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DEGENERATION AND CURVES ON K3 SURFACES

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Abstract. We study curves on K3 surfaces. A folklore conjecture concerning rational curves on K3 surfaces states that all K3 surfaces contain infinite number of irreducible rational curves. It is known that all K3 surfaces, except those contained in the countable union of hypersurfaces in the moduli space of K3 surfaces satisfy this property. We present a new approach for constructing curves using degenerations, and apply this technique to the above problem. In particular, we prove that there is a Zariski open dense subset in the moduli space of K3 surfaces whose members satisfy the conjecture.

1. Introduction

A folklore conjecture concerning rational curves on K3 surfaces is the following (see [4, Section 13] for more details about the historical development related to this conjecture).

Conjecture 1. Every polarized K3 surface \((X, H)\) over an algebraically closed field contains infinitely many integral rational curves linearly equivalent to some multiple of \(H\).

Mori and Mukai [7] showed (attributed to Bogomorov and Mumford) that every complex polarized K3 surface \((X, H)\) contains at least 1 rational curve which belongs to the linear system \(|H|\). One subtle point about this problem is the difference between generic and general. Here generic means a property held by members in a non-empty Zariski open subset of the moduli space of K3 surfaces. While general means a property held by members in the complement of the countable union of proper Zariski closed subsets (typically hypersurfaces). Using this terminology, one can deduce from Mori and Mukai’s argument that a general polarized complex K3 surface \((X, H)\) contains infinitely many irreducible rational curves linearly equivalent to some \(nH\) (see [4, Corollary 1.2, Section 13]).

In the late 90’s, Chen [2] proved the existence of infinitely many irreducible nodal rational curves on general K3 surfaces. More recently, Bogomolov-Hassett-Tschinkel [1] and Li-Liedtke [6] proved stronger result for general K3 surfaces based on sophisticated arithmetic geometric argument as well as deep geometry of K3 surfaces.

We introduce new technique to construct curves on varieties via degeneration. Our method is essentially elementary, constructive, and does not much depend on the special properties of K3 surfaces. In fact, it can also be used to produce many kinds of holomorphic curves in variety of situations. For example, the same ideas can be applied to construct rational curves on Calabi-Yau quintic 3-folds and holomorphic disks on hypersurfaces in \(\mathbb{P}^n\) ([8]), and holomorphic curves on Abelian surfaces ([10]).

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2. Degeneration of K3 surfaces and obstruction

In this section, we recall some calculations in [8]. There we considered a degeneration of a quartic K3 surface
\[ xyzw + tf(x, y, z, w) = 0, \]
where \( x, y, z, w \) are homogeneous coordinates of \( \mathbb{P}^3 \), \( t \in \mathbb{C} \) is the parameter of the degeneration and \( f \) is a generic homogeneous quartic polynomial of \( x, y, z, w \). Let
\[ X \subset \mathbb{P}^3 \times \mathbb{C} \]
be the variety defined by the above equation (the total space of the degeneration), and \( X_0 \) be the central fiber. Let
\[ i_0: X_0 \to X \]
be the inclusion. The space \( X_0 \) is the union of 4 projective planes glued along projective lines:
\[ X_0 = \bigcup_{i=1}^4 \mathbb{P}_i^2 \]
Each \( \mathbb{P}_i^2 \) has a natural structure of a toric variety, in which the lines mentioned above are the toric divisors. Let
\[ \ell_1, \ldots, \ell_6 \]
be these projective lines. Let
\[ L = \bigcup_{i=1}^6 \ell_i \]
be the union of them. Each \( \ell_i \) has 2 distinguished points, which are the triple-intersections of the projective planes. We write by \( \ell_i^0 \) the complement of these 2 points.

Since \( f \) is generic, the total space \( X \) has 24 singular points, and for each \( i \), 4 of them lie on \( \ell_i^0 \). Let
\[ S \subset X_0 \]
be the set of these singular points.

We consider the following problem. Let \( C_0 \) be a pre-stable curve and \( \varphi_0: C_0 \to X_0 \) be a stable map. Then the problem is, whether the map \( \varphi_0 \) has a smoothing or not. More precisely, we ask whether there is a family of pre-stable curves \( \mathcal{C} \to \mathbb{C} \) which coincides with \( \varphi_0 \) on \( C_0 \).

2.1. Pre-log curves and pre-smoothable curves. As is pointed out in [5] (see also [9] in the context of tropical curves), for an immersed stable map \( \varphi_0: C_0 \to X_0 \), if the image is away from the singularities of \( X \), it is necessary for \( \varphi_0 \) to satisfy the pre-log condition ([9, Definition 4.3]) to solve the above problem. We now recall the pre-log condition.

**Definition 2** ([9, Definition 4.1]). Let \( Y \) be a toric variety. A holomorphic curve \( C \subset Y \) is torically transverse if it is disjoint from all toric strata of codimension greater than 1. A stable map \( \phi: C \to Y \) is torically transverse if \( \phi^{-1}(\text{int}Y) \subset C \) is dense and \( \phi(C) \subset Y \) is a torically transverse curve. Here \( \text{int}Y \) is the complement of the union of toric divisors.
Let $\pi: \mathcal{Y} \to \mathbb{C}$ be a flat family of varieties such that
- The fibers $Y_t = \pi^{-1}(t)$, $t \neq 0$ are nonsingular irreducible varieties.
- The special fiber $Y_0$ is a union of toric varieties $Y_0 = \bigcup_{i=1}^{k} Y_{0,i}$ such that for any different $i, j \in \{1, \ldots, k\}$, the intersection $Y_{0,i} \cap Y_{0,j}$ is a toric stratum of both $Y_{0,i}$ and $Y_{0,j}$.
- For each point $p \in Y_0$, there is an analytic neighborhood $U \subset \mathcal{Y}$ with the property that $U$ is analytically isomorphic to an open subset of a toric variety and the restriction $U \rightarrow \mathbb{C}$ of $\pi$ to $U$ has a natural (up to isomorphisms) structure of an open subset of a degeneration of toric varieties whose central fiber is $U \cap Y_0$.

In other words, any point on $Y_0$ has a neighborhood modeled on a toric degeneration of toric varieties.

**Definition 3.** Let $C_0$ be a prestable curve. A pre-log curve on $Y_0$ is a stable map $\varphi_0: C_0 \rightarrow Y_0$ with the following properties.

(i) For any component $Y_{0,i}$ of $Y_0$, the restriction $C \times_Y Y_{0,i} \rightarrow Y_{0,i}$ is a torically transverse stable map.

(ii) Let $p \in C_0$ be a point which is mapped to the singular locus of $Y_0$. Then $C_0$ has a node at $p$, and $\varphi_0$ maps the two branches $(C_0', p), (C_0'', p)$ of $C_0$ at $P$ to different irreducible components $Y_{0,i'}, Y_{0,i''} \subset Y_0$. Moreover, if $w'$ is the intersection index of the restriction $(C_0', p) \rightarrow (Y_{0,i'}, D')$ with the toric divisor $D' \subset Y_{0,i'}$, and $w''$ accordingly for $(C_0'', p) \rightarrow (Y_{0,i''}, D'')$, then $w' = w''$.

Let $T$ be the singular locus of the total space $\mathcal{Y}$. Note that $T$ is contained in the singular locus (in particular, in the union of toric strata of positive codimension) of $Y_0$. Then we generalize the notion of pre-log curves to this case.

As we noted above, if the image of a map $\varphi_0: C_0 \rightarrow Y_0$ intersects a singular point of $Y_0$ not contained in $T$, the map $\varphi_0$ must satisfy the pre-log condition above for the existence of smoothings. In particular, if a smooth point of $C_0$ is mapped to a toric boundary of a component of $Y_0$, then it must be mapped into the singular subset $T$. We record this point as a lemma.

**Lemma 4.** Let $\varphi_0$ be as above. Also assume the map $\varphi_0$ admits a smoothing $\Phi$ in the above sense. Let $x \in C_0$ be a smooth point which is mapped to a toric boundary of a component of $Y_0$ by $\varphi_0$. Then, the image $\varphi_0(x)$ must be contained in the set $T$. $\square$

We give a slight generalization of the notion of pre-log curves, to account for the singularity of the total space.

**Definition 5.** We call a map $\varphi_0: C_0 \rightarrow Y_0$ pre-smoothable if it satisfies the following conditions.

- The set $\varphi_0^{-1}(T)$ is a finite set consisting of regular points of $C_0$.
- The restriction of the map $\varphi_0$ to $C_0 \setminus \varphi_0^{-1}(T)$ is a pre-log curve.

In our situation, the degeneration $\mathcal{X} \rightarrow \mathbb{C}$ satisfies the above condition for $\pi: \mathcal{Y} \rightarrow \mathbb{C}$. By Lemma 4, if the map $\varphi_0$ admits a smoothing, then it must be pre-smoothable. The curves we study are rather simple ones among pre-smoothable curves.
Definition 6. Let \( \varphi_0 : C_0 \to X_0 \) be a pre-smoothable curve. If the map \( \varphi_0 \) satisfies the condition

- the image of any irreducible component of \( C_0 \) by \( \varphi_0 \) is a curve of degree 1 in some component of \( X_0 = \bigcup_{i=1}^{d} \mathbb{P}^2_i \),

then we call \( \varphi_0 \) simply pre-smoothable.

2.2. Log structures on a neighborhood of a curve and log normal sheaves of simply pre-smoothable curves. Now we recall the calculation in [8] of the sheaves which control the deformation of \( \varphi_0 \). This uses the notion of log structures. Let \( \mathbb{P}^2_i \) be a component of \( X_0 \). Let \( \ell \) be one of the toric divisors of \( \mathbb{P}^2_i \). Let \( x \in \ell \) be a point on \( \ell \) which is not contained in the set \( S \). Then there is a neighbourhood of the point \( x \) in \( X \) which is isomorphic to a neighbourhood of the origin of the variety defined by the equation

\[ z_1 z_2 + t = 0 \]

in \( \mathbb{C}^3 \times \mathbb{C} \) with coordinates \((z_1, z_2, z_3, t)\). This variety has a natural structure of a toric variety over \( \mathbb{C} = \text{Spec} \mathbb{C}[t] \) (which is also seen as a toric variety) and we put a natural log structure coming from the toric structure.

On the other hand, if \( x \in \ell \) lies in the set \( S \), then \( X \) is locally isomorphic to a neighborhood of the origin of the variety defined by the equation

\[ z_1 z_2 + tz_3 = 0. \]

This also has a natural toric structure, and we put a log structure coming from it.

Remark 7. It is important to notice that we do not need to put a log structure on whole \( X \), but only around the curve \( \varphi_0(C_0) \). The local log structures above may not extend globally to \( X \), but it does not matter in our argument. We put log structures around the intersection of \( \varphi_0(C_0) \) and the toric divisors of the components of \( X_0 \) in the way described above, and around other points of \( \varphi_0(C_0) \), we put a strict log structure pulled back from the standard log structure on \( C \) as a toric variety. This is possible when the subvariety of \( X \) locally defined by the equation \( z_3 = 0 \) in the above notation around the singular locus of \( X \) intersects the image \( \varphi_0(C_0) \) transversally. For general \( f \) (the defining polynomial of the degenerating K3 surface), we can always assume this condition in the argument below.

We put a log structure on \( C_0 \) over \( \text{Spec} \mathbb{C}[t] \) which is described in [9, Proposition 7.1] on neighborhoods of the nodes, and in [8, Section 2.1] on neighborhoods of points in the inverse image of the set \( S \), and put a strict log structure over \( \text{Spec} \mathbb{C}[t] \) otherwise. Then the map \( \varphi_0 \) can be equipped with a structure of a map between log schemes so that the composition with the projection to \( \text{Spec} \mathbb{C}[t] \) is log smooth in an essentially unique way.

Log smooth deformations of \( \varphi_0 \) are controlled by its log normal sheaf \( \mathcal{N}_{\varphi_0} = \varphi_0^* \Theta_U / \Theta_{C_0} \). Here \( U \) is a neighborhood of \( \varphi_0(C_0) \) in \( X \) (see Remark 7 above), \( \Theta_U \) and \( \Theta_{C_0} \) are the log tangent sheaves with respect to the above log structures. In [8, Subsection 2.1.1], we computed this sheaf for the case of a line on \( X_0 \). The computation for general pre-smoothable curves with arbitrary genus is done by gluing the restrictions of \( \mathcal{N}_{\varphi_0} \) to irreducible components of \( C_0 \), which reduces the computation to the case of a line above. So the computation is essentially the same as that in [8, Subsection 2.1.1] and we obtain the following.
Proposition 8. Let $\varphi_0 : C_0 \to X_0$ be a simply pre-smoothable curve. Then the log normal sheaf $N_{\varphi_0}$ is an invertible sheaf, and if $C_{0,i}$ is a component of $C_0$, then the restriction of $N_{\varphi_0}$ to $C_{0,i}$ is isomorphic to

$$N_{C_{0,i}/\mathbb{P}^2}(-\sum_i x_i),$$

here $N_{C_{0,i}/\mathbb{P}^2}$ is the usual (non log) normal sheaf of $\varphi_0|_{C_{0,i}}$, as a map to a component ($\cong \mathbb{P}^2$) of $X_0$, and the set $\{x_i\}$ is inverse image of the intersection $\varphi_0|_{C_{0,i}}(C_{0,i}) \cap S$. □

The Zariski tangent space of the space of log-smooth deformations of $\varphi_0$ is given by

$$H^0(C_0, N_{\varphi_0})$$

and the obstruction is given by

$$H^1(C_0, N_{\varphi_0}).$$

Lemma 9. Let $\varphi_0 : C_0 \to X_0$ be a simply pre-smoothable curve. Then we have the isomorphism

$$H^1(C_0, N_{\varphi_0}) \cong \mathbb{C}.$$

Remark 10. Note that Lemma 9 applies regardless of the genus of $C_0$. Also note that each component of $C_0$ is a rational curve with 3 special points (that is, the points mapped to the toric divisors of the components of $X_0$), and so belongs to the unique isomorphism class. Given 2 simply pre-smoothable curves $\varphi_0 : C_0 \to X_0$ and $\varphi'_0 : C'_0 \to X_0$, consider components $C_{0,v}$ and $C_{0,v'}$ of $C_0$ and $C'_0$ which are mapped to the same component of $X_0$. Then under this isomorphism, a generator of the groups $H^0(C_0, N_{\varphi_0}^v \otimes \omega_{C_0})$ and $H^0(C'_0, N_{\varphi'_0}^{v'} \otimes \omega_{C'_0})$ restricts to the same meromorphic 1-form (up to a constant multiple) on $C_{0,v}$ and $C_{0,v'}$. Here the curves $C_0, C'_0$ need not be isomorphic (in particular, the genera of them may be different).

We also need to consider the case where the stable map is not pre-smoothable. The main example of such a curve is a limit of degenerate higher genus curves. An important point is that the image of such a curve is the same as the image of a map from a degenerate rational curve. Another important point is that although such a map may not fit in the formalism of log smooth deformation theory, the generator of $H^0(C_0, N_{\varphi_0}^v \otimes \omega_{C_0})$ makes sense.

3. Calculation of the obstruction in the simplest case and the existence of the smoothing

3.1. Remarks on calculation of the obstruction. Given a simply pre-smoothable curve $\varphi_0 : C_0 \to X_0$, we calculated the generator of the dual obstruction class $H^0(C_0, N_{\varphi_0}^v \otimes \omega_{C_0})$ in Lemma 9. We write this generator by $\eta$. On the other hand, the obstruction class in $H^1(C_0, N_{\varphi_0})$ is calculated as follows.

Namely, we can take a suitable covering $\{U_i\}_i$ of $C_0$ so that on $U_i$ there is a local lift of the map $\varphi_0$. Such a covering exists since the projection $X \to \mathbb{C}$ is log smooth. These local lifts are torsor over the group of sections of $N_{\varphi_0}$, and so the difference of the lifts on the intersections $U_i \cap U_j$ defines a $N_{\varphi_0}$-valued Čech 1-cocycle. This is the obstruction class in $H^1(C_0, N_{\varphi_0})$. 

It is clear that the class does not depend on the choices of local lifts, since by construction the cocycles defined by two lifts differ only by coboundary. It is not easy in general to directly identify the obstruction cohomology class (in particular, determine whether it is zero or nonzero) from the presentation as a Čech cohomology class. More efficient way is to calculate the coupling between $H^1(C_0, \mathcal{N}_{\phi_0})$ and its dual $H^0(C_0, \mathcal{N}_{\phi_0}^\vee \otimes \omega_{C_0})$, and it can be reduced to the calculation of appropriate residues at the nodes, see [8].

In this note we do not really need to do the actual calculation of the obstruction class since we can show that it vanishes by another reason. However, since the actual calculation will help understand the later argument, we perform a calculation of the obstruction in a simple case in the next subsection.

3.2. Degenerate curves of degree 4 and their obstructions. Our construction of rational curves on a quartic K3 surface will be done by deforming a rational curve on the degenerate space $X_0$ to general fibers $X_t$, $t \neq 0$. So the starting point is the construction of a rational curve on $X_0$. This can be done by considering intersections with other particular surfaces.

Let us start from the simpler case of general curves of degree 4. Such a curve on a quartic K3 surface or its degeneration is obtained by taking the intersection with a general hyperplane $H$. In particular, we assume that the image of the map $\phi_0: C_0 \to X_0$ does not intersect the singular locus $\mathcal{S}$ of $X$. Thus, the map $\phi_0$ is a pre-log curve in the sense of [9]. On the degenerate space $X_0$, the intersection will be the union of 4 lines (see Figure 1).

![Figure 1. Picture of a general curve of degree 4 on $X_0$. Here $X_0$ is the union of 4 $\mathbb{P}^2$s. The tetrahedron is the intersection complex of $X_0$, so each triangle corresponds to $\mathbb{P}^2$, and the triangle can be regarded as the moment polytope of $\mathbb{P}^2$. On the other hand, the graph on the tetrahedron is the dual intersection graph of a curve on $X_0$.](image)

Now we turn to the construction of degenerate rational curves. As we mentioned in Section 2, the total space $\mathcal{X}$ of the degeneration contains the singular set $\mathcal{S}$ consisting of 24 points. On the tetrahedron of Figure 2, every edge contains 4 of these singular points.

If we choose a hyperplane $H$ so that $H$ intersects 3 singular points which are not contained in a single component of $X_0$, we can regard the intersection of $H$ and $X_0$ as the image of a map from a nodal rational curve, and by Lemma 4, we can ask whether this map can be lifted to a non-zero fiber $X_t$, $t \neq 0$, giving a rational curve there, see Figure 3. Let us write by

$$\psi_0: B_0 \to X_0$$
the stable map from a nodal rational curve to $X_0$ obtained in this way.

On the other hand, we can also choose a family of hyperplanes $H_s$, $s \in D$ parametrized by a small disk $D$ around the origin of $\mathbb{C}$ whose central fiber $H_0$ is the hyperplane $H$ chosen above. The intersection between $H_s \times \mathbb{C} \subset \mathbb{P}^3 \times \mathbb{C}$ and $X$ gives a family of degenerating families of curves of degree 4. In particular, the intersection between $H_0 \times \mathbb{C}$ and $X$ also gives a degenerating family whose general fiber is a smooth curve of genus 3. Thus, the image $\psi_0(B_0)$ can also be seen as a degenerate genus 3 curve. We write it by

$$\varphi_{0,0} : C_{0,0} \to X_0,$$

see Figure 4. We also write by

$$\varphi_{0,s} : C_{0,s} \to X_0$$

the degenerate curve obtained as the intersection between $H_s$ and $X_0$. Note that all the domain curves $C_{0,s}$ are in fact isomorphic, including $C_{0,0}$.

Compared with the map $\psi_0 : B_0 \to X_0$ above, the map $\varphi_{0,0}$ has the different domain curve, but their images are the same. We do not calculate the obstruction cohomology group of the map $\varphi_{0,0}$. However, the generator $\eta$ of the obstruction to lift the maps $\varphi_{0,s}$,
$s \neq 0$ calculated in Lemma 9 can be naturally defined on $C_{0,0}$, since all $C_{0,s}$ are isomorphic (see Remark 10).

Also, we can construct a family of obstruction Čech 1-cocycles on $C_{0,s}$ by using a family version of local lifts considered in Subsection 3.1, parametrized by $s$. These cocycles couple with the above generator $\eta$, and the result is an analytic function of $s$. This analytic function is identically 0.

The important point is that although the domain curves are different, the generator $\eta$ of the obstruction class on $C_{0,s}$ and $B_0$ are identical after partial normalization of $C_{0,s}$ (see Remark 10).

Moreover, in the calculation of the obstruction in Subsection 3.1, clearly it is possible to choose the covering $\{U_i\}$ of $C_{0,s}$ so that for any node of the image $\varphi_{0,s}(C_{0,s})$, there is only one $U_i$ which contains the inverse image of that node. The covering $\{U_i\}$ gives rise to a covering $\{\hat{U}_i\}$ of $B_0$ in an obvious way. That is, if $p : B_0 \to C_0$ is the partial normalization, then $\hat{U}_i$ is the inverse image $p^{-1}(U_i)$. Note that some $\hat{U}_i$ has 2 components.

Choosing the covering in this way, the Čech 1-cocycle defined by the local lifts on each covering is supported on open subsets of $C_{0,s}$ or $B_0$ which do not contain a node. When there is a local lift of $C_{0,0}$ on a neighborhood of each node such that it can also be seen as a local lift of $B_0$, the following holds. Namely, if the cohomology class of the Čech 1-cocycle defined by the local lifts on $C_{0,0}$ is 0, then the cohomology class defined by the same Čech 1-cocycle on $B_0$ is also 0, since if the class is given as the coboundary of a Čech 0-cochain $\{\eta_i\}$ on $C_{0,0}$ associated to the covering $\{U_i\}$, it naturally determines a Čech 0-cochain on $B_0$ associated to the covering $\{\hat{U}_i\}$, and its coboundary is the given Čech 1-cocycle on $B_0$.

The condition that the cohomology class of the Čech 1-cocycle defined by the local lifts on $C_{0,0}$ is 0 can be assured when such lifts can be extended to a family of lifts on $C_{0,s}$, since the corresponding cohomology classes on $C_{0,s}$, $s \neq 0$ is 0 as we noted above.

Based on this observation, we can prove the following.

**Proposition 11.** The map $\psi_0$ has smoothings up to any order. □

4. Construction of infinitely many rational curves

Now we turn to the construction of infinitely many rational curves on generic quartic K3 surfaces. The idea is the same as the previous section. Namely:
• Construct a degenerate curve on the special fiber $X_0$ of the degeneration $\mathfrak{X}$. In general, such a curve is seen as a degeneration of smooth curves of high genus. But there are some particular curves which are seen as degenerations of nodal rational curves.

• Compute the obstruction cohomology classes to deform these degenerate curves. An important point is that although the genus is different in general, the obstruction classes can be identified in a natural way (see Remark 10).

• Compute the actual obstruction. In the case of higher genus curves, one sees that the obstruction automatically vanishes since one can construct actual families of curves which degenerate to the given degenerate curves. Then the case of a degenerate rational curve is a limit of them, and since the obstruction can be seen as an analytic function of suitable parameters which is identically 0 when these parameters are nonzero, the obstruction vanishes also in the case of the rational curve.

The construction of degenerate curves in the case of higher genus curves is simple. If we want a degeneration of general curves of degree $4m$, then take a product

$$\prod_{i=1}^{m} (a_i x + b_i y + c_i z + d_i w)$$

of $m$ general linear functions. This gives a degenerate curve on $X_0$ which is a nodal union of $4m$ rational curves. Then take a general deformation of this polynomial over $\mathbb{C}[t]$, and consider the intersection of its zero and $\mathfrak{X}$. In this way one obtains a family of smooth curves which degenerates to the above degenerate curve.

Now we turn to the construction of degenerate rational curves. We start from the degree 4 rational curve $\psi_0 : C_0 \to X_0$ in the previous section. We choose a node, see Figure 5. Note that in the case of rational curves, we think that the intersection points between the curve and the singular locus of $X$ are not nodes of the curve, see Figure 3. So the chosen point is away from the singular locus of $\mathfrak{X}$.

![Figure 5. The node circled is chosen.](image)

Then consider another degenerate curve $\sigma_0 : D_0 \to X_0$, whose image shares the chosen node with $\psi_0(C_0)$, and intersects the singular locus of $\mathfrak{X}$ at 2 points, see Figure 6.

This is a degenerate genus 1 curve. Then take a finite covering of the domain curve $D_0$. In the tropicalized picture, this corresponds to cut the loop of the graph and graft some copies of it, and make a loop again, see Figure 7. On the side of holomorphic curves, this is again a degenerate genus 1 curve.
Figure 6. Tropicalized picture of a degenerate genus 1 curve on $X_0$ and its domain $D_0$ of the map. The ends of the edges connected by the dotted curve are glued so these edges are merged into 1 edge, which corresponds to a node of the holomorphic curve.

Figure 7. 3-fold covering of $D_0$. The edges connected by dotted curves are glued into 1 edge, each of which corresponds to a node on the side of holomorphic curves. The nodes corresponding to the edges marked by circles intersect the degenerate rational curve.

Let us write by $\tilde{D}_0$ the resulting genus 1 curve. To obtain a degenerate rational curve, we cut the curve $\tilde{D}_0$ at 2 of its intersection with the degenerate rational curve $C_0$ at the chosen node. This divides the curve $\tilde{D}_0$ into 2 pieces. Then similarly cut the degenerate rational curve at the corresponding node, and graft parts of the degenerate rational curve to one of the components of the divided $\tilde{D}_0$, see Figure 8, 9 and 10.

Figure 8. Cut $\tilde{D}_0$ at 2 nodes (picture on the left), and pick one of the resulting connected components (picture on the right).
Similarly, cut the degenerate rational curve into 2 pieces. The result is a degenerate rational curve $D_{0,r}$.

When the degree of the covering of $D_0$ we first take is $r$, we write the resulting rational curve by this process by $D_{0,r}$. We have an obvious map

$$\psi_{0,r}: D_{0,r} \rightarrow X_0$$

induced from $\varphi_0$ and $\sigma_0$. The image of $\psi_{0,r}$ is a curve of degree $4r$.

Similarly to Lemma 9, the obstruction cohomology classes of these curves are 1 dimensional, and the generators of them can be identified in a natural way. Then Proposition 11 can also be extended to higher degree curves.

**Theorem 12.** The degenerate rational curve $\psi_{0,r}$ has smoothings up to any order. □

Since the defining polynomial $f$ of a $K3$ surface can be taken generic, we have the following.

**Theorem 13.** The moduli space of quartic $K3$ surfaces contains a Zariski open subset whose members contain infinitely many irreducible rational curves. □

### 5. General K3 surfaces

The argument so far can be extended to more general $K3$ surfaces provided nice degenerations are given. In fact, such degenerations are constructed in [3]. Namely, a $K3$ surface $X \subset \mathbb{P}^n$ degenerates into $Q_1 \cup Q_2$, where

$$\begin{cases} Q_1 \cong \mathbb{P}^1 \times \mathbb{P}^1, & n: \text{odd} \\ Q_1 \cong \mathbb{F}_1, & n: \text{even.} \end{cases}$$

The intersection $Q_1 \cap Q_2$ is an anti-canonical divisor (elliptic curve).
Using this degeneration, we can construct degenerate rational and higher genus curves by the same method as in the quartic case. Namely, a rational curve can be taken in the central fiber in the following way:

\[ Q_1 \cap Q_2 \]

**Figure 11.** The straight line is the intersection \( Q_1 \cap Q_2 \) (elliptic curve) and the cross marks on it are the singular points of the total space.

An elliptic curve can be taken in the central fiber in the following way:

\[ Q_1 \cap Q_2 \]

**Figure 12**

Higher degree rational curves are obtained by grafting:

\[ \text{Higher degree rational curves} \]

**Figure 13**

These curves also have 1 dimensional obstructions, and the vanishing of them follows from the trivial vanishing of the obstruction to deform higher genus curves as before. Thus, we have the following.

**Theorem 14.** For each component of the moduli space of polarized K3 surfaces, there is a Zariski open subset whose members contain infinitely many irreducible rational curves. □

**References**


