CHARACTERIZATION OF NON-F-SPLIT DEL PEZZO SURFACES

NATSUO SAITO

1. Introduction

Let k be an algebraically closed field of characteristic p. We work over k throughout this report.

The notion of Frobenius splitting is very important for analyzing algebraic varieties in positive characteristic. For a scheme X over k, the Frobenius morphism is defined as the morphism $F : X \to X$ such that it is the identity map on the underlying topological space, while the map of sheaves of rings $F^{\#} : \mathscr{O}_X \to F_*\mathscr{O}_X$ is the p-th power map. Now we say that a scheme X is *Frobenius split*, or *F-split* for short, if the Frobenius map $\mathscr{O}_X \to F_*\mathscr{O}_X$ splits as a map of $\mathscr{O}_X$-modules.

In this report, we investigate F-splitting of smooth del Pezzo surfaces. In particular, we would like to characterize non-F-split del Pezzo surfaces. For the detail, see [7].

Our starting point is the following result in [3]:

Theorem 1.1 (Hara). *Let X be a smooth del Pezzo surface over k. Then X is F-split unless*

(1) $K_X^2 = 3$, $p = 2$;
(2) $K_X^2 = 2$, $p = 2, 3$;
(3) $K_X^2 = 1$, $p = 2, 3, 5$.

Especially, if X is a del Pezzo surface of degree 3, then X is not F-split if and only if X is a Fermat cubic surface in characteristic 2.

As is well known, smooth del Pezzo surfaces are obtained as the blow-up σ of $(9-d)$-points $P_1, \dots, P_{9-d}$ on $\mathbb{P}^2$ in general position, where d is the degree of the surface. In the case $d = 3$, it is a cubic surface in $\mathbb{P}^3$. Homma gives a characterization of the Fermat cubic surface in characteristic 2 ([5]):

Theorem 1.2 (Homma). *Let X be a smooth del Pezzo surface of degree 3 over k. Suppose that X is obtained as the blow-up of $\mathbb{P}^2$ with centers $P_1, \dots, P_6$. Then the following conditions are equivalent:*

(i) *$p = 2$ and the anticanonical embedding of X is projectively equivalent to the Fermat cubic surface;*

(ii) $p > 0$ *and each smooth member of* $|-K_X|$ *is a supersingular elliptic curve;*

(iii) *Each point of* $P_1, \ldots, P_6$ *is the concurrent point of all tangent lines of the conic passing through the remaining five points. The set* $\{P_1, \ldots, P_6\}$ *is projectively equivalent to*

$$\{[1,0,0],[0,1,0],[0,0,1],[1,1,1],[1,\alpha,\alpha^2],[1,\alpha^2,\alpha]\},$$

where $\alpha \in \mathbb{F}_4$ *with* $\alpha^2 + \alpha + 1 = 0$.

Thus this also gives a characterization of non-F-split del Pezzo surfaces of degree 3.

Now we would like to have "degree 2 version" of Homma's result. Let X be a smooth del Pezzo surface of degree 2 over k. The anticanonical system $|-K_X|$ induces a double cover $\pi_{|-K_X|} : X \to \mathbb{P}^2$. The branch curve C on $\mathbb{P}^2$ is a smooth quartic curve if $p \neq 2$, while it is a possibly degenerate conic if $p = 2$:

$$\begin{array}{ccc} & & X \\ & \overset{\sigma}{\swarrow} & \downarrow \pi_{|-K_X|} : \text{deg 2 finite morphism} \\ \mathbb{P}^2 & & \mathbb{P}^2 \supset C : \text{branch curve.} \end{array}$$

We have the following theorem:

Theorem 1.3. *Let* X *be a smooth del Pezzo surface of degree* 2 *over an algebraically closed field* k *of characteristic* p. *Then the following conditions are equivalent:*

(i) $p = 2, 3$, *and the branch locus* C *of the double cover* $\pi : X \to \mathbb{P}^2$ *is isomorphic to*
  (a) *Fermat quartic curve in characteristic* 3,
  (b) *Double line in characteristic* 2.

(ii) X *is not F-split.*

(iii) *Each conic passing through 5 points in* $\{P_1, \ldots, P_7\}$ *is tangent to the line passing through remaining 2 points.*

(iv) *Each smooth member of* $|-K_X|$ *is a supersingular elliptic curve.*

In particular, when $p = 3$, *the following condition is also equivalent to the above:*

(v) *the set* $\{P_1, \ldots, P_7\}$ *is projectively equivalent to*

$$\mathscr{P}_0 = \{[1,0,0],[0,1,0],[0,0,1],[1,1,1],$$
$$[1,-1,\alpha],[1,-\alpha^3,\alpha^3],[1,-\alpha,-1]\},$$

where $\alpha \in \mathbb{F}_9$ *with* $\alpha^2 - \alpha - 1 = 0$.

Remark 1.4. (1) $\mathscr{P}_0$ in Theorem 1.3 (v) is characterized as the only complete 7-arc over $\mathbb{F}_9$ (See Hirschfeld [4]). We say a finite set $\mathscr{P}$ of points in $\mathbb{P}^2$ over $\mathbb{F}_q$ is a *complete k-arc* if
 (a) $\mathscr{P}$ consists of k points and no three of them are collinear, and
 (b) $\mathscr{P}$ is not contained in any $(k+1)$-arc.

Note that the set of 6 points in Theorem 1.2 (iii) is characterized as the only complete 6-arc over $\mathbb{F}_4$.

A group $G(\mathscr{P})$ acting on a k-arc $\mathscr{P}$ is called the *projective group for $\mathscr{P}$* if $G(\mathscr{P})$, which is a subgroup of the projective general linear group, fixes $\mathscr{P}$ as a set.

(2) Each non-F-split del Pezzo surface of degree 2 in characteristic 2 is obtained as the blow-up of a Fermat cubic surface at a general point.

2. Preliminaries

2.1. **Facts about Frobenius splittings.** To check whether a given ring is F-split or not, one of the very useful tools is the Fedder's criterion:

Lemma 2.1 (Fedder's criterion)**.** *Let I be an ideal in $k[x_1,\dots,x_n]$ defined by $I=\langle f_1,\dots,f_m\rangle$. Then $k[x_1,\dots,x_n]/\langle f_1,\dots,f_m\rangle$ is F-split at a point $\mathfrak{m}\in \operatorname{Spec} k[x_1,\dots,x_n]$ if and only if $I^{[p]}:I\not\subset \mathfrak{m}^{[p]}$, where we denote $\langle f_1^p,\dots,f_m^p\rangle$ by $I^{[p]}$.*

Lemma 2.1 states the criterion in the local settings. Thanks to the following theorem (cf. [9]), we can apply Fedder's criterion to a projective variety:

Theorem 2.2. *Let X be any projective scheme over a perfect field. The followings are equivalent:*
 (i) *X is Frobenius split;*
 (ii) *the ring $S_{\mathscr{L}}=\oplus_{n\in\mathbb{N}}H^0(X,\mathscr{L}^n)$ is Frobenius split for all invertible sheaves $\mathscr{L}$;*
 (iii) *the section ring $S_{\mathscr{L}}=\oplus_{n\in\mathbb{N}}H^0(X,\mathscr{L}^n)$ is Frobenius split for some ample invertible sheaf $\mathscr{L}$.*

By Grothendieck duality for $F:X\to X$ where X is a scheme over k, we have an isomorphism $\mathscr{H}om_{\mathscr{O}_X}(F_*\mathscr{O}_X,\mathscr{O}_X)\cong F_*\omega_X^{1-p}$, which implies that if X is Frobenius split, then there exists a section $s\in H^0(X,\omega_X^{1-p})$, called a *splitting section*, corresponding to the F-splitting. The following useful criterion for Frobenius splitting is proved by Mehta and Ramanathan ([6, Proposition 8]):

Proposition 2.3. *Let X be a smooth projective variety of dimension n. X is Frobenius split if there is a point P in X and a global section $s \in H^0(X, \omega_X^{-1})$ with divisor of zeros*

$$(s)_0 = Y_1 + \cdots + Y_n + Z,$$

where $Y_1, \ldots, Y_n$ are prime divisor intersection transversally at P, and Z is an effective divisor not containing P. The section s is a splitting section.

2.2. **Aronhold sets.** Let X be a smooth del Pezzo surface of degree 2. If $p \neq 2$, the branch locus of the double cover $\pi_{|-K_X|} : X \to \mathbb{P}^2$ is a smooth quartic curve C. Over the complex numbers, it is well known that a plane quartic curve has 28 bitangents. This also holds in positive characteristic if $p \neq 2$. The problem is that which 7 of them correspond to the blow-ups.

Definition 2.4. Let C be a smooth plane quartic curve. A set $\mathscr{K} = \{\ell_1, \ldots, \ell_7\}$ of 7 bitangents of C among 28 ones is called an *Aronhold set* if for each subtriple $\{\ell_i, \ell_j, \ell_k\} \subset \mathscr{K}$ there no exists a conic passing through the 6 contact points of $\ell_i \cup \ell_j \cup \ell_k$ with C.

Note that two of the contact points may be infinitely near. Aronhold sets can also be defined by using the theory of theta characteristics. There are 288 Aronhold sets for each smooth plane quartic curve. For more details, see [2].

For a del Pezzo surface of degree 2, there is a one-to-one correspondence between each Aronhold set of a quartic curve which is a branch curve of the double cover induced by anticanonical divisor and a set of 7 points, the centers of the blow-up, on the dual projective plane:

Proposition 2.5. *Let $\sigma : X \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ with centers $\{P_1, \ldots, P_7\}$, and C a branch quartic curve of the double cover $\pi_{|-K_X|} : X \to \mathbb{P}^2$. Then 7 bitangents which are the images of (-1)-curves $\sigma^{-1}(P_i)$ form an Aronhold set of C. Conversely, for each Aronhold set $\mathscr{K}$ of C, there exists a set of 7 points on $\mathbb{P}^2$ such that each (-1)-curve $\sigma^{-1}(P_i)$ corresponds to a member in $\mathscr{K}$.*

3. Outline of the Proof of the Main Theorem

We sketch the outline of the proof of Theorem 1.3. For the detail, see [7].

If $p = 2$, the equivalences of (i)-(iv) easily follows from [1] and some easy calculation. Thus from now on, we assume $p \neq 2$.

(i)$\Rightarrow$(ii)$\Leftrightarrow$(iii): Apply Lemma 2.1 and Theorem 2.2 and some known facts.

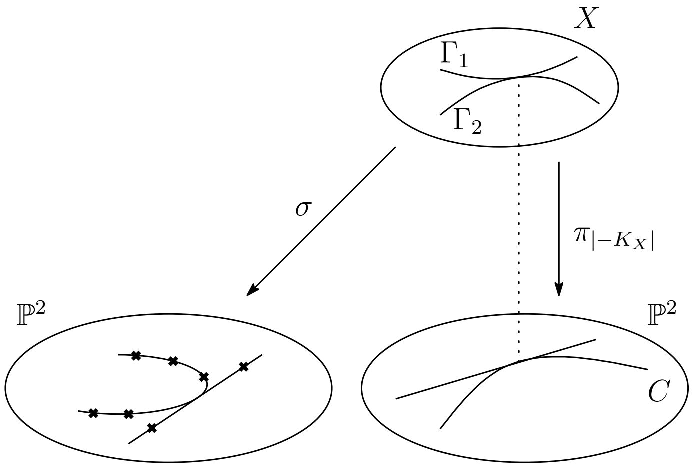
(ii)⇒(iv): It is known that a quartic curve has 28 bitangents. If ℓ is a bitangent of a quartic branch curve of $\pi : X \to \mathbb{P}^2$, then

$$\pi^*\ell = \Gamma_1 + \Gamma_2 \in |-K_X|,$$

where Γ_1, Γ_2 are (-1)-curves with $\Gamma_1.\Gamma_2 = 2$. X has 56 (-1)-curves paired into 28 pairs. Moreover, these pairs are the proper inverse transforms of the following curves:

21 pairs: Each pair consists of the conic through 5 points in $\{P_1, \ldots, P_7\}$ and the line through the remaining 2 points.
7 pairs: Each pair consists of the cubic through the 7 points with a double point at P_i and the exceptional curve $\sigma^{-1}(P_i)$.

If there exists a pair of (-1)-curves such that they intersect transversally, then X is F-split by Proposition 2.3. Thus if X is not F-split, then each pair of a conic and a line through the 7 points must intersect tangentially. Hence the condition (iv) holds.



(iv)⇒(i): If the condition (iv) holds, then 21 bitangents among 28 touch C quadruply. This implies that C has 21 hyperflexes. Then thanks to the following theorem ([10]), we see that C is a Fermat quartic:

Theorem 3.1 (Stöhr-Voloch). *A smooth quartic curve in $\mathbb{P}^2$ has 28 hyperflexes if and only if it is a Fermat quartic curve in characteristic 3. Moreover, if it is not, then the number of hyperflexes is less than 12.*

(v)⇒(iii): We can check that the set of 7 points described in (v) satisfies the condition (iii) by direct calculation.

(i)⇒(v): Let $C \subset \mathbb{P}^2$ be a Fermat quartic curve in characteristic 3. Consider a set of 7 points

$$\mathscr{P}'_0 = \{[0,\alpha,1],[0,\alpha-1,1],[-\alpha,0,1],[\alpha-1,0,1],\\ [\alpha,1,0],[-\alpha+1,1,0],[1,1,1]\}$$

on C, where $\alpha \in \mathbb{F}_9$ with $\alpha^2-\alpha-1=0$. We can check that there exists no conic passing through any three points in $\mathscr{P}'_0$ with double multiplicities. Hence $\mathscr{P}'_0$ is an Aronhold set for C. Moreover, there exists a projective transformation $T : \mathscr{P}'_0 \to \mathscr{P}_0$ given by

$$\begin{bmatrix} 1 & \alpha-1 & \alpha \\ \alpha-1 & \alpha & 1 \\ \alpha & 1 & \alpha-1 \end{bmatrix} \in PGL(3,\mathbb{F}_9).$$

Hence $\mathscr{P}'_0$ is projectively equivalent to $\mathscr{P}_0$. Thus $\mathscr{P}'_0$ is the complete 7-arc.

It is known that the automorphism group of C is isomorphic to the projective unitary group $U_3(3)$ (see [8] for example). The projective group $G(\mathscr{P}'_0)$ for $\mathscr{P}'_0$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/7\mathbb{Z}$, in which $\mathbb{Z}/3\mathbb{Z}$ is induced by changing cyclic coordinates, and $\mathbb{Z}/7\mathbb{Z}$ is a cyclic transformation of the 7 points given by

$$\begin{bmatrix} 1 & \alpha+1 & -\alpha+1 \\ \alpha+1 & \alpha & 1 \\ -\alpha+1 & 1 & -1 \end{bmatrix} \in PGL(3,\mathbb{F}_9).$$

It is easy to see that $G(\mathscr{P}'_0)$ is a subgroup of $U_3(3)$. Thus the number of orbits of $\mathscr{P}'_0$ under the action of $U_3(3)$ on C is

$$\frac{|U_3(3)|}{|G(\mathscr{P}'_0)|} = \frac{6048}{21} = 288,$$

which implies the set of Aronhold sets is transitive for the action of $U_3(3)$. Hence by Proposition 2.5, each 7-arc which is obtained as the centers of the blow-ups is projectively equivalent to each other.

4. On non-F-split del Pezzo surfaces of degree 1

We have a few comments on non-F-split del Pezzo surfaces of degree 1. If X is a smooth del Pezzo surface of degree 1, the linear system $|-2K_X|$ induces a finite map $\pi_{|-2K_X|} : X \to Q \subset \mathbb{P}^3$ of degree 2 ramified along a curve of degree 6, where Q is a quadric cone. X can also be expressed as a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,2,3)$.

Unlikely the case that the degree of a del Pezzo surface X is 3 or 2, we cannot expect a characterization of non-F-split del Pezzo surfaces like

Theorem 1.3 if $\deg(X) = 1$. By Theorem 1.1, all del Pezzo surfaces are F-split unless $p = 2, 3$ or 5. We can easily see that del Pezzo surfaces defined by

$$y^2 = x^3 + t_0^5 f_1(t_0, t_1) + t_1^5 g_1(t_0, t_1) \qquad (p = 5),$$
$$y^2 = x^3 + f_4(t_0, t_1)x + f_6(t_0, t_1) \qquad (p = 3),$$
$$y^2 + f_3(t_0, t_1)y = x^3 + f_2(t_0, t_1)x^2 + f_4(t_0, t_1)x + f_6(t_0, t_1) \quad (p = 2)$$

are not F-split by Lemma 2.1, where t_0, t_1, x, y are variables of weights $1, 1, 2$ and 3 respectively, and $f_i(t_0, t_1)$ and $g_i(t_0, t_1)$ are some homogeneous polynomials of degree i. Note that non-F-split del Pezzo surfaces of degree 2 are projectively equivalent to a double cover of $\mathbb{P}^2$ ramified along a Fermat quartic curve if $p = 3$. In contrast, those of degree 1 are not uniquely determined even in $p = 5$. For example, we can see that two del Pezzo surfaces X_1, X_2 in $p = 5$ defined by

$$X_1 : y^2 = x^3 + t_0 t_1 (t_0^4 - t_1^4),$$
$$X_2 : y^2 = x^3 + t_0^6 + t_1^6$$

are not projectively equivalent, while they are both non-F-split.

Moreover, the equivalence of the conditions (ii) and (iv) in Theorem 1.3 also fails if $\deg(X) = 1$. For example, assuming $p \geq 5$, consider a del Pezzo surface defined by

$$y^2 = x^3 + f_6(t_0, t_1)$$

in $\mathbb{P}(1, 1, 2, 3)$. Then we can see that each smooth member of $|-K_X|$ is a supersingular elliptic curve if $p \equiv 2 \pmod 3$.

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