Bertini theorems for canonical or klt 3-folds in positive characteristic

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§1. Introduction

Question

Let X be a quasi-projective normal variety over $k = \bar{k}$.

(Q1) If X has klt singularities, does the general hyperplane section $H \subseteq X$ have klt singularities ?

(Q2) How about canonical or terminal singularities?

Known results

- When $\dim(X) = 2$, both (Q1) and (Q2) hold.
- When X is locally complete intersection and $\dim(X) = 3$, (Q2) holds by using the theory of "MJ-canonical.(cf. [IR])
- When $\mathrm{char}(k) = 0$, both (Q1) and (Q2) hold.
- When $\mathrm{char}(k) > 0$, a positive characteristic analogue of (Q1) holds: The general hyperplane section of a strongly F-regular variety is strongly F-regular.

Remark about "strongly F-regular singularities"

- Strongly F-regular singularity is defined in terms of Frobenius morphisms and considered as a positive characteristic version of klt singularity.
- If a variety X is strongly F-regular, then X has klt singularities. The converse is not true in general.

Main Theorem

Let X be a quasi-projective normal variety over $k = \bar{k}$. Assume $\mathrm{char}(k) = p > 0$ and $\dim(X) = 3$.

(1) if $p > 5$, then (Q1) holds.

(2) for all $p > 0$, (Q2) holds.

§2. Proof of the Main Theorem

Proposition.1

Let $(R, \mathfrak{m})$ be a 2-dimensional normal local ring of $\mathrm{char}(R) = p > 0$ and F-finite (that is, $R^p \subseteq R$ is finite).

(1) When $p > 5$, R is strongly F-regular if and only if R has klt singularities.

(2) If R has canonical (resp. terminal) singularities, then R is complete intersection (resp. regular).

Remark

When the residue field $k(R) := R/\mathfrak{m}$ satisfies $k(R) = \overline{k(R)}$, (2) is well-known and (1) is proved in [Har]. By using essentially the same proof, we can show (2). For the proof of (1), see §3 below.

Proposition.2

Let X be a 3-dimensional normal variety oner $k = \bar{k}$.

(1) If X has klt singularities and $\mathrm{char}(k) > 5$, then we can take 0-dimensional closed set $Z \subseteq X$ such that $U := X \setminus Z$ is strongly F-regular.

(2) If X has canonical (resp. terminal) singularities, then we can take 0-dimensional closed set $Z \subseteq X$ such that $U := X \setminus Z$ is locally complete intersection (resp. regular).

proof of Proposition.2

(1) Take the *strongly F-regular locus* $U := \{P \in X \mid \mathcal{O}_{X,P} \text{is strongly } F\text{-regular}\} \subseteq X$. Then, U is open. By Proposition.1 (1), U contain all codimension 2 points. Hence, the dimension of $X \setminus U$ is at most 0.

(2) Take the *locally complete intersection locus* (resp. regular locus) $U \subseteq X$ and use Proposition.1 (2). □

proof of Main Theorem

(1)Take U and Z as in Proposition.2 (1). Since $\dim Z = 0$, the general hyperplane $H \subseteq X$ is the general hyperplane of U. By Bertini Theorem for strongly F-regular singularities, H is strongly F-regular and hence klt.

(2)Take U as in Proposition.2 (2). Since U is locally complete intersection and canonical (resp. regular), general $H \subseteq U$ has canonical singularities (resp. regular). □

§3. Proof of Proposition.1 (1)

To prove Proposition.1 (1), we use the *dual graph* of surface singularities. Let $(R, \mathfrak{m}, k = R/\mathfrak{m})$ be a 2-dimensional F-finite normal local ring. Take minimal resolution $f : Y \to X := \mathrm{Spec}\ (R)$ and $\mathrm{Exc}(f) = \bigcup_{i=1}^{n} E_i$. We define the weighted dual graph Γ_X as below:

vertex $E_1, \ldots, E_n$

edge the number of edges between E_i and E_j is the intersection number $(E_i \cdot E_j)$.

weights each vertex E_i has 3 weights: $r_i := \dim_k(H^0(\mathcal{O}_{E_i}))$, $b_i := (E_i^2)$, $g_i := 1 - \dim_k(H^1(\mathcal{O}_{E_i}))/r_i$.

Fact([Kol] §3)

There is the list of weighted dual graphs such that X is klt if and only if Γ_X is in the list.

sketch of proof of Proposition.1

By completion, we may assume that R contains k. When Γ is non-twisted (ie. $\forall i, r_i = 1$), we can show by essentially the same proof as [Har]. Assume that Γ is twisted. By extending the base field k, we can take an étale finite extension $R \subset S$ such that $p | \deg(S/R)$ and Γ_S is non-twisted graph. Hence we can reduce to the non-twisted case. □

Reference

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