DEGENERATION AND CURVES ON K3 SURFACES

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Abstract. We study curves on K3 surfaces. A folklore conjecture concerning rational curves on K3 surfaces states that all K3 surfaces contain infinite number of irreducible rational curves. It is known that all K3 surfaces, except those contained in the countable union of hypersurfaces in the moduli space of K3 surfaces satisfy this property. We present a new approach for constructing curves using degenerations, and apply this technique to the above problem. In particular, we prove that there is a Zariski open dense subset in the moduli space of K3 surfaces whose members satisfy the conjecture.

1. Introduction

A folklore conjecture concerning rational curves on K3 surfaces is the following (see [4, Section 13] for more details about the historical development related to this conjecture).

Conjecture 1. Every polarized K3 surface (X, H) over an algebraically closed field contains infinitely many integral rational curves linearly equivalent to some multiple of H.

Mori and Mukai [7] showed (attributed to Bogomorov and Mumford) that every complex polarized K3 surface (X, H) contains at least 1 rational curve which belongs to the linear system $|H|$. One subtle point about this problem is the difference between *generic* and *general*. Here generic means a property held by members in a non-empty Zariski open subset of the moduli space of K3 surfaces. While general means a property held by members in the complement of the countable union of proper Zariski closed subsets (typically hypersurfaces). Using this terminology, one can deduce from Mori and Mukai's argument that a general polarized complex K3 surface (X, H) contains infinitely many irreducible rational curves linearly equivalent to some nH (see [4, Corollary 1.2, Section 13]).

In the late 90's, Chen [2] proved the existence of infinitely many irreducible nodal rational curves on general K3 surfaces. More recently, Bogomolov-Hassett-Tschinkel [1] and Li-Liedtke [6] proved stronger result for general K3 surfaces based on sophisticated arithmetic geometric argument as well as deep geometry of K3 surfaces.

We introduce new technique to construct curves on varieties via degeneration. Our method is essentially elementary, constructive, and does not much depend on the special properties of K3 surfaces. In fact, it can also be used to produce many kinds of holomorphic curves in variety of situations. For example, the same ideas can be applied to construct rational curves on Calabi-Yau quintic 3-folds and holomorphic disks on hypersurfaces in $\mathbb{P}^n$ ([8]), and holomorphic curves on Abelian surfaces ([10]).

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2. Degeneration of K3 surfaces and obstruction

In this section, we recall some calculations in [8]. There we considered a degeneration of a quartic K3 surface

$$xyzw + tf(x, y, z, w) = 0,$$

where x, y, z, w are homogeneous coordinates of $\mathbb{P}^3$, $t \in \mathbb{C}$ is the parameter of the degeneration and f is a generic homogeneous quartic polynomial of x, y, z, w. Let

$$\mathfrak{X} \subset \mathbb{P}^3 \times \mathbb{C}$$

be the variety defined by the above equation (the total space of the degeneration), and X_0 be the central fiber. Let

$$i_0 \colon X_0 \to \mathfrak{X}$$

be the inclusion. The space X_0 is the union of 4 projective planes glued along projective lines:

$$X_0 = \cup_{i=1}^4 \mathbb{P}_i^2$$

Each $\mathbb{P}_i^2$ has a natural structure of a toric variety, in which the lines mentioned above are the toric divisors. Let

$$\ell_1, \dots, \ell_6$$

be these projective lines. Let

$$L = \cup_{i=1}^6 \ell_i$$

be the union of them. Each ℓ_i has 2 distinguished points, which are the triple-intersections of the projective planes. We write by ℓ_i° the complement of these 2 points.

Since f is generic, the total space $\mathfrak{X}$ has 24 singular points, and for each i, 4 of them lie on ℓ_i°. Let

$$\mathcal{S} \subset X_0$$

be the set of these singular points.

We consider the following problem. Let C_0 be a pre-stable curve and

$$\varphi_0 \colon C_0 \to X_0$$

be a stable map. Then the problem is, whether the map φ_0 has a smoothing or not. More precisely, we ask whether there is a family of pre-stable curves

$$\mathcal{C} \to \mathbb{C}$$

such that the fiber over 0 is C_0, and a family of stable maps

$$\Phi \colon \mathcal{C} \to \mathfrak{X}$$

over $\mathbb{C}$ which coincides with φ_0 on C_0.

2.1. **Pre-log curves and pre-smoothable curves.** As is pointed out in [5] (see also [9] in the context of tropical curves), for an immersed stable map $\varphi_0 \colon C_0 \to X_0$, if the image is away from the singularities of $\mathfrak{X}$, it is necessary for φ_0 to satisfy the *pre-log* condition ([9, Definition 4.3]) to solve the above problem. We now recall the pre-log condition.

Definition 2 ([9, Definition 4.1])**.** Let Y be a toric variety. A holomorphic curve $C \subset Y$ is *torically transverse* if it is disjoint from all toric strata of codimension greater than 1. A stable map $\phi \colon C \to Y$ is torically transverse if $\phi^{-1}(intY) \subset C$ is dense and $\phi(C) \subset Y$ is a torically transverse curve. Here $intY$ is the complement of the union of toric divisors.

Let $\pi\colon \mathfrak{Y} \to \mathbb{C}$ be a flat family of varieties such that

- The fibers $Y_t = \pi^{-1}(t)$, $t \neq 0$ are nonsingular irreducible varieties.
- The special fiber Y_0 is a union of toric varieties $Y_0 = \cup_{i=1}^k Y_{0,k}$ such that for any different $i, j \in \{1, \dots, k\}$, the intersection $Y_{0,i} \cap Y_{0,j}$ is a toric stratum of both $Y_{0,i}$ and $Y_{0,j}$.
- For each point $p \in Y_0$, there is an analytic neighborhood $U \subset \mathfrak{Y}$ with the property that U is analytically isomorphic to an open subset of a toric variety and the restriction $U \to \mathbb{C}$ of π to U has a natural (up to isomorphisms) structure of an open subset of a degeneration of toric varieties whose central fiber is $U \cap Y_0$.

In other words, any point on Y_0 has a neighborhood modeled on a toric degeneration of toric varieties.

Definition 3. Let C_0 be a prestable curve. A *pre-log curve* on Y_0 is a stable map $\varphi_0\colon C_0 \to Y_0$ with the following properties.

(i) For any component $Y_{0,i}$ of Y_0, the restriction $C \times_{Y_0} Y_{0,i} \to Y_{0,i}$ is a torically transverse stable map.

(ii) Let $p \in C_0$ be a point which is mapped to the singular locus of Y_0. Then C_0 has a node at p, and φ_0 maps the two branches $(C_0', p), (C_0'', p)$ of C_0 at P to different irreducible components $Y_{0,i'}, Y_{0,i''} \subset Y_0$. Moreover, if w' is the intersection index of the restriction $(C_0', p) \to (Y_{0,i'}, D')$ with the toric divisor $D' \subset Y_{0,i'}$, and w'' accordingly for $(C_0'', p) \to (Y_{0,i''}, D'')$, then $w' = w''$.

Let $\mathcal{T}$ be the singular locus of the total space $\mathfrak{Y}$. Note that $\mathcal{T}$ is contained in the singular locus (in particular, in the union of toric strata of positive codimension) of Y_0. Then we generalize the notion of pre-log curves to this case.

As we noted above, if the image of a map $\varphi_0\colon C_0 \to Y_0$ intersects a singular point of Y_0 not contained in $\mathcal{T}$, the map φ_0 must satisfy the pre-log condition above for the existence of smoothings. In particular, if a smooth point of C_0 is mapped to a toric boundary of a component of Y_0, then it must be mapped into the singular subset $\mathcal{T}$. We record this point as a lemma.

Lemma 4. *Let φ_0 be as above. Also assume the map φ_0 admits a smoothing Φ in the above sense. Let $x \in C_0$ be a smooth point which is mapped to a toric boundary of a component of Y_0 by φ_0. Then, the image $\varphi_0(x)$ must be contained in the set $\mathcal{T}$.* □

We give a slight generalization of the notion of pre-log curves, to account for the singularity of the total space.

Definition 5. We call a map $\varphi_0\colon C_0 \to Y_0$ *pre-smoothable* if it satisfies the following conditions.

- The set $\varphi_0^{-1}(\mathcal{T})$ is a finite set consisting of regular points of C_0.
- The restriction of the map φ_0 to $C_0 \setminus \varphi_0^{-1}(\mathcal{T})$ is a pre-log curve.

In our situation, the degeneration $\mathfrak{X} \to \mathbb{C}$ satisfies the above condition for $\pi\colon \mathfrak{Y} \to \mathbb{C}$. By Lemma 4, if the map φ_0 admits a smoothing, then it must be pre-smoothable. The curves we study are rather simple ones among pre-smoothable curves.

Definition 6. Let $\varphi_0 : C_0 \to X_0$ be a pre-smoothable curve. If the map φ_0 satisfies the condition

- the image of any irreducible component of C_0 by φ_0 is a curve of degree 1 in some component of $X_0 = \cup_{i=1}^4 \mathbb{P}_i^2$,

then we call φ_0 *simply pre-smoothable.*

2.2. **Log structures on a neighborhood of a curve and log normal sheaves of simply pre-smoothable curves.** Now we recall the calculation in [8] of the sheaves which control the deformation of φ_0. This uses the notion of log structures. Let $\mathbb{P}_i^2$ be a component of X_0. Let ℓ be one of the toric divisors of $\mathbb{P}_i^2$. Let $x \in \ell$ be a point on ℓ° which is not contained in the set $\mathcal{S}$. Then there is a neighbourhood of the point x in $\mathfrak{X}$ which is isomorphic to a neighbourhood of the origin of the variety defined by the equation

$$z_1 z_2 + t = 0$$

in $\mathbb{C}^3 \times \mathbb{C}$ with coordinates (z_1, z_2, z_3, t). This variety has a natural structure of a toric variety over $\mathbb{C} = \mathit{Spec}\, \mathbb{C}[t]$ (which is also seen as a toric variety) and we put a natural log structure coming from the toric structure.

On the other hand, if $x \in \ell$ lies in the set $\mathcal{S}$, then $\mathfrak{X}$ is locally isomorphic to a neighborhood of the origin of the variety defined by the equation

$$z_1 z_2 + t z_3 = 0.$$

This also has a natural toric structure, and we put a log structure coming from it.

Remark 7. *It is important to notice that we do not need to put a log structure on whole $\mathfrak{X}$, but only around the curve $\varphi_0(C_0)$. The local log structures above may not extend globally to $\mathfrak{X}$, but it does not matter in our argument. We put log structures around the intersection of $\varphi_0(C_0)$ and the toric divisors of the components of X_0 in the way described above, and around other points of $\varphi_0(C_0)$, we put a strict log structure pulled back from the standard log structure on $\mathbb{C}$ as a toric variety. This is possible when the subvariety of $\mathfrak{X}$ locally defined by the equation $z_3 = 0$ in the above notation around the singular locus of $\mathfrak{X}$ intersects the image $\varphi_0(C_0)$ transversally. For general f (the defining polynomial of the degenerating $K3$ surface), we can always assume this condition in the argument below.*

We put a log structure on C_0 over $\mathit{Spec}\,\mathbb{C}[t]$ which is described in [9, Proposition 7.1] on neighborhoods of the nodes, and in [8, Section 2.1] on neighborhoods of points in the inverse image of the set $\mathcal{S}$, and put a strict log structure over $\mathit{Spec}\,\mathbb{C}[t]$ otherwise. Then the map φ_0 can be equipped with a structure of a map between log schemes so that the composition with the projection to $\mathit{Spec}\,\mathbb{C}[t]$ is log smooth in an essentially unique way.

Log smooth deformations of φ_0 are controlled by its log normal sheaf $\mathcal{N}_{\varphi_0} = \varphi_0^* \Theta_U / \Theta_{C_0}$. Here U is a neighborhood of $\varphi_0(C_0)$ in $\mathfrak{X}$ (see Remark 7 above), Θ_U and Θ_{C_0} are the log tangent sheaves with respect to the above log structures. In [8, Subsection 2.1.1], we computed this sheaf for the case of a line on X_0. The computation for general pre-smoothable curves with arbitrary genus is done by gluing the restrictions of $\mathcal{N}_{\varphi_0}$ to irreducible components of C_0, which reduces the computation to the case of a line above. So the computation is essentially the same as that in [8, Subsection 2.1.1] and we obtain the following.

Proposition 8. *Let $\varphi_0\colon C_0 \to X_0$ be a simply pre-smoothable curve. Then the log normal sheaf $\mathcal{N}_{\varphi_0}$ is an invertible sheaf, and if $C_{0,i}$ is a component of C_0, then the restriction of $\mathcal{N}_{\varphi_0}$ to $C_{0,i}$ is isomorphic to*

$$\mathcal{N}_{C_{0,i}/\mathbb{P}^2}(-\sum_i x_i),$$

here $\mathcal{N}_{C_{0,i}/\mathbb{P}^2}$ is the usual (non log) normal sheaf of $\varphi_0|_{C_{0,i}}$ as a map to a component ($\cong \mathbb{P}^2$) of X_0, and the set $\{x_i\}$ is inverse image of the intersection $\varphi_0|_{C_{0,i}}(C_{0,i}) \cap \mathcal{S}$. □

The Zariski tangent space of the space of log-smooth deformations of φ_0 is given by

$$H^0(C_0, \mathcal{N}_{\varphi_0})$$

and the obstruction is given by

$$H^1(C_0, \mathcal{N}_{\varphi_0}).$$

Lemma 9. *Let $\varphi_0\colon C_0 \to X_0$ be a simply pre-smoothable curve. Then we have the isomorphism*

$$H^1(C_0, \mathcal{N}_{\varphi_0}) \cong \mathbb{C}.$$

□

Remark 10. *Note that Lemma 9 applies regardless of the genus of C_0. Also note that each component of C_0 is a rational curve with 3 special points (that is, the points mapped to the toric divisors of the components of X_0), and so belongs to the unique isomorphism class. Given 2 simply pre-smoothable curves $\varphi_0\colon C_0 \to X_0$ and $\varphi_0'\colon C_0' \to X_0$, consider components $C_{0,v}$ and $C_{0,v'}$ of C_0 and C_0' which are mapped to the same component of X_0. Then under this isomorphism, a generator of the groups $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})$ and $H^0(C_0', \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0'})$ restricts to the same meromorphic 1-form (up to a constant multiple) on $C_{0,v}$ and $C_{0,v'}$. Here the curves C_0, C_0' need not be isomorphic (in particular, the genera of them may be different).*

We also need to consider the case where the stable map is not pre-smoothable. The main example of such a curve is a limit of degenerate higher genus curves. An important point is that the image of such a curve is the same as the image of a map from a degenerate rational curve. Another important point is that although such a map may not fit in the formalism of log smooth deformation theory, the generator of $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})$ makes sense.

3. Calculation of the obstruction in the simplest case and the existence of the smoothing

3.1. **Remarks on calculation of the obstruction.** Given a simply pre-smoothable curve $\varphi_0 : C_0 \to X_0$, we calculated the generator of the dual obstruction class $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})$ in Lemma 9. We write this generator by η. On the other hand, the obstruction class in $H^1(C_0, \mathcal{N}_{\varphi_0})$ is calculated as follows.

Namely, we can take a suitable covering $\{U_i\}_i$ of C_0 so that on U_i there is a local lift of the map φ_0. Such a covering exists since the projection $\mathfrak{X} \to \mathbb{C}$ is log smooth. These local lifts are torsor over the group of sections of $\mathcal{N}_{\varphi_0}$, and so the difference of the lifts on the intersections $U_i \cap U_j$ defines a $\mathcal{N}_{\varphi_0}$-valued Čech 1-cocycle. This is the obstruction class in $H^1(C_0, \mathcal{N}_{\varphi_0})$.

It is clear that the class does not depend on the choices of local lifts, since by construction the cocycles defined by two lifts differ only by coboundary. It is not easy in general to directly identify the obstruction cohomology class (in particular, determine whether it is zero or nonzero) from the presentation as a Čech cohomology class. More efficient way is to calculate the coupling between $H^1(C_0, \mathcal{N}_{\varphi_0})$ and its dual $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})$, and it can be reduced to the calculation of appropriate residues at the nodes, see [8].

In this note we do not really need to do the actual calculation of the obstruction class since we can show that it vanishes by another reason. However, since the actual calculation will help understand the later argument, we perform a calculation of the obstruction in a simple case in the next subsection.

3.2. **Degenerate curves of degree 4 and their obstructions.** Our construction of rational curves on a quartic K3 surface will be done by deforming a rational curve on the degenerate space X_0 to general fibers X_t, $t \neq 0$. So the starting point is the construction of a rational curve on X_0. This can be done by considering intersections with other particular surfaces.

Let us start from the simpler case of general curves of degree 4. Such a curve on a quartic $K3$ surface or its degeneration is obtained by taking the intersection with a general hyperplane H. In particular, we assume that the image of the map $\varphi_0 \colon C_0 \to X_0$ does not intersect the singular locus $\mathcal{S}$ of $\mathfrak{X}$. Thus, the map φ_0 is a pre-log curve in the sense of [9]. On the degenerate space X_0, the intersection will be the union of 4 lines (see Figure 1).

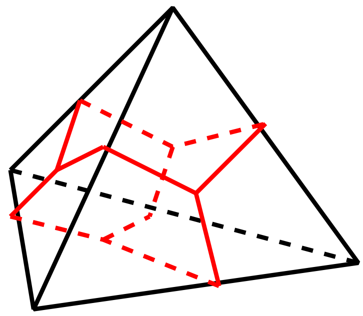

FIGURE 1. Picture of a general curve of degree 4 on X_0. Here X_0 is the union of 4 $\mathbb{P}^2$s. The tetrahedron is the intersection complex of X_0, so each triangle corresponds to $\mathbb{P}^2$, and the triangle can be regarded as the moment polytope of $\mathbb{P}^2$. On the other hand, the graph on the tetrahedron is the *dual* intersection graph of a curve on X_0.

Now we turn to the construction of degenerate rational curves. As we mentioned in Section 2, the total space $\mathfrak{X}$ of the degeneration contains the singular set $\mathcal{S}$ consisting of 24 points. On the tetrahedron of Figure 2, every edge contains 4 of these singular points.

If we choose a hyperplane H so that H intersects 3 singular points which are not contained in a single component of X_0, we can regard the intersection of H and X_0 as the image of a map from a nodal rational curve, and by Lemma 4, we can ask whether this map can be lifted to a non-zero fiber X_t, $t \neq 0$, giving a rational curve there, see Figure 3. Let us write by

$$\psi_0 \colon B_0 \to X_0$$

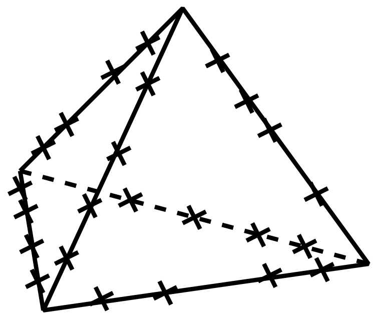

FIGURE 2. The cross marks mean the singular points of the total space $\mathfrak{X}$ of the degeneration.

the stable map from a nodal rational curve to X_0 obtained in this way.

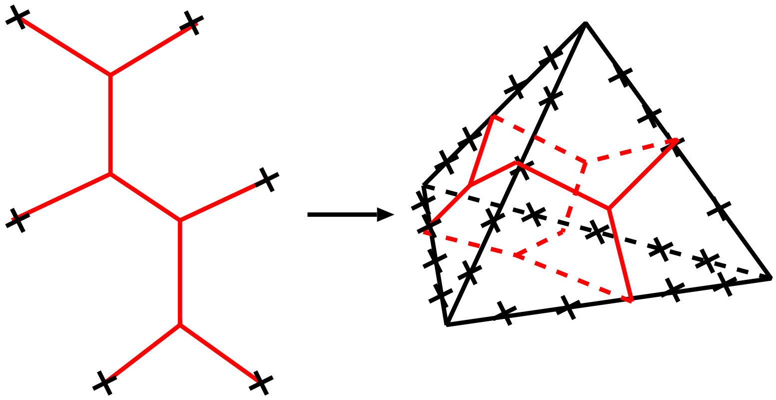

FIGURE 3. Picture of a degenerate rational curve $\psi_0(B_0)$ on X_0. The picture on the left is the dual intersection graph of the nodal rational curve which is the domain of the map ψ_0. The cross marks on the graph corresponds to regular points on the rational curve mapped to the singular locus of $\mathfrak{X}$.

On the other hand, we can also choose a family of hyperplanes H_s, $s \in D$ parametrized by a small disk D around the origin of $\mathbb{C}$ whose central fiber H_0 is the hyperplane H chosen above. The intersection between $H_s \times \mathbb{C} \subset \mathbb{P}^3 \times \mathbb{C}$ and $\mathfrak{X}$ gives a family of degenerating families of curves of degree 4. In particular, the intersection between $H_0 \times \mathbb{C}$ and $\mathfrak{X}$ also gives a degenerating family whose general fiber is a smooth curve of genus 3. Thus, the image $\psi_0(B_0)$ can also be seen as a degenerate genus 3 curve. We write it by

$$\varphi_{0,0} : C_{0,0} \to X_0,$$

see Figure 4. We also write by

$$\varphi_{0,s} : C_{0,s} \to X_0$$

the degenerate curve obtained as the intersection between H_s and X_0. Note that all the domain curves $C_{0,s}$ are in fact isomorphic, including $C_{0,0}$.

Compared with the map $\psi_0 \colon B_0 \to X_0$ above, the map $\varphi_{0,0}$ has the different domain curve, but their images are the same. We do not calculate the obstruction cohomology group of the map $\varphi_{0,0}$. However, the generator η of the obstruction to lift the maps $\varphi_{0,s}$,

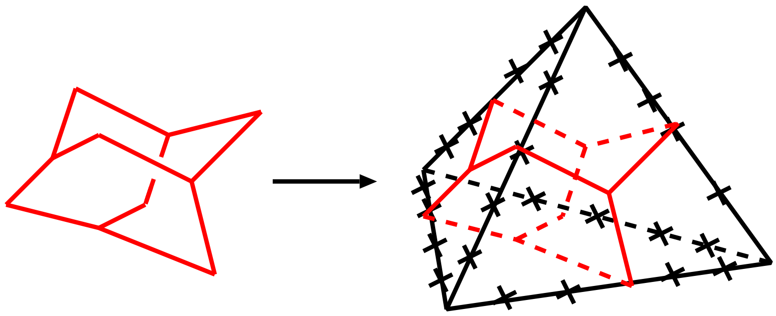

Figure 4. The intersection $H \cap X_0$ seen as a degenerate genus 3 curve $\varphi_{0,0}(C_{0,0})$.

$s \neq 0$ calculated in Lemma 9 can be naturally defined on $C_{0,0}$, since all $C_{0,s}$ are isomorphic (see Remark 10).

Also, we can construct a family of obstruction Čech 1-cocycles on $C_{0,s}$ by using a family version of local lifts considered in Subsection 3.1, parametrized by s. These cocycles couple with the above generator η, and the result is an analytic function of s. This analytic function is identically 0.

The important point is that although the domain curves are different, the generator η of the obstruction class on $C_{0,s}$ and B_0 are identical after partial normalizaton of $C_{0,s}$ (see Remark 10).

Moreover, in the calculation of the obstruction in Subsection 3.1, clearly it is possible to choose the covering $\{U_i\}$ of $C_{0,s}$ so that for any node of the image $\varphi_{0,s}(C_{0,s})$, there is only one U_i which contains the inverse image of that node. The covering $\{U_i\}$ gives rise to a covering $\{\tilde{U}_i\}$ of B_0 in an obvious way. That is, if $p : B_0 \to C_0$ is the partial normalization, then $\tilde{U}_i$ is the inverse image $p^{-1}(U_i)$. Note that some $\tilde{U}_i$ has 2 components.

Choosing the covering in this way, the Čech 1-cocycle defined by the local lifts on each covering is supported on open subsets of $C_{0,s}$ or B_0 which do not contain a node. When there is a local lift of $C_{0,0}$ on a neighborhood of each node such that it can also be seen as a local lift of B_0, the following holds. Namely, if the cohomology class of the Čech 1-cocycle defined by the local lifts on $C_{0,0}$ is 0, then the cohomology class defined by the same Čech 1-cocycle on B_0 is also 0, since if the class is given as the coboundary of a Čech 0-cochain $\{\eta_i\}$ on $C_{0,0}$ associated to the covering $\{U_i\}$, it naturally determines a Čech 0-cochain on B_0 associated to the covering $\{\tilde{U}_i\}$, and its coboundary is the given Čech 1-cocycle on B_0.

The condition that the cohomology class of the Čech 1-cocycle defined by the local lifts on $C_{0,0}$ is 0 can be assured when such lifts can be extended to a family of lifts on $C_{0,s}$, since the corresponding cohomology classes on $C_{0,s}$, $s \neq 0$ is 0 as we noted above.

Based on this observation, we can prove the following.

Proposition 11. *The map ψ_0 has smoothings up to any order.* □

4. Construction of infinitely many rational curves

Now we turn to the construction of infinitely many rational curves on generic quartic K3 surfaces. The idea is the same as the previous section. Namely:

- Construct a degenerate curve on the special fiber X_0 of the degeneration $\mathfrak{X}$. In general, such a curve is seen as a degeneration of smooth curves of high genus. But there are some particular curves which are seen as degenerations of nodal rational curves.
- Compute the obstruction cohomology classes to deform these degenerate curves. An important point is that although the genus is different in general, the obstruction classes can be identified in a natural way (see Remark 10).
- Compute the actual obstruction. In the case of higher genus curves, one sees that the obstruction automatically vanishes since one can construct actual families of curves which degenerate to the given degenerate curves. Then the case of a degenerate rational curve is a limit of them, and since the obstruction can be seen as an analytic function of suitable parameters which is identically 0 when these parameters are nonzero, the obstruction vanishes also in the case of the rational curve.

The construction of degenerate curves in the case of higher genus curves is simple. If we want a degeneration of general curves of degree $4m$, then take a product

$$\prod_{i=1}^{m}(a_i x + b_i y + c_i z + d_i w)$$

of m general linear functions. This gives a degenerate curve on X_0 which is a nodal union of $4m$ rational curves. Then take a general deformation of this polynomial over $\mathbb{C}[t]$, and consider the intersection of its zero and $\mathfrak{X}$. In this way one obtains a family of smooth curves which degenerates to the above degenerate curve.

Now we turn to the construction of degenerate rational curves. We start from the degree 4 rational curve $\psi_0 : C_0 \to X_0$ in the previous section. We choose a node, see Figure 5. Note that in the case of rational curves, we think that the intersection points between the curve and the singular locus of $\mathfrak{X}$ are not nodes of the curve, see Figure 3. So the chosen point is away from the singular locus of $\mathfrak{X}$.

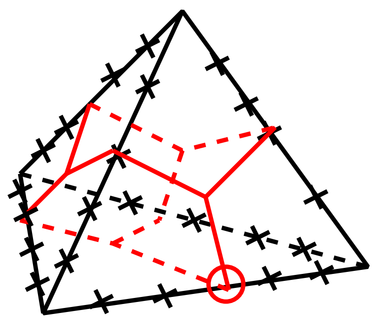

FIGURE 5. The node circled is chosen.

Then consider another degenerate curve $\sigma_0 : D_0 \to X_0$, whose image shares the chosen node with $\psi_0(C_0)$, and intersects the singular locus of $\mathfrak{X}$ at 2 points, see Figure 6.

This is a degenerate genus 1 curve. Then take a finite covering of the domain curve D_0. In the tropicalized picture, this corresponds to cut the loop of the graph and graft some copies of it, and make a loop again, see Figure 7. On the side of holomorphic curves, this is again a degenerate genus 1 curve.

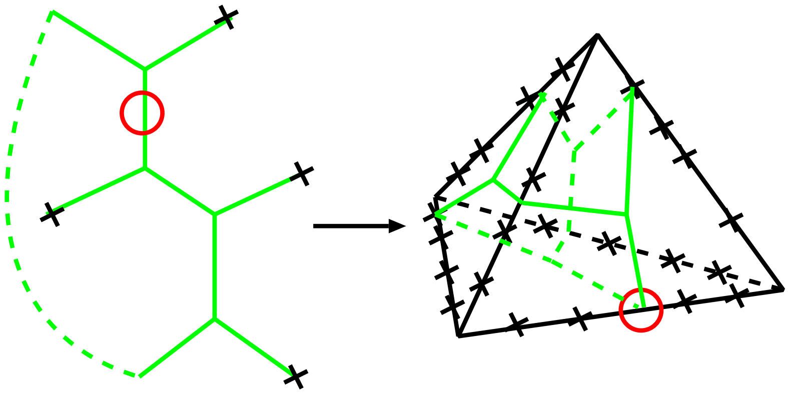

FIGURE 6. Tropicalized picture of a degenerate genus 1 curve on X_0 and its domain D_0 of the map. The ends of the edges connected by the dotted curve are glued so these edges are merged into 1 edge, which corresponds to a node of the holomorphic curve.

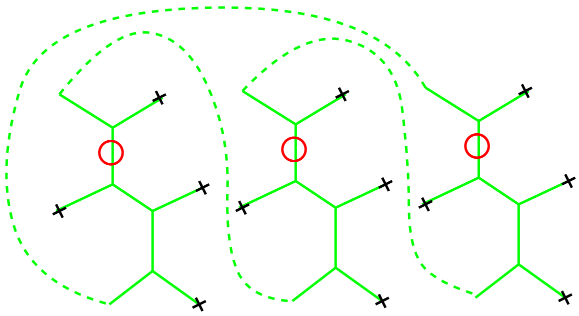

FIGURE 7. 3-fold covering of D_0. The edges connected by dotted curves are glued into 1 edge, each of which corresponds to a node on the side of holomorphic curves. The nodes corresponding to the edges marked by circles intersect the degenerate rational curve.

Let us write by $\tilde{D}_0$ the resulting genus 1 curve. To obtain a degenerate rational curve, we cut the curve $\tilde{D}_0$ at 2 of its intersection with the degenerate rational curve C_0 at the chosen node. This divides the curve $\tilde{D}_0$ into 2 pieces. Then similarly cut the degenerate rational curve at the corresponding node, and graft parts of the degenerate rational curve to one of the components of the divided $\tilde{D}_0$, see Figure 8, 9 and 10.

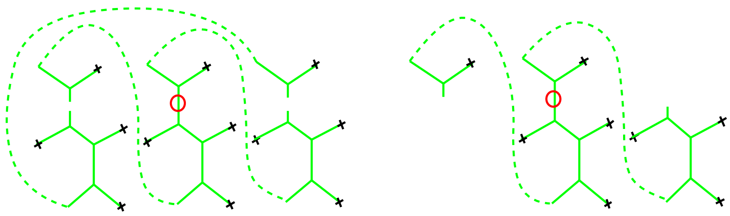

FIGURE 8. Cut $\tilde{D}_0$ at 2 nodes (picture on the left), and pick one of the resulting connected components (picture on the right).

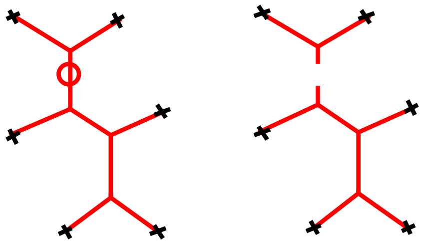

FIGURE 9. Similarly, cut the degenerate rational curve into 2 pieces.

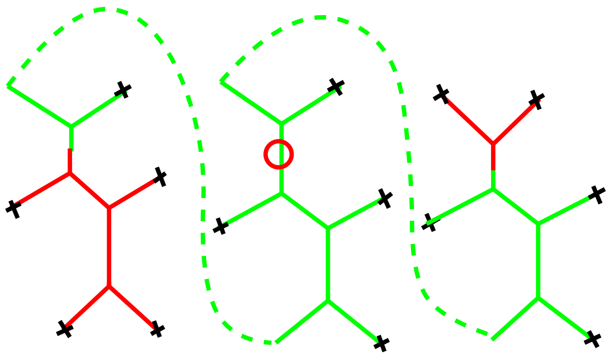

FIGURE 10. Graft the pieces of the rational curve to a part of divided $\tilde{D}_0$. The result is a degenerate rational curve $D_{0,r}$.

When the degree of the covering of D_0 we first take is r, we write the resulting rational curve by this process by $D_{0,r}$. We have an obvious map

$$\psi_{0,r} : D_{0,r} \to X_0$$

induced from φ_0 and σ_0. The image of $\psi_{0,r}$ is a curve of degree $4r$.

Similarly to Lemma 9, the obstruction cohomology classes of these curves are 1 dimensional, and the generators of them can be identified in a natural way. Then Proposition 11 can also be extended to higher degree curves.

Theorem 12. *The degenerate rational curve $\psi_{0,r}$ has smoothings up to any order.* □

Since the defining polynomial f of a $K3$ surface can be taken generic, we have the following.

Theorem 13. *The moduli space of quartic K3 surfaces contains a Zariski open subset whose members contain infinitely many irreducible rational curves.* □

5. General K3 surfaces

The argument so far can be extended to more general K3 surfaces provided nice degenerations are given. In fact, such degenerations are constructed in [3]. Namely, a K3 surface $X \subset \mathbb{P}^n$ degenerates into $Q_1 \cup Q_2$, where

$$\begin{cases} Q_i \cong \mathbb{P}^1 \times \mathbb{P}^1, & n\text{: odd} \\ Q_i \cong \mathbb{F}_1, & n\text{: even.} \end{cases}$$

The intersection $Q_1 \cap Q_2$ is an anti-canonical divisor (elliptic curve).

Using this degeneration, we can construct degenerate rational and higher genus curves by the same method as in the quartic case. Namely, a rational curve can be taken in the central fiber in the following way:

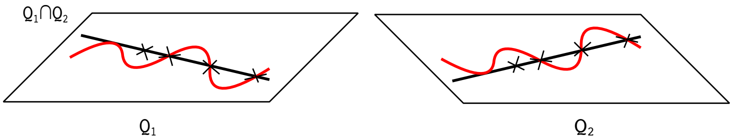

FIGURE 11. The straight line is the intersection $Q_1 \cap Q_2$ (elliptic curve) and the cross marks on it are the singular points of the total space.

An elliptic curve can be taken in the central fiber in the following way:

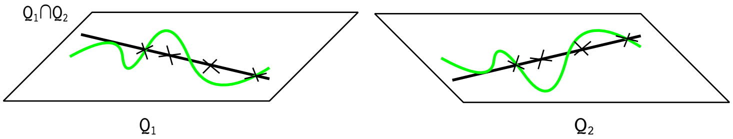

FIGURE 12

Higher degree rational curves are obtained by grafting:

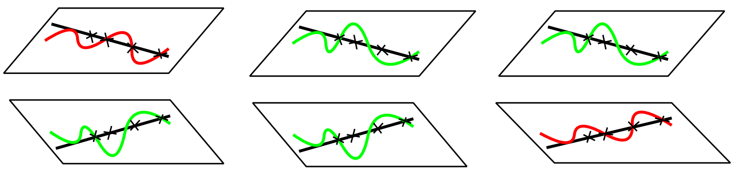

FIGURE 13

These curves also have 1 dimensional obstructions, and the vanishing of them follows from the trivial vanishing of the obstruction to deform higher genus curves as before. Thus, we have the following.

Theorem 14. *For each component of the moduli space of polarized K3 surfaces, there is a Zariski open subset whose members contain infinitely many irreducible rational curves.* □

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