RECENT PROGRESS OF SUBADDITIVITY OF KODAIRA DIMENSIONS IN POSITIVE CHARACTERISTIC

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Abstract. Subadditivity of Kodaira dimensions is one of fundamental problem of birational geometry over the field of complex numbers. In this note, we survey the recent progress of subadditivity problem over an algebraically closed field of positive characterisitc.

1. Introduction

The following conjecture due to Iitaka is one of fundamental problems of the classification theory of algebraic varieties over $\mathbb{C}$, the field of complex numbers:

Conjecture 1.1 ($C_{n,m}$). *Let $f : X \to Z$ be a surjective morphism of proper, smooth varieties over $\mathbb{C}$, where $n = \dim X$ and $m = \dim Z$. Assuming the generic geometric fibre $X_{\bar{\eta}}$ of f is connected, we have the following inequality for the Kodaira dimension:*

$$\kappa(X) \geq \kappa(X_{\bar{\eta}}) + \kappa(Z).$$

This conjecture is proved in the following cases:

(1) $\dim F = 1, 2$ by Viehweg ([Vie1], [Vie4]);
(2) Z is of general type by Kawamata and Viehweg ([Ka81] Theorem 3, [Vie3]);
(3) $m = 1$ by Kawamata ([Ka82]);
(4) F has a good minimal model by Kawamata ([Ka85]);
(5) F is of general type by Kollár ([Ko]);
(6) Z is of maximal Albanese dimension by J. A. Chen and Hacon ([CH]);, and Birkar and J. A. Chen ([BC])
(7) F is of maximal Albanese dimension by Fujino ([Fu1]);
(8) $n \leq 6$ by Birkar ([Bi]);
(9) $m = 2$ by Cao ([C15]).

In this note, we summarize the recent results of subadditivity of Kodaira dimension on an algebraically closed field k of positive characteristic, and briefly introduce their methods.

We call a projective surjective morphism $f : X \to Z$ of two varieties a fibration, if $f_*\mathcal{O}_X = \mathcal{O}_Z$. We say a fibration $f : X \to Z$ is separable if the field extension $f^* : k(Z) \to k(X)$ between the rational function fields is separable and $k(Z)$ is algebraically closed in $k(X)$. Then general fiber of a separable fibration is geometrically integral ([Ba] Theorem 7.1).

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Different from the case of $\mathbb{C}$, even if we assume that $f : X \to Z$ is a separable fibration of smooth projective varieties over k, the generic fiber is possibly singular. For example, Raynaud's counter-example of Kodaira vanishing ([Ra]). So there are some ambiguity in the definition of Kodaira dimension of singular varieties.

1.2. **Kodaira dimension.** Let X be a projective variety, L a line bundle on X and $N(L)$ the set of all positive integers m such that $|mL| \neq \emptyset$. For an integer $m \in N(L)$, let $\Phi_{|mL|}$ be the rational map defined by the linear system $|mL|$. The L-dimension $\kappa(X, L)$ is defined as

$$\kappa(X, L) = \begin{cases} -\infty & \text{if} \quad N(L) = \emptyset \\ \max\{\dim \Phi_{|mL|}(X) | m \in N(L)\}, & \text{if} \quad N(L) \neq \emptyset \end{cases}$$

If X is a smooth projective variety over k, the Kodaira dimension $\kappa(X) := \kappa(X, \omega_X)$, where ω_X is the canonical sheaf of X.

Definition 1.3 Let X be a projective variety. We say that X' is a *smooth model* of X, if X' is smooth projective and X' is birational to X. If X has a smooth model X', then the Kodaira dimension $\kappa(X)$ is defined by $\kappa(X) := \kappa(X')$.

Remark 1.4 1) Resolution of singularities in a positive characteristic field hasn't been settled yet, so the existence of a smooth model is not clear. However, if $\dim X \leq 3$, smooth models of X always exist.

2) If smooth models of X exist, then the definition of Kodaira dimension is independent of choice of the smooth models.

Another way to define Kodaira dimension is to use the dualizing sheaf.

Definition 1.5 Let X be a projective variety with the invertible dualizing sheaf ω_X. Then $\kappa_1(X) := \kappa(X, \omega_X)$.

Remark 1.6 1) For a fibration $f : X \to Z$ between smooth quasi-projective varieties, the dualizing sheaf of a general fiber and the generic fiber is a line bundle.

2) If X is Gorenstein and projective, then ω_X is an invertible sheaf. If X is smooth, then the dualizing sheaf coincides with the canonical sheaf, thus $\kappa_1(X) = \kappa(X)$.

Example 1.7 Let C' be a cuspidal projective curve in $\mathbb{P}^2_k$. Then $\kappa(C') = -\infty$ and $\kappa_1(C') = 0$. For a projective curve C over k, $\kappa_1(C) \geq \kappa(C)$.

In general, we have the following Proposition.

Proposition 1.8 ([Pa13] Corollary B.3)**.** *Let X be a smooth projective variety over an algebraically closed field k. If X has a smooth model and ω_X, then $\kappa(X) \leq \kappa_1(X) = \kappa(X, \omega_X)$.*

So in positive characteristic, we can ask if the following subadditivity of Kodaira dimension holds in positive characteristic field.

Problem 1.9 ($C_{n,m}$)**.** *Let $f : X \to Z$ be a separable fibration of smooth projective varieties over an algebraically closed field k of positive characteristic. Does the following inequality hold?*

$$\kappa(X) \geq \kappa(X_{\bar{\eta}}, \omega_{X_{\bar{\eta}}}) + \kappa(Z),$$

where $X_{\bar{\eta}}$ is the geometric generic fiber of f, $n = \dim X, m = \dim Z$.

Another weaker subadditivity is:

Problem 1.10. *Let $f : X \to Z$ be a separable fibration of smooth projective varieties over an algebraically closed field k of positive characteristic. If the geometric generic fiber $X_{\bar{\eta}}$ has a smooth model, does the following inequality hold?*

$$\kappa(X) \geq \kappa(X_{\bar{\eta}}) + \kappa(Z),$$

where $X_{\bar{\eta}}$ is the geometric generic fiber of f, $n = \dim X, m = \dim Z$.

2. Recent progress of the subadditivity of Kodaira dimension

In the remaining part of the note, we denote k an algebraically closed field of positive characteristic.

In 2012, Patakfalvi studied the semi-positivity of direct image in positive characteristic, which will be mentioned in detail later. As an application, he got

Theorem 2.1 ([Pa] Corollary 4.6)**.** *Let $f : X \to Z$ be a separable fibration of smooth projective varieties over k. If Z is an S_2, G_1, equidimensional projective variety with K_Z $\mathbb{Q}$-Cartier and big, generic giber F is sharply F-pure, $K_{X/Z}$ is f-semi-ample and $K_F|_F$ is big for the generic fiber F, then $K_X + A$ is big.*

Roughly speaking, if Z is of general type and the generic fiber is of mild singularities and general type, then X is of general type.

In 2013, Zhang and myself considered relative dimension 1 case. We can prove a weaker form of Problem 1.9.

Theorem 2.2 ([CZ13] Theorem 1.2)**.** *Let $f : X \to Z$ be a separable fibration of relative dimension 1 between smooth projective varieties over an algebraically closed field k of positive characteristic. Then*

$$\kappa(X) \geq \kappa(Z) + \kappa(X_{\bar{\eta}}), \tag{2.2.1}$$

In 2013, Patakfalvi proved the following theorem.

Theorem 2.3 ([Pa13] Theorem 1.1)**.** *Let $f : X \to Z$ be a separable fibration of smooth projective varieties over k. Further assume that $\kappa(Z) = \dim Z$, and the Hasse-Witt matrix of the geometric generic fiber X_η is not nilpotent (including that it is not the zero matrix). Then*

$$\kappa(X) \geq \kappa(X_\eta, \omega_{X_\eta}) + \kappa(Z)$$

In 2015, Bikar, Zhang and myself consider the subadditivity of threefolds over the algebraic closure of finite field.

Theorem 2.4 ([BCZ15] Corollary 1.3)**.** *Let $f : X \to Z$ be a fibration from a smooth projective threefold to a smooth projective curve over $\bar{\mathbb{F}}_p, p > 5$. Then*

$$\kappa(X) \geq \kappa(X_\eta) + \kappa(Z).$$

In 2015, Ejiri proves

Theorem 2.5 ([E15] Theorem 1.5)**.** *Let $f : X \to Z$ be a separable fibration from a smooth projective variety X of dimension three to a smooth projective curve Z over k. If the geometric generic fiber $X_{\bar{\eta}}$ is a smooth projective surface of general type and char $k = p \geq 7$, then*

$$\kappa(X) \geq \kappa(Z) + \kappa(X_{\bar{\eta}}).$$

It is worth to mention that only known case of Problem 1.9 without any assumption of the generic fibers is $C_{2,1}$. That is $f : S \to C$ from a smooth surface to a curve C ([CZ13] Theorem 1.3).

3. SEMIPOSITIVITY OF DIRECT IMAGE

It is known that the subadditivity of Kodaira dimension is closely related to the semi-positivity of direct image over $\mathbb{C}$. However, the semi-positivity of direct image is very subtle in positive characteristics. It fails in general. Even worse, Raynaud's counterexample of Kodaira vanishing shows that $f_*\omega^m_{S/C}$ is NOT nef for any $m \geq 1$. The example shows that the semi-positivity is NOT expected if general fibers of f are very singular. Even if $f : S \to C$ is a semi-stable fibration of genus 2 curves, where S is a smooth projective surface and C is a smooth projective curve, $f_*\omega_{S/C}$ is possibly not nef. For example, the well known Moret-Bailly's example ([MB81]). It shows that we can not get the semi-positivity of $f_*\omega^m_{X/Z}$ from the semi-positivity of $f_*\omega_{X/Z}$ by covering trick, as it is known over $\mathbb{C}$ by Viehweg, Kawamata and others.

It seems natural to expect the semi-positivity of $f_*\omega^m_{X/Z}$ for large enough m if the general fibers of f is not very singular. Patakafalvi and Ejiri obtain their semi-positivity results and then apply to the subadditvity problem. I am curious that is there some other phenomenon? For example, is there an example of surface fibration $f : S \to C$ such that $f_*\omega^{2k+1}_{S/C}$ are NOT nef for $k \in \mathbb{N}$, but $f_*\omega^{2k+2}_{S/C}$ are nef for $k \in \mathbb{N}$.

Theorem 3.1 ([Pa] Theorem 1.1)**.** *Let $f : X \to C$ be a fibration from a normal variety to a smooth projective curve C with sharply F-pure generic fiber. Further assume that rK_X is Cartier for some $(p, r) = 1$. If $K_{X/C}$ is f-ample, then $f_*\mathcal{O}_X(mrK_{X/C})$ is a nef vector bundle for $m \gg 0$.*

Theorem 3.2 ([E15] Theorem 1.1)**.** *Let $f : X \to Z$ be a separable fibration between smooth projective varieties, let Δ be an effective $\mathbb{Q}$-divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p, and let $\bar{\eta}$ be the geometric generic point of Y . Assume that*

(i) the $k(\eta)$-algebra $\oplus_{m\geq 0} H^0(X_{\bar{\eta}}, m(aK_{X_{\bar{\eta}}} + (a\Delta)_{\bar{\eta}}))$ is finitely generated, and

(ii) there exists an integer $m_0 > 0$ such that for each $m \geq m_0$,

$$S^0(X_{\bar\eta}, \Delta_{\bar\eta}, m(aK_{X_{\bar\eta}} + (a\Delta)_{\bar\eta})) = H^0(X_{\bar\eta}, m(aK_{X_{\bar\eta}} + (a\Delta)_{\bar\eta})).$$

Then $f_*\mathcal{O}_X(am(K_{X/Z} + \Delta))$ is weakly positive for each $m \geq m_0$.

Both Patakfalvi and Ejiri's semi-positivity results assume mild singularities of general fibers. For the terminologies and notation, please refer to Patakfalvi's and Ejiri's paper ([Pa],[Pa13],[E15]).

4. Moduli of curves

In the joint work with Lei Zhang, we didn't use semi-positivity of direct image. Instead, we use Viehweg's ideal of proving $C_{n,n-1}$ over $\mathbb{C}$ by moduli of curves and comparison of change of relative dualizing sheaves.

In positive characteristic, resolution of singularities is unknown. However, resolution argument can be replaced by de Jong's alteration. The difficulties are how to compare the change of relative dualizing sheaves if the diagram is not a flat base change. We have the following comparison, which is the Key lemma in our proof.

Theorem 4.1 ([CZ13] Theorem 2.4)**.** *Let X, X', Z, Z' be projective varieties. Assume there exists the following commutative diagram*

$$\begin{array}{ccccc} X' & \xrightarrow{\sigma} & \bar{X} \hookrightarrow \bar{X}' = X \times_Z Z' & \xrightarrow{\pi_1} & X \\ {\scriptstyle h}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle f}\downarrow \\ Z' & \xrightarrow{\mathrm{id}_{Z'}} & Z' & \xrightarrow{\pi} & Z \end{array}$$

where $f : X \to Z$ is a fibration of relative dimension r, $\pi : Z' \to Z$ a generically finite surjective morphism, π_1 and g' the projections, $\bar{X}$ the unique irreducible component of $\bar{X}'$ dominating over X, and $\sigma : X' \to \bar{X}$ a birational morphism.

Assume moreover that $f^!\mathcal{O}_Z \simeq \omega^o_{X/Z}[r]$, $h^!\mathcal{O}_{Z'} \simeq \omega^o_{X'/Z'}[r]$, and $\omega^o_{X/Z}, \omega^o_{X'/Z'}$ are invertible sheaves on X and X' respectively. Then there exists an effective σ-exceptional divisor E on X' such that

$$\omega^o_{X'/Z'} \leq \sigma^*\pi_1^*\omega^o_{X/Z} + E.$$

Applying de Jong's idea of alterations, we have the following commutative diagram.

$$\begin{array}{ccccccccc} U & \xleftarrow{\rho_1} & X'' = U \times_M Z' & \xleftarrow{\rho'} & X' & \xrightarrow{\sigma} & \bar{X}' = X \times_Z Z' & \xrightarrow{\pi_1} & X \\ {\scriptstyle h}\downarrow & & {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle f}\downarrow \\ M & \xleftarrow{\rho} & Z' & \xleftarrow{\mathrm{id}_{Z'}} & Z' & \xrightarrow{\mathrm{id}_{Z'}} & Z' & \xrightarrow{\pi} & Z \end{array} \tag{4.1.1}$$

where

1) $\pi : Z' \to Z$ is an alteration where Z' is smooth;
2) $\sigma : X' \to \bar{X}'$ is a birational morphism onto the strict transformation of X under $Z' \to Z$, and X' is smooth;

3) $\rho' : X' \to X''$ is a birational morphism such that $R\rho'_*\mathcal{O}_{X'} \cong \mathcal{O}_{X''}$;
4) $h : U \to M$ with M normal and projective, is a family of stable curves if $g \geq 2$, or a family of 1-point stable curves if $g = 1$. Moreover, the natural morphism $M \to \bar{M}_g$ if $g \geq 2$ (or $M \to \bar{M}_{1,1}$ if $g = 1$) is a finite morphism.

The remaining proof, we just compare the change of dualizing sheaves and get the weak subadditivity of Kodaira dimension. From the proof, we can see that we can not prove $C_{n,n-1}$ so far, because the first step of alteration change singular fibers to smooth fibers.

5. Base point freeness

Over $\mathbb{C}$, Kawamata proves that $C_{n,m}$ holds if assuming MMP conjecture and Abundance conjecture. In the joint work of Birkar and Zhang, we apply the recent results of MMP for threefolds char $k > 5$ by Hacon-Xu and Birkar; base point freeness results of Keel and Tanaka in positive characteristic to study $C_{3,1}$ over $\bar{\mathbb{F}}_p$ for $p > 5$. The case of $\bar{\mathbb{F}}_p$ is very special, since any point on $\mathrm{Pic}^0(X)$ is torsion over $\bar{\mathbb{F}}_p$. Besides, we also prove some base point free results for surface over arbitrary field and some other results in MMP.

Theorem 5.1 ([BCZ15] Theorem 1.5)**.** *Let (X, B) be a projective klt pair of dimension two over a field k where B is a $\mathbb{Q}$-boundary. Assume $K_X + B$ is nef and that $\kappa(K_X + B) \geq 0$. Then $K_X + B$ is semi-ample.*

Based on base point freeness results, we can get some natural fibrations, such as Iitaka fibration. The the subadditivity problem is reduced to lower dimension. In the case if fibers are of general type, we can use canonical model and then apply Patakfalvi's results to get subadditivity. At that time, Ejiri's semi-positivity results are not known. After Ejiri got his semi-positivity results, he apply it to the case general fibers are of general type case and proved the subadditivity of Kodaira dimension in his case over an algebraically closed field of positive characteristics.

The hardest case for "$C_{3,1}$" is the case base curve is elliptic curve, general fibers are of general type, $\deg f_*\omega^m_{X/C} = 0$ for $m \gg 0$. In the case, we need to application classification of vector bundles over elliptic curves in positive characteristic. Then show $f_*\omega^m_{X/C}$ has global sections for $m \gg 0$. This problem is relatively easy for $\bar{\mathbb{F}}_p, p > 5$, and harder for char $k > 5$ case. Ejiri makes more efforts to solve the case. For remaining cases of $C_{3,1}$, we study case by case, and consider the natural fibration. Among them, we also apply Patafalvi's nefness results of relative dualizing sheaf.

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