Supersingular K3 surfaces have an automorphism of Salem degree 22

Author(s)
Simon, Brandhorst

Citation
代数幾何学シンポジウム記録 2016: 148-148

Issue Date
2016

URL
http://hdl.handle.net/2433/218280

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Supersingular K3 surfaces have an automorphism of Salem degree 22

Simon Brandhorst
Institute of Algebraic Geometry, Leibniz University Hanover

Salem Numbers

We call a monic, irreducible polynomial \( s(x) \in \mathbb{Z}[x] \) a Salem polynomial if its complex roots are of the form

\[
\lambda, \lambda^{-1}, \alpha_1, \pi_1, \ldots, \alpha_d, \pi_d
\]

where \( |\alpha| = 1 \) and \( 1 < \lambda \in \mathbb{R} \). We call \( \lambda \) a Salem number and \( \deg(s(x)) \) its degree.

Lehmer’s number (1932) \( \lambda_0 \approx 1.17628 \) defined by

\[
S(x) = x^8 + x^7 - x^6 - x^4 - x^3 - x^2 - x + 1
\]

is conjectured to be the smallest Salem number.

Theorem 1. (Esnault - Srinivas) Let \( X \) be a projective surface over an algebraically closed field \( k \) and \( f : X \to X \) an automorphism. Then the characteristic polynomial \( \chi(f^*|H^2(X, \mathbb{Q})) \) (1 ≠ char \( k \)) factors as \( s(z)z^d \) where \( s(z) \) is a Salem polynomial, and \( s(z) \) is called the Salem factor of \( X \).\( \square \)

Supersingular K3 surfaces

A K3 surface is a smooth projective variety \( X/k \) such that \( \Omega_X \cong \omega_X \) and \( h^1(X, \mathcal{O}_X) = 0 \).

Common examples are smooth quartics in \( \mathbb{P}^3 \), e.g., Fermat’s quartic

\[
X_4: \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subseteq \mathbb{P}^3.
\]

In characteristic zero the Picard number \( \rho(X) \) of NS \( X = \rho(X) \leq h^{1,2} = 20 \). However in positive characteristic \( p \) there are so called supersingular K3 surfaces with \( \rho(X) = 22 \). Then det NS \( X = -p^9 \) by the Artin invariant \( \sigma \in \{1, \ldots, 10\} \) and for given Artin invariant there is a \( \sigma = 1 \) dimensional family. The supersingular K3 surface over \( k = \overline{\mathbb{Q}} \) of Artin invariant \( \sigma = 1 \) is unique up to isomorphism. For example \( X_3/\mathbb{F}_p \) is supersingular of Artin invariant \( \sigma = 1 \) for \( p \equiv 3 \mod 4 \).

The proof

Theorem 4. (Lieblich - Maulik) Let \( X \) be a K3 surface over an algebraically closed field of characteristic \( p > 3 \). Let \( \Gamma(X) \subseteq O(\text{NS} X) \) be the subgroup preserving the nef cone. Then the natural map \( \text{Aut}(X) \to \Gamma(X) \) has finite kernel and cokernel.

Since any power of a Salem number of degree \( d \) remains a Salem number of this degree, passing to a finite index subgroup does not change the maximal occurring Salem degree of a matrix group. Combining this with Theorem 4, we get that the maximum occurring Salem degree of an automorphism of \( X \) depends only on \( \Gamma(X) \). Now \( \Gamma(X) \) depends up to conjugation by an element of the Weyl group only on the isometry class of \( X \). In particular the maximal Salem degree of an automorphism of \( X \) depends only on \( X \).

The proof

Lemma 5. Let \( N \subseteq L \) be two lattices of the same rank and \( G \subseteq O(L) \) a subgroup. Then

\[
[G : O(N) \cap G] < \infty
\]

where we view \( O(N) \) and \( O(L) \) as subgroups of \( O(\text{NS} N) \).

Proof. Since the ranks coincide, the index \( n = [L : N] \) is finite and \( nL \subseteq N \subseteq L \).

Any isometry of \( L \) preserves \( nL \), hence we get a map \( \varphi : G \to \text{Aut}(L/nL) \).

Set \( K = \ker \varphi \), which is a finite index subgroup of \( G \). So to show that \( K \subseteq O(N) \) as well recall that an isometry \( f \) of \( O(L) \) extends to \( O(N) \) iff \( f(N/nL) = N/nL \). Indeed, \( f|L/nL = \text{id}|L/nL \) for \( f \in K \) by definition.

Theorem 6. Let \( X, Y \) be two K3 surfaces over an algebraically closed field \( k \) and \( \kappa > 1 \). Suppose that \( \rho(X) = \rho(Y) \) and there is an isometric embedding \( \iota : \text{NS} X \to \text{NS} Y \).

Then \( \text{sdeg}(X) \leq \text{sdeg}(Y) \) where \( \text{sdeg}(X) = \max(\text{Salem degree of } f|L = \text{Aut}(X)) \).

Proof. Denote by \( \text{Nef}(X) \) and \( \text{Nef}(Y) \) the nef cones of \( X \) and \( Y \). Any chamber of the positive cone of \( \text{NS} X \) is contained in the image of a unique chamber of the positive cone of \( \text{NS} Y \). Since the Weyl group acts transitively on the chambers, we can find an element \( r \in W(\text{NS} Y) \) of the Weyl group such that \( \text{Nef}(X) \subseteq r \mathcal{U}(\text{Nef}(Y)) \). To ease notation we identify \( \text{NS} Y \) with its image under \( r \). By the preceding Lemma \( \Gamma(Y) \cap \Gamma(X) \subseteq O(\text{NS} Y) \) is finite, and since \( \text{Nef}(X) \subseteq \text{Nef}(Y) \), we get that \( \Gamma(X) \cap O(\text{NS} Y) \subseteq \Gamma(Y) \).

Now, by Theorem 4 and the discussion after it we can conclude with

\[
\text{sdeg}(X) = \text{sdeg}(\Gamma(X)) = \text{sdeg}(\Gamma(X) \cap O(\text{NS} Y)) \leq \text{sdeg}(\Gamma(Y)) \leq \text{sdeg}(Y).
\]

Proof of Theorem 3. It is well known that if \( X/k \) and \( Y/k \) are supersingular K3 surfaces with \( \sigma(X) \leq \sigma(Y) \), then \( \text{NS} Y \to \text{NS} X \). Combining the \( \sigma = 1 \) case and the previous theorem we get that \( 12 = \text{sdeg}(X) \leq \text{sdeg}(Y) \leq 22 \).

References


