise, and $\epsilon(m)$ is a bounded function that can be expressed as a certain sum over the divisors $d \mid m$ such that $\gcd(d,6) > 1$ and $d \leqslant 180$. Note that the last three terms vanish whenever $\gcd(m,6) = 1$.

2.2. **Counting projective spaces.** Assume that $d = 2k$ is even and pick an unordered partition

$$J = \bigl\{\{p_0,q_0\},\ldots,\{p_k,q_k\}\bigr\} \tag{2.3}$$

of the index set $\{0,\ldots,d+1\}$ into $(k+1)$ unordered pairs. Then, for each sequence $\eta = (\eta_0,\ldots,\eta_k)$ of m-th roots of (-1), the projective d-space

$$L_{J,\eta} := \{z_{p_i} = \eta_i z_{q_i},\ i = 0,\ldots,k\} \tag{2.4}$$

lies in Φ^d_m. Varying J and η, we obtain $(2k+1)!!\,m^{k+1}$ distinct subspaces; their classes generate a certain subgroup $\mathbf{S}^d_m \subset N\!S(X)$.

If $d = 2$, the above spaces are lines and, for $m \geqslant 3$, it can easily be shown that there are no other lines in Φ_m^2 (see. *e.g.*, [2]). In this special case, analyzing the intersection matrix (see, *e.g.*, [14]), one can also show that

$$\operatorname{rk} \mathbf{S}_m^2 = 3(m-1)(m-2) + \delta_m + 1. \tag{2.5}$$

(Alternatively, this rank can be found using Theorem 3.6 below.) Comparing this to (2.2), we arrive at the following statement.

Theorem 2.6 (see [17]). *If $m \leqslant 5$ or $\gcd(m, 6) = 1$, then $\mathbf{S}_m^2 \otimes \mathbb{Q} = NS(\Phi_m^2; \mathbb{Q})$.*

If $d \geqslant 4$, a similar statement can be obtained by other means (induction rather than direct counting).

Theorem 2.7 (see [13, 16]). *If $m = 4$ or m is prime, then $\mathbf{S}_m^d \otimes \mathbb{Q} = NS(\Phi_m^d; \mathbb{Q})$.*

Hence, a natural question, first raised in [1], is whether, under the assumptions of Theorem 2.6 or 2.7, we have the equality $\mathbf{S}_m^2 = NS(\Phi_m^d)$, *i.e.*, whether the classes of the projective subspaces contained in Φ_m^d generate $NS(\Phi_m^d)$ over the integers given that they do so over the rationals. To answer this question, we will study the torsion group $\mathbf{T}_m^d := \operatorname{Tors}\bigl(H_d(\Phi_m^d)/\mathbf{S}_m^d\bigr)$. (Throughout the paper, the notation $\operatorname{Tors} A$ always stands for the integral torsion of the abelian group A, even if the latter is also a module over another ring.)

2.3. **Delsarte surfaces.** A *Delsarte surface* $\Phi_A \subset \mathbb{P}^3$ is a surface given by a four-term equation of the form

$$\sum_{i=0}^{3} \prod_{j=0}^{3} z_j^{a_{ij}} = 0, \tag{2.8}$$

see [7, 18], where the exponent matrix $A := [a_{ij}]$ satisfies the following conditions:

(1) each entry a_{ij}, $0 \leqslant i, j \leqslant 3$, is a non-negative integer;
(2) each column of A has at least one zero;
(3) $(1, 1, 1, 1)$ is an eigenvector of A, *i.e.*, $\sum_{j=0}^{3} a_{ij} = \lambda = \operatorname{const}(i)$;
(4) A is non-degenerate, *i.e.*, $\det A \neq 0$.

Condition (2) asserts that the surface does not contain a coordinate plane, and (3) makes (2.8) homogeneous, the degree being the eigenvalue λ.

In general, this surface is singular and we silently replace Φ_A with its resolution of singularities. The particular choice of the resolution is not important as we will only deal with birational invariants.

Following [18], introduce the cofactor matrix $A^* := (\det A)A^{-1}$ and let

$$d := \gcd(a_{ij}^*), \quad m := |\det A|/d, \quad B = [b_{ij}] := mA^{-1} = \pm d^{-1}A^*.$$

Then, we have maps

$$\Phi_m^2 \xrightarrow{\pi_B} \Phi_A \xrightarrow{\pi_A} \Phi := \Phi_1^2$$

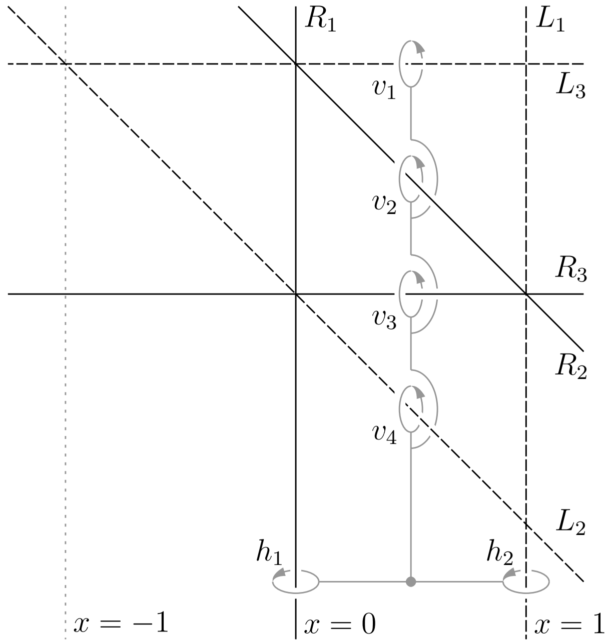

FIGURE 1. The divisor $V := L + R \subset \Phi$

given by

$$\pi_B \colon (z_i) \mapsto \left(\prod_{j=0}^{3} z_j^{b_{ij}}\right), \quad \pi_A \colon (z_i) \mapsto \left(\prod_{j=0}^{3} z_j^{a_{ij}}\right).$$

Both maps are ramified coverings; π_A and $\pi_B \circ \pi_A \colon (z_i) \mapsto (z_i^m)$ are ramified over the union $R := R_0 + R_1 + R_2 + R_3 \subset \Phi$ of the traces of the coordinate planes, $R_i := \Phi \cap \{z_i = 0\}$. The $3m^2$ lines in Φ_m^2 (see §2.2) project to the three lines

$$L_{ij} := \Phi \cap \{z_i = z_j\}, \quad 0 \leqslant i < j \leqslant 3.$$

(Obviously, $L_{ij} = L_{kl}$ whenever i, j, k, l are pairwise distinct, *i.e.*, the L-lines are indexed by partitions J as in (2.3).) Together, R and $L := L_{01} + L_{02} + L_{03}$ form the so-called *Ceva-7* arrangement in the projective plane Φ, see Figure 1 (where R_0 is the missing line at infinity).

Since R is a nodal curve, the fundamental group $\mathbb{G} := \pi_1(\Phi \smallsetminus R)$ is abelian: it has four generators t_i dual to $[R_i]$, $i = 0, \ldots, 3$, that are subject to the relation $t_0 t_1 t_2 t_3 = 1$. The finite ramified coverings π_A as above are in a natural one-to-one correspondence with *finite quotients* of $\mathbb{G}$, *i.e.*, epimorphisms $\alpha \colon \mathbb{G} \twoheadrightarrow G$ onto a finite group G. Henceforth, we can disregard the original matrix A and speak about the Delsarte surface $\Phi[\alpha]$, which is defined as (any) smooth analytic compactification of the covering of $\Phi \smallsetminus R$ corresponding to α. In this notation, $\Phi_m^2 = \Phi[m]$, where an integer $m \in \mathbb{Z}$ is regarded as a map $m \colon \mathbb{G} \twoheadrightarrow \mathbb{G}/m\mathbb{G}$.

Found in [18] is an algorithm making use of (2.2) and computing the Picard rank (or rather corank, which is a birational invariant) of $\Phi[\alpha]$ in terms of α. On the other hand, there is an "obvious" divisor $V[\alpha] := \pi_A^*(R+L)$ in $\Phi[\alpha]$ that plays the rôle of lines in Φ_m^2; let $\mathbf{S}[\alpha] \subset \mathit{NS}(\Phi[\alpha])$ be the subgroup generated by the components of $V[\alpha]$. One of the outcomes of [18] is the equality $\mathbf{S}[\alpha] \otimes \mathbb{Q} = \mathit{NS}(\Phi[\alpha]; \mathbb{Q})$ that holds whenever $\gcd(m, 6) = 1$ and the natural question whether, under the same assumption, one also has $\mathbf{S}[\alpha] = \mathit{NS}(\Phi[\alpha])$ over the integers, *i.e.*, whether the group

$$\mathbf{T}[\alpha] := \operatorname{Tors}\bigl(\mathit{NS}(\Phi[\alpha])/\mathbf{S}[\alpha]\bigr)$$

is trivial. Note that, in the case of a Fermat surface, this question is equivalent to the original one raised in [1], see the end of §2.1. Indeed, in this case, each divisorial pull-back $\pi_A^* R_i$, $i = 0, \ldots, 3$, is a reduced irreducible Fermat curve and, fixing J and η_1, η_2, η_3 in (2.4), we obtain m lines whose classes sum up to $[\pi_A^* R_i]$. Hence, whenever $\alpha = m \in \mathbb{N}_+$, we have $\mathbf{S}[m] = \mathbf{S}_m^2$ and $\mathbf{T}[m] = \mathbf{T}_m^2$.

3. The topological reduction

3.1. **The torsion group.** Given a divisor D in a smooth compact surface X, let $\mathbf{S}\langle D\rangle \subset \mathit{NS}(X)$ be the subgroup generated by the irreducible components of D. Here and below, the Néron–Severi lattice $\mathit{NS}(X)$ is the image of $\operatorname{Pic} X$ in the free abelian group $H_2(X)/\operatorname{Tors}$, which is canonically identified with $H^2(X)/\operatorname{Tors}$ *via* Poincaré duality. The homomorphism is given by $D \mapsto [D]$ in the language of divisors and homology or by $\mathcal{L} \mapsto c_1(\mathcal{L})$ in the language of line bundles and cohomology. Thus,

$$\mathbf{S}\langle D\rangle := \operatorname{Im}\bigl[\iota_*\colon H_2(D) \to H_2(X)/\operatorname{Tors}\bigr],$$

where $\iota\colon D \hookrightarrow X$ is the inclusion. We will also consider the groups

$$\mathbf{K}\langle D\rangle := \operatorname{Ker}\bigl[\iota_*\colon H_2(D) \to H_2(X)\bigr], \qquad \mathbf{T}\langle D\rangle := \operatorname{Tors}\bigl(\mathit{NS}(X)/\mathbf{S}\langle D\rangle\bigr).$$

The following statement is essentially the definition of Ext and Poincaré duality.

Theorem 3.1 (see [3, 4]). *For $D \subset X$ as above, let*

$$K(X, D) := \operatorname{Ker}\bigl[\kappa_*\colon H_1(X \smallsetminus D) \to H_1(X)\bigr]$$

be the kernel of the homomorphism κ_* induced by the inclusion $\kappa\colon X \smallsetminus D \hookrightarrow X$. Then there are canonical isomorphisms

$$\operatorname{Tors} K(X, D) = \operatorname{Hom}(\mathbf{T}\langle D\rangle, \mathbb{Q}/\mathbb{Z}), \qquad K(X, D)/\operatorname{Tors} = \operatorname{Hom}(\mathbf{K}\langle D\rangle, \mathbb{Z}).$$

Indeed, κ_* is Poincaré dual to the homomorphism rel in the exact sequence

$$\longrightarrow H^2(X) \xrightarrow{\iota^*} H^2(D) \longrightarrow H^3(X, D) \xrightarrow{\mathrm{rel}} H^3(X) \longrightarrow .$$

Thus, $K(X, D) = \operatorname{Coker} \iota^*$. The abelian group $H^2(D)$ is free and, modulo torsion in $H^2(X)$, the homomorphism ι^* is the adjoint of ι_* in the free resolution

$$\longrightarrow H_2(D) \xrightarrow{\iota_*} H_2(X)/\operatorname{Tors} \longrightarrow T \longrightarrow 0,$$

where $T := H_2(X)/\big(\operatorname{Tors} H_2(X) + \mathbf{S}\langle D\rangle\big)$. Thus,

$$\operatorname{Coker}\iota^* = \operatorname{Ext}(T,\mathbb{Z}) = \operatorname{Hom}(\operatorname{Tors} T, \mathbb{Q}/\mathbb{Z}).$$

On the other hand, the quotient $H_2(X)/c_1(\operatorname{Pic} X)$ is known to be torsion free and we have $\operatorname{Tors} T = \mathbf{T}\langle D\rangle$.

Now, let $X = \Phi[\alpha]$ be a Delsarte surface and $D = V[\alpha]$. To avoid excessive nested parentheses, we will abbreviate

$$\mathbf{S}[\alpha] := \mathbf{S}\langle V[\alpha]\rangle, \quad \mathbf{T}[\alpha] := \mathbf{T}\langle V[\alpha]\rangle, \quad \mathbf{K}[\alpha] := \mathbf{K}\langle V[\alpha]\rangle$$

and use the shortcut $\Phi^\circ[\alpha] := \Phi[\alpha] \smallsetminus V[\alpha]$. The group $H_1(\Phi[\alpha]) = \pi_1(\Phi[\alpha])$ is finite abelian (see [4] or (3.5) below) and the homomorphism κ_* in Theorem 3.1 factors through the free abelian group

$$H_1(\Phi[\alpha] \smallsetminus \pi_A^{-1}R) = \pi_1(\Phi[\alpha] \smallsetminus \pi_A^{-1}R) = \operatorname{Ker}\alpha \subset \mathbb{G}.$$

Hence, Theorem 3.1 can be recast in a simpler form

$$\operatorname{Tors} H_1(\Phi^\circ[\alpha]) = \operatorname{Hom}(\mathbf{T}[\alpha], \mathbb{Q}/\mathbb{Z}), \qquad H_1(\Phi^\circ[\alpha])/\operatorname{Tors} = \operatorname{Hom}(\mathbf{K}[\alpha], \mathbb{Z}). \tag{3.2}$$

Note also that, as long as the torsion is concerned, $H_1(\Phi^\circ[\alpha])$ can be replaced with $C_1/\operatorname{Im}\partial_2$, where (C_*, ∂) is the cellular complex (with respect to any *CW*-structure) computing the homology of $\Phi^\circ[\alpha]$. Indeed, the quotient $(C_1/\operatorname{Im}\partial_2)/H_1(\Phi^\circ[\alpha])$ is a subgroup of the free abelian group C_0.

3.2. **The Alexander module.** Given a topological space X and an epimorphism $\alpha\colon \pi_1(X) \twoheadrightarrow G$ onto an abelian group G, the homology of the covering $\tilde{X} \to X$ defined by α are naturally $\mathbb{Z}[G]$-modules, G acting by the deck translations of the covering. The $\mathbb{Z}[G]$-module $H_1(\tilde{X})$ is called the *Alexander module* of X or, more precisely, of pair (X, α). The Alexander module depends only on the group $\pi_1(X)$ and epimorphism α; algebraically, it is the abelian group $\operatorname{Ker}\alpha/[\operatorname{Ker}\alpha, \operatorname{Ker}\alpha]$ on which $G = \pi_1(X)/\operatorname{Ker}\alpha$ acts by conjugation.

We employ the concept of Alexander module to compute $H_1(\Phi^\circ[\alpha])$. First, let α be the identity map $0\colon \mathbb{G} \twoheadrightarrow \mathbb{G}/0\mathbb{G}$ (awkward as it seems, this notation agrees with our convention; we also have $1\colon \mathbb{G} \twoheadrightarrow \mathbb{G}/\mathbb{G} = \{1\}$) and consider the ring $\Lambda := \mathbb{Z}[\mathbb{G}]$. Note that, unlike $\Phi[0]$, the unramified covering $\Phi^\circ[0]$ still makes sense. The group $\pi_1(\Phi^\circ[1])$ is computed using Zariski–van Kampen theorem [19] (this computation is essentially shown in Figure 1, see [4] for the relations and further details), and the map $\pi_1(\Phi^\circ[1]) \twoheadrightarrow \mathbb{G}$ is $h_1 \mapsto t_1$, $v_2 \mapsto t_2$, $v_3 \mapsto t_3$, $v_1, v_4, h_2 \mapsto 1$. Then, the Λ-module $H_1(\Phi^\circ[0])$ is found by means of the Fox free calculus [8]. It is more convenient to work with the module

$$\mathrm{A}[0] := C_1[0]/\operatorname{Im}\partial_2,$$

where $(C_*[0], \partial)$ is an appropriate cellular complex computing the homology (or even just the fundamental group) of $\Phi^\circ[0]$; as explained in §3.1, that would suffice for our purposes. As a Λ-module, $\mathrm{A}[0]$ is generated by six elements a_1, a_2, a_3, c_1,

c_2, c_3 (corresponding, in the order listed, to the six generators h_1, v_2, v_3, h_2, v_4, v_1 of $\pi_1(\Phi^\circ[1])$ shown in Figure 1), which are subject to the six relations

$$\begin{gathered}(t_2t_3-1)c_1=(t_1t_3-1)c_2=(t_1t_2-1)c_3=0,\\ (t_3-1)c_1+(t_3-1)a_2-(t_2-1)a_3=0,\\ (t_3-1)c_2+(t_3-1)a_1-(t_1-1)a_3=0,\\ (t_1-1)c_3+(t_1-1)a_2-(t_2-1)a_1=0.\end{gathered}$$

Recall that we also have $t_0t_1t_2t_3=1$ in Λ.

Now, given a finite quotient $\alpha\colon \mathbb{G}\to G$, the group ring $\Lambda[\alpha]:=\mathbb{Z}[G]$ is naturally a Λ-module and the complex $(C_*[\alpha],\partial)$ is obviously $(C_*[0],\partial)\otimes_\Lambda\Lambda[\alpha]$. Hence, the new $\Lambda[\alpha]$-module $\mathrm{A}[\alpha]$ is $\mathrm{A}[0]\otimes_\Lambda\Lambda[\alpha]$. In some cases, it is more convenient to work with the submodule $\mathrm{B}[\alpha]\subset\mathrm{A}[\alpha]$ generated by c_1,c_2,c_3. (Note though that it is not always easy to find the defining relations for $\mathrm{B}[\alpha]$.) Since the complex

$$0\longrightarrow \mathrm{A}[\alpha]/\mathrm{B}[\alpha]\longrightarrow\Lambda[\alpha]\longrightarrow 0$$

computes the homology $H_0=\mathbb{Z}$ and $H_1=\operatorname{Ker}\alpha\subset\mathbb{G}$ of the space $\Phi[\alpha]\smallsetminus\pi_A^{-1}R$, the two modules have the same integral torsion. Summarizing, we arrive at the following algebraic description of $\mathbf{K}[\alpha]$ and $\mathbf{T}[\alpha]$.

Theorem 3.3 (see [3]). *For any finite quotient $\alpha\colon\mathbb{G}\to G$ one has*

$$\begin{gathered}\mathbf{T}[\alpha]=\operatorname{Ext}_\mathbb{Z}(\mathrm{A}[\alpha],\mathbb{Z})=\operatorname{Ext}_\mathbb{Z}(\mathrm{B}[\alpha],\mathbb{Z}),\\ \operatorname{rk}_\mathbb{Z}\mathbf{K}[\alpha]=\operatorname{rk}_\mathbb{Z}\mathrm{A}[\alpha]-|G|+1=\operatorname{rk}_\mathbb{Z}\mathrm{B}[\alpha]+3.\end{gathered}$$

In other words, the torsion $\mathbf{T}[\alpha]$ in question is isomorphic to the integral torsion of either of the two modules $\mathrm{A}[\alpha]$ or $\mathrm{B}[\alpha]$.

Note also that, even if $\mathbf{T}[\alpha]\ne0$, a sufficiently good description of this group would still let one recover the complete structure of $NS(\Phi[\alpha])$. For example, one can use the technique of discriminant forms introduced in [11]. From this point of view, Theorem 3.3 does give us a suitable description of $\mathbf{T}[\alpha]$, as it actually places this group to the discriminant group $\mathbf{S}[\alpha]^\vee/\mathbf{S}[\alpha]$.

3.3. **Vanishing and bounds.** Numeric experiments with random matrices show that, typically, $\mathbf{T}[\alpha]\ne0$, even if $\gcd(m,6)=1$ (see §2.3). However, the vanishing of the group $\mathbf{T}[\alpha]$ can be established in several important special cases. We have the following theorem.

Theorem 3.4 (see [3, 4]). *In each of the following three cases, one has $\mathbf{T}[\alpha]=0$:*

(1) $\Phi[\alpha]$ *is a Fermat surface,* i.e., $\alpha=m\in\mathbb{N}_+$;
(2) $\Phi[\alpha]$ *is* cyclic, i.e., *the image G of α is a cyclic group*;
(3) $\Phi[\alpha]$ *is* unramified at ∞, i.e., $\alpha(t_0)=1$.

Statement (3) in Theorem 3.4 was a toy example considered in [4]. Statement (2) is proved in [3] by comparing the dimensions $\dim_\Bbbk\mathrm{A}[\alpha]\otimes\Bbbk$, where $\Bbbk$ is either $\mathbb{C}$ or a finite field $\mathbb{F}_p$: if G is cyclic, $\Lambda[\alpha]\otimes\Bbbk=\Bbbk[G]$ is a principal ideal domain and the dimension of a module can be computed algorithmically using elementary divisors of the matrix of relations.

Statement (1) is more involved. In [4], it is proved by considering an appropriate rather long filtration

$$0 = A_0 \subset A_1 \subset \ldots \subset A_7 = \mathrm{A}[\alpha]$$

and *estimating* from above the length $\ell(A_{i+1}/A_i)$ of each quotient. (Recall that the *length* $\ell(A)$ of an abelian group A is the minimal number of generators of A, whereas its *rank* $\operatorname{rk} A$ is the maximal number of linearly independent elements. Always $\operatorname{rk} A \leqslant \ell(A)$, and a finitely generated abelian group A is free if and only if $\ell(A) = \operatorname{rk} A$.) Luckily, these estimates sum up to the expected rank $\operatorname{rk} \mathrm{A}[\alpha]$ given by (2.5) and Theorem 3.3; hence, each quotient A_{i+1}/A_i is a free abelian group, and so is the original module $\mathrm{A}[\alpha]$.

The same approach can be used in the general case, but the counts no longer match; hance, we only obtain an estimate on the size of $\mathbf{T}[\alpha]$. To state the next theorem, we need to introduce a few invariants measuring the non-uniformity of the homomorphism α. (Note that the group $\mathbb{G}$ is to be considered in its canonical generating set t_0, t_1, t_2, t_3 introduced in §2.3; the only automorphisms allowed are permutations of the generators. This rigidity explains also why we are using four generators instead of three.) First, consider the following subgroups of $\mathbb{G}$:

- $\mathbb{G}_{ij}$, generated by t_i and t_j, $i, j = 0, 1, 2, 3$;
- $\mathbb{G}_i$, generated by $t_i t_j$ and $t_i t_k$, $i = 1, 2, 3$ and $\{i, j, k\} = \{1, 2, 3\}$;
- $\mathbb{G}_= := \sum_i \mathbb{G}_i$, generated by $t_1 t_2$, $t_1 t_3$, and $t_2 t_3$.

In more symmetric terms, $\mathbb{G}_i$ depends only on the partition $\{\{0, i\}, \{j, k\}\}$ of the index set, see (2.3), and $\mathbb{G}_=$ is generated by all products $t_i t_j$, $i, j = 0, 1, 2, 3$; one has $[\mathbb{G} : \mathbb{G}_=] = 2$. Now, for a finite quotient $\alpha \colon \mathbb{G} \twoheadrightarrow G$, denote $G_* := G/\alpha(\mathbb{G}_*)$ (where the subscript $*$ is one of the symbols ij, i, or $=$ as above) and define $\delta[\alpha] := |G_=| - 1 \in \{0, 1\}$. Let, further, $\exp G$ be the minimal positive integer m such that $mG = 0$. (This notion applies to any abelian group G; in our case, it is also the minimal positive integer m such that $m\mathbb{G} \subset \operatorname{Ker} \alpha$).

In this notation, the fundamental group $\pi_1(\Phi[\alpha])$ found in [4] is given by

$$\pi_1(\Phi[\alpha]) = H_1(\Phi[\alpha]) = \operatorname{Ker} \alpha \Big/ \prod (\mathbb{G}_{ij} \cap \operatorname{Ker} \alpha), \tag{3.5}$$

the product running over all pairs $0 \leqslant i < j \leqslant 3$. This group is trivial in any of the three special classes considered in Theorem 3.4. In general, as shown in [3], the group $\pi_1(\Phi[\alpha])$ is cyclic and its order $|\pi_1(\Phi[\alpha])|$ divides the *height* $\operatorname{ht} \alpha := \exp G/n$, where n is the maximal integer such that $\operatorname{Ker} \alpha \subset n\mathbb{G}$.

Theorem 3.6 (see [3]). *For any finite quotient $\alpha \colon \mathbb{G} \twoheadrightarrow G$, one has*

$$\operatorname{rk} \mathbf{K}[\alpha] = \sum_{0 \leqslant i < j \leqslant 3} |G_{ij}| + \sum_{1 \leqslant i \leqslant 3} |G_i| - 3 - \delta[\alpha].$$

Besides, $\ell(\mathbf{T}[\alpha]) \leqslant 6 + \delta[\alpha]$ and $\exp \mathbf{T}[\alpha]$ divides $(\exp G)^3/|G|$.

The bound on $\ell(\mathbf{T}[\alpha])$ is sharp, whereas that on $\exp \mathbf{T}[\alpha]$ is probably not.

Analyzing the proof, one can also establish the almost vanishing of the torsion in the case of *Brieskorn surfaces* (called *diagonal Delsarte surfaces* in [3]), *i.e.*, those given by an affine equation of the form

$$x^{m_1} + y^{m_2} + z^{m_3} = 1,$$

so that α is the projection $\mathbb{G} \twoheadrightarrow \mathbb{G}/(t_1^{m_1} = t_2^{m_2} = t_3^{m_3} = 1)$. For such surfaces, $\mathbf{T}[\alpha]$ is cyclic: one has $\ell(\mathbf{T}[\alpha]) \leqslant \delta[\alpha]$ and the order $|\mathbf{T}[\alpha]|$ divides the ratio

$$h(m_1, m_2, m_3) := \frac{\operatorname{lcm}_{1\leqslant i<j\leqslant 3}\bigl(\gcd(m_i, m_j)\bigr)}{\gcd(m_1, m_2, m_3)} = \sqrt{\frac{m_1 m_2 m_3}{\gcd(m_1, m_2, m_3)^3}}.$$

(Observe that $h(m_1, m_2, m_3) = 1$ if $m_1 = m_2 = m_3$, *i.e.*, in the case of classical Fermat surfaces.) Examples show that these bounds are also sharp.

4. Higher dimensions

4.1. **The reduction.** Now, let us consider a Fermat variety $\Phi := \Phi_m^d$ of even dimension $d = 2k \geqslant 4$. Denote by $\mathcal{J} := \mathcal{J}(d)$ the set of partitions as in (2.3); each element $J \in \mathcal{J}$ gives rise to m^{k+1} subspaces in Φ. To put the statements in a slightly more general form, we will pick a nonempty subset $\mathcal{K} \subset \mathcal{J}$ and denote by $V_\mathcal{K} \subset \Phi$ the union of all subspaces $L_{J,\eta}$, see (2.4), with $J \in \mathcal{K}$ and η running over all sequences of roots of (-1). Denoting by $\iota\colon V_\mathcal{K} \hookrightarrow \Phi$ the inclusion, consider the groups

$$\mathbf{S}_\mathcal{K} := \operatorname{Im}\bigl[\iota_*\colon H_d(V_\mathcal{K}) \to H_d(\Phi)\bigr], \qquad \mathbf{T}_\mathcal{K} := \operatorname{Tors}\bigl(H_d(\Phi)/\mathbf{S}_\mathcal{K}\bigr).$$

Clearly, $\mathbf{S}_\mathcal{J} = \mathbf{S}_m^d$ and $\mathbf{T}_\mathcal{J} = \mathbf{T}_m^d$.

In what follows, we always regard $H_d(\Phi)$ as a unimodular lattice by means of the Poincaré duality isomorphism $H_d(\Phi) \to H^d(\Phi) = H_d(\Phi)^\vee$. (Here and below, we denote by $A^\vee := \operatorname{Hom}(A, \mathbb{Z})$ the dual of an abelian group A.) Consider the subspace $Y := \Phi \smallsetminus \{z_0 = 0\}$ and let $H_\circ \subset H_d(\Phi)$ be the image of the inclusion homomorphism $H_d(Y) \to H_d(\Phi)$; it coincides with the orthogonal complement of the class $h \in H_d(\Phi)$ of the intersection of Φ with a generic $(d+1)$-plane. Since the lattice $H_d(\Phi)$ is unimodular, the composition $H_d(\Phi) = H_d(\Phi)^\vee \twoheadrightarrow H_\circ^\vee$ is surjective; in fact, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}h \longrightarrow H_d(\Phi) \longrightarrow H_\circ^\vee \longrightarrow 0.$$

Let $\mathbf{S}'_\mathcal{K} \subset H_\circ^\vee$ be the image of $\mathbf{S}_\mathcal{K}$. Since $h \in \mathbf{S}_\mathcal{K}$, we have $H_d(\Phi)/\mathbf{S}_\mathcal{K} = H_\circ^\vee/\mathbf{S}'_\mathcal{K}$ and $\operatorname{rk} \mathbf{S}'_\mathcal{K} = \operatorname{rk} \mathbf{S}_\mathcal{K} - 1$. (Recall that we assume that $\mathcal{K} \neq \varnothing$; hence, fixing $J \in \mathcal{K}$ and all but one η_i in (2.4), we obtain m spaces whose classes sum up to h.)

Let $\Lambda := \mathbb{Z}[G_m]$, see (2.1). To make the notation more conventional, we rename the canonical generators $(1, \ldots, \exp(2\pi i/m), \ldots, 1)$ of G_m into $t_0, \ldots, t_{d+1}$ and regard Λ as the quotient of the ring $\mathbb{Z}[t_0^{\pm 1}, \ldots, t_{d+1}^{\pm 1}]$ of Laurent polynomials by the ideal generated by $t_0 \ldots t_{d+1} - 1$ and $t_i^m - 1$, $i = 0, \ldots, d+1$. Since G_m acts on Φ, all homology groups involved are naturally Λ-modules and, G_m acting identically on the fundamental class $[\Phi]$, the lattice structure on $H_d(\Phi)$ is Λ-invariant.

For further statements, we need to prepare several polynomials. Let

$$\varphi(t) := \sum_{i=0}^{m-1} t^i = \frac{t^m - 1}{t - 1}, \qquad \rho(x, y) := \sum_{0 \leqslant i \leqslant j \leqslant m-2} x^j y^i$$

and, for $J \in \mathcal{J}$ as in (2.3), denote

$$\tau_J := \prod_{i=0}^{k} (t_{q_i} - 1), \quad \psi_J := \tau_J \prod_{i=1}^{k} \varphi(t_{p_i} t_{q_i}), \quad \rho_J := \prod_{i=1}^{k} \rho(t_{p_i}, t_{q_i}).$$

Also, for any quotient ring R of Λ, including Λ itself, we will denote by $\bar{R}$ its "reduced" version, *viz.* $\bar{R} := R \big/ \sum_{i=0}^{d+1} R\varphi(t_i)$.

The advantage of using Y instead of Φ is the fact that this space has extremely simple homology, which have been extensively studied as the vanishing cycles of a Pham–Brieskorn singularity. Fix a number $\zeta \in \mathbb{C}$ such that $\zeta^m = -1$ and consider the topological simplex

$$\Delta := \left\{ (s_1, \ldots, s_{d+1})\zeta \;\middle|\; s_s \in [0, 1],\ s_1^m + \ldots + s_{d+1}^m = 1 \right\} \subset Y.$$

Then, the so-called *Pham polyhedron*

$$\Sigma := (1 - t_1^{-1}) \ldots (1 - t_{d+1}^{-1})\Delta$$

is a cycle in Y; in fact, Σ is a topological sphere.

Theorem 4.1 (see [12])**.** *The group $H_d(Y)$ is the free $\bar{\Lambda}$-module generated by Σ.*

Therefore, $H_\circ^\vee$ is an ideal in $\Lambda = \Lambda^\vee$ (where all groups dual to Λ-modules are regarded as Λ-modules with respect to the contragredient G-action) and, in order to find the image $\mathbf{S}'_{\mathcal{K}} \subset H_\circ^\vee \subset \Lambda$, it suffices to compute the algebraic intersection of each space $L_{J,\eta}$ with Σ. This is done in [6], and, omitting intermediate details, the result can be stated as follows.

Theorem 4.2 (see [6])**.** *For each $J \in \mathcal{J}$, one has $\mathbf{S}'_J = \Lambda\psi_J \subset \Lambda$. Hence, for a subset $\mathcal{K} \subset \mathcal{J}$, one has $\mathbf{S}'_{\mathcal{K}} = \sum_J \Lambda\psi_J \subset \Lambda$, the summation running over $J \in \mathcal{K}$.*

4.2. **Partial vanishing statements.** Using Theorem 4.2 and employing various dualities and torsion-free quotients, we can obtain several expressions for $\mathbf{T}_{\mathcal{K}}$. In the statement below, for $J \in \mathcal{J}$ as in (2.3), we use the ring

$$\Lambda_J := \Lambda \big/ \textstyle\sum_i \Lambda(t_{p_i} t_{q_i} - 1), \quad i = 0, \ldots, k+1,$$

and 1_J stands for the unit in Λ_J or $\bar{\Lambda}_J$.

Theorem 4.3 (see [6])**.** *Let $\mathcal{K} \subset \mathcal{J}$ be a nonempty subset. Then, the torsion $\mathbf{T}_{\mathcal{K}}$ is isomorphic to the integral torsion of any of the following modules*:

(1) *the ring $\Lambda \big/ \sum_J \Lambda\psi_J$, $J \in \mathcal{K}$;*
(2) *the ring $\bar{\Lambda} \big/ \sum_J \bar{\Lambda}\rho_J$, $J \in \mathcal{K}$;*

(3) *the* Λ*-module* $M_{\mathcal{K}} := \bigl(\bigoplus_J \Lambda_J\bigr)/\Lambda\tau$*, where* $\tau := \sum_J \tau_J 1_J$ *and* $J \in \mathcal{K}$*;*
(4) *the* $\bar{\Lambda}$*-module* $\bar{M}_{\mathcal{K}} := \bigl(\bigoplus_J \bar{\Lambda}_J\bigr)/\bar{\Lambda}1$*, where* $1 := \sum_J 1_J$ *and* $J \in \mathcal{K}$*.*

Denoting by T the integral torsion of the respective module in Theorem 4.3, in Statements (1) and (2) we have canonical isomorphisms $\mathbf{T}_{\mathcal{K}} = T$, whereas in (3) and (4), the isomorphisms are $\mathbf{T}_{\mathcal{K}} = \operatorname{Hom}(T, \mathbb{Q}/\mathbb{Z})$.

If $d = 2$, the module $M_{\mathcal{J}}$ in Theorem 4.3(3) coincides with the module $\mathrm{B}[m]$ introduced in §3.2. Both $M_{\mathcal{J}}$ and $\bar{M}_{\mathcal{J}}$ appear as intermediate quotients of the filtration used in the proof of Theorem 3.4(1).

Conjecture 4.4. For the full set $\mathcal{K} = \mathcal{J}$, one has $\mathbf{T}_{\mathcal{J}} = 0$.

This conjecture holds true in small dimensions $d = 0$ (obvious) and $d = 2$ (see Theorem 3.4) and is supported by some numerical evidence: by a computer aided computation, we managed to establish the vanishing of $\mathbf{T}_{\mathcal{J}}$ for the values

$$(d, m) = (4, m),\ 3 \leqslant m \leqslant 12, \quad (6,3),\ (6,4),\ (6,5),\ \text{and } (8,3).$$

Unfortunately, we failed to prove the conjecture in full generality. It is not difficult to show (see [6]) that $\gcd(|\mathbf{T}_{\mathcal{K}}|, p) = 1$ for each prime $p \nmid m$. One can also show that $\mathbf{T}_{\mathcal{K}} = 0$ for some special subsets $\mathcal{K} \subset \mathcal{J}$. As an important example (which can probably be used as a base for induction), fixing the degree m and denoting by $(\cdot)$ the dependence on the dimension, consider the natural inclusions $\mathcal{J}(s) \subset \mathcal{J}(d)$, $s = 2l \leqslant d = 2k$, extending each partition $J \in \mathcal{J}(s)$ *identically beyond* s, *i.e.*, attaching the pairs $\{2i, 2i+1\}$, $i = l+1, \ldots, k$. Then, given $\mathcal{K} \subset \mathcal{J}(s)$, for the module $\bar{M}_{\mathcal{K}}$ in Theorem 4.3(4) one can easily see that

$$\bar{M}_{\mathcal{K}}(d) = \bar{M}_{\mathcal{K}}(s) \otimes_{\mathbb{Z}} \bar{\Delta}_{s+2} \otimes_{\mathbb{Z}} \bar{\Delta}_{s+4} \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} \bar{\Delta}_d,$$

where $\bar{\Delta}_i := \mathbb{Z}[t_i^{\pm 1}]/\phi(t_i)$. Hence, we have stabilization

$$\mathbf{T}_{\mathcal{K}}(d) = \mathbf{T}_{\mathcal{K}}(s) \otimes_{\mathbb{Z}} \bar{\Delta}_{s+2} \otimes_{\mathbb{Z}} \bar{\Delta}_{s+4} \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} \bar{\Delta}_d. \tag{4.5}$$

In particular, $\mathbf{T}_{\mathcal{K}}(d) = 0$ if and only if $\mathbf{T}_{\mathcal{K}}(s) = 0$. This observation applies to the image $\mathcal{J}(s, d)$ of $\mathcal{J}(s)$ in $\mathcal{J}(d)$; thus $\mathbf{T}_{\mathcal{J}(s,d)} = 0$ for all dimensions $d \geqslant s$ if and only if $\mathbf{T}_{\mathcal{J}(s)} = 0$. As a consequence, for any d,

$$\mathbf{T}_{\mathcal{J}(0,d)} = \mathbf{T}_{\mathcal{J}(2,d)} = 0; \tag{4.6}$$

the former is obvious, and the latter follows from Theorem 3.4(1).

4.3. **Other classes of varieties.** The last two statements (4.5), (4.6) have a geometric interpretation. Given $s = 2l < d = 2k$, consider the *partial Fermat variety* $\Phi_m^{s,d}$ given by an equation of the form

$$f_0(z_0, z_1) + f_1(z_2, z_3) + \ldots + f_k(z_d, z_{d+1}) = 0,$$

where each f_i is a homogeneous bivariate polynomial of degree m and

$$f_0(u, v) = f_1(u, v) = \ldots = f_l(u, v) = u^m + v^m,$$

whereas all other terms are generic, pairwise distinct, and other than $u^m + v^m$. Arguing as in §2.2, one can see that $\Phi_m^{s,d}$ contains several group of d-spaces: each group consists of m^{k+1} spaces, and the groups are indexed by the members of $\mathcal{J}(s,d)$. Now, observe that the proof of Theorem 4.3 is purely topological; hence, we can deform $\Phi_m^{s,d}$ to Φ_m^d (followed by a deformation of the d-spaces in $\Phi_m^{s,d}$ to some of those in Φ_m^d) and apply Theorem 4.3, obtaining the following corollary.

Corollary 4.7 (see [6])**.** *The subspaces contained in $\Phi_m^{s,d}$ generate a primitive subgroup in the Néron–Severi lattice $NS(\Phi_m^{s,d})$ for any dimension $d \geqslant s$ if and only if they do so for $d = s$,* i.e., *if $\mathbf{T}_m^s = 0$.*

Corollary 4.8 (see [6])**.** *For $s = 0$ or 2, the subspaces contained in $\Phi_m^{s,d}$ generate a primitive subgroup in the Néron–Severi lattice $NS(\Phi_m^{s,d})$.*

If $s = 0$, we can also choose f_0 generic, retaining the m^{k+1} spaces contained in $\Phi_m^{0,d}$. In this case, if $d = 2$ (lines in surfaces) and m is prime, the m^2 lines contained in $\Phi_m^{0,2}$ are known to generate the rational Néron–Severi lattice $NS(\Phi_m^{0,2}; \mathbb{Q})$, see, *e.g.*, [2]. Corollary 4.8 implies that these lines generate $NS(\Phi_m^{0,2})$ over $\mathbb{Z}$ (see [4]).

5. Open problems

Apart from Conjecture 4.4, there are a few other interesting open questions that may be worth stating explicitly.

As explained in §3.3, typically, for a Delsarte surface $\Phi[\alpha]$ one has $\mathbf{T}[\alpha] \neq 0$. Naturally, one may ask if there are other classes of surfaces for which one can assert that $\mathbf{T}[\alpha] = 0$ or obtain a bound on the size of this group better than that given by Theorem 3.6. In Theorem 3.4, the Delsarte surfaces are treated according to the complexity (or non-uniformity) of the finite quotient $\alpha\colon \mathbb{G} \twoheadrightarrow G$. However, there are other taxonomies which, from many points of view, may seem much more natural. For example, one can classify Delsarte surfaces according to the singularities of the original (not yet resolved) projective hypersurface given by (2.8). Thus, it is known that there are ten families (one of them being Fermat) of nonsingular Delsarte surfaces, see [10], and 83 families of those with **A**–**D**–**E** singularities, see [9]. The Picard ranks for these families were computed in [9, 10].

Problem 5.1. Does the vanishing $\mathbf{T}[\alpha] = 0$ hold for all nonsingular Delsarte surfaces $\Phi[\alpha]$? For those with **A**–**D**–**E** singularities?

Problem 5.2. Are there sharper bounds on the size (length, order, exponent) of the group $\mathbf{T}[\alpha]$ in terms of the singularities of $\Phi[\alpha]$?

As another generalization, one can consider a Fermat surface Φ_m^2 of a degree m not prime to 6, so that the lines do *not* generate $NS(\Phi_m^2; \mathbb{Q})$. In some cases, there are explicit lists of additional generators. Thus, found in [1], there is a list of relatively simple curves, lying in quadrics, cubics, and quartics, that compensate for the terms $24(m/3)^*$ and $48(m/2)^*$ in (2.2). As in §2.2, the generating property is established by comparing the ranks; hence, the question whether these curves (together with the lines) generate the integral group $NS(\Phi_m^2)$ remains open.

Problem 5.3 (T. Shioda)**.** For the known explicit generating sets $\mathcal{S}$ of the group $NS(\Phi_m^2;\mathbb{Q})$, is it true that the curves constituting $\mathcal{S}$ also generate $NS(\Phi_m^2)$ over the integers? In other words, is it true that the subgroup $\mathbf{S}\langle\mathcal{S}\rangle := \sum \mathbb{Z}[C]$, $C \in \mathcal{S}$, is primitive in $H_2(\Phi_m^2)$? If not, what is the torsion of $H_2(\Phi_m^2)/\mathbf{S}\langle\mathcal{S}\rangle$?

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Department of Mathematics, Bilkent University, 06800 Ankara, Turkey
E-mail address: degt@fen.bilkent.edu.tr