

# Constructing $\mathbb{Q}$ -Fano 3-folds à la Prokhorov & Reid

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## Introduction

A  $\mathbb{Q}$ -Fano 3-fold  $X$  is a normal projective 3-dimensional variety over  $\mathbb{C}$  with  $-K_X$  ample, at worst  $\mathbb{Q}$ -factorial terminal singularities and Picard rank  $\rho_X = 1$ . The ( $\mathbb{Q}$ -Fano) index of  $X$  is:

$$q_X := \max \{q \in \mathbb{Z}_{\geq 1} : \exists A \in \text{Cl}(X), -K_X = qA\}$$

and, given a Weil divisor  $A$  for which  $-K_X = q_X A$ , we consider  $X$  to be **polarised by  $A$** , i.e. with an embedding into weighted projective space given by Proj of the graded ring

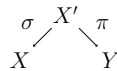
$$R(X, A) = \bigoplus_{k \geq 0} H^0(X, \mathcal{O}_X(kA)).$$

Some of the basic numerical invariants of  $(X, A)$  are the **codimension** of this embedding, the **index**  $q_X$  and the **degree**  $A^3$ .

The **graded ring database** contains a list of 1964 possible Hilbert series for a  $\mathbb{Q}$ -Fano 3-fold  $(X, A)$  of index  $\geq 2$ . However it is not known if a  $\mathbb{Q}$ -Fano 3-fold actually exists with each Hilbert series.

## Main result [2]

For each case in Table 1 we can construct a **Sarkisov link**:



where  $\sigma$  is a divisorial extraction from a certain kind of **irreducible singular curve**  $\Gamma \subset X$  and  $\pi$  is the **Kawamata blowdown** of a divisor  $E' \subset X'$  to a terminal cyclic quotient singularity.

**Table 1: The  $\mathbb{Q}$ -Fano 3-folds  $Y$  we can construct.**

deg $\Gamma$	$E$	$X$	$Y$
7	$\mathbb{P}(1, 2, 1)$	$\mathbb{P}^3$	$Y \subset \mathbb{P}(1^4, 2^2, 3)$
5	$\mathbb{P}(1, 2, 1)$	$X_2 \subset \mathbb{P}^4$	$Y \subset \mathbb{P}(1^5, 2^2, 3)$
14	$\mathbb{P}(1, 3, 2)$	$\mathbb{P}(1^3, 2)$	$Y \subset \mathbb{P}(1^3, 2^2, 3, 4, 5)$
9	$\mathbb{P}(1, 2, 3)$	$X_4 \subset \mathbb{P}(1^2, 2^2, 3)$	$Y \subset \mathbb{P}(1^2, 2^2, 3^2, 4, 5)$

The first two cases were **constructed by Prokhorov & Reid** [3]. We generalise their construction to get the remaining cases. We can also get two more examples, however in these cases  $\Gamma$  is necessarily **reducible** and hence  $Y$  will have **large Picard rank**  $\rho_Y > 1$ .

## A generalisation of Prokhorov & Reid's construction

(1) Embed  $E = \mathbb{P}(1, r, ra - 1)$  into a  $\mathbb{Q}$ -Fano 3-fold  $(X, A)$  such that  $A|_E = \mathcal{O}_E(r)$ . The point of considering such an embedding is that  $E$  has a  $\frac{1}{r}(1, -1)$  **type  $A_{r-1}$  singularity**  $P \in E$  which is supported at a **smooth point**  $P \in X$ .

**Lemma.** *Suppose that  $(X, A)$  admits an embedding  $E \subset X$  such that  $E \in |eA|$  and  $A|_E = \mathcal{O}_E(r)$ . Then  $X$  is either of the form  $\mathbb{P}(1, 1, a, ra - 1)$  or  $X_{ra} \subset \mathbb{P}(1, 1, a, ra - 1, e)$ . By an explicit classification there are precisely **10 cases** with terminal singularities.*

(2) Let  $\Gamma \subset E \subset X$  be an irreducible curve of degree  $d$  passing through  $P \in X$  which is contained in the smooth locus of  $X$ .

**Key claim:** *If  $\Gamma$  is has an 'appropriately singular' point at  $P \in X$ , then there exists a terminal divisorial extraction*

$$\sigma: (F \subset X') \rightarrow (\Gamma \subset X)$$

*such that  $\sigma$  induces an isomorphism  $E' \cong E$ , where  $E'$  is the birational transform of  $E$ .*

If  $\sigma$  exists then it is given by the blowup of the **symbolic powers** of the ideal sheaf  $\mathcal{I}_{\Gamma/X}$ . We take  $\sigma: X' \rightarrow X$  to be the left-hand side of our Sarkisov link.

(3) If we make the clever choice  $d = qr - 1$  then, following the 2-ray game that starts with  $\sigma$ , we find a **nef divisor**  $B'$  which is **numerically trivial** along  $E'$ . We check that the corresponding morphism  $\pi: X \rightarrow Y$  contracts  $E'$  to a  $\frac{1}{ra+r-1}(1, r, ra - 1)$  singularity.

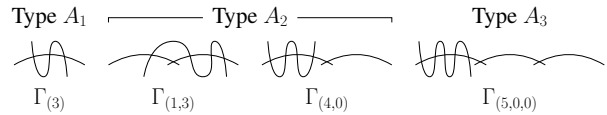
**Conclusion.** This construction is valid provided the divisorial extraction  $\sigma$  exists as in the **Key claim**. In this case we construct a Sarkisov link from  $(X, A)$  to a  $\mathbb{Q}$ -Fano 3-fold  $(Y, B)$  of index  $q_Y = q - e$  and degree  $B^3 = \frac{d}{ra+r-1}A^3$ .

## Divisorial extractions from singular curves

A type  $A_{r-1}$  Du Val singularity  $P \in E$  has a resolution given by a **chain of  $(-2)$ -curves**. We call  $P \in \Gamma \subset E$  a **curve singularity of type  $\Gamma_{(a_1, \dots, a_{r-1})}$**  if the strict transform of  $\Gamma$  on this resolution is smooth with  $a_i$  branches intersecting the  $i$ th exceptional divisor transversely.

We now explain what **'appropriately singular'** means:

**Proposition.** *For the 10 cases found in the Lemma,  $P \in E$  is either an  $A_1, A_2$  or  $A_3$  Du Val singularity. If a divisorial extraction  $\sigma$  from  $\Gamma \subset X$  exists as in the Key claim, then  $P \in \Gamma$  has one of the following singularity types (up to a degeneration):*

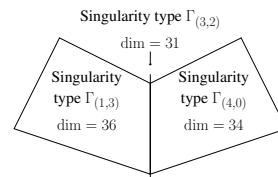


This follows from the unprojection method for constructing divisorial extractions explicitly [1] and by excluding cases according to  $\text{deg } \Gamma$ .

Now we can check that the only cases admitting one of these singularity types (for  $\Gamma$  **irreducible**) are the four cases of Table 1. In the third case of Table 1 **both** of the  $A_2$  singularity types are possible. In all other cases the singularity type is **unique**.

## Unprojection construction for $Y$

We can construct  $Y$  explicitly using **unprojection**. Prokhorov & Reid [3] did this for the first two cases in Table 1. The third case  $Y \subset \mathbb{P}(1^3, 2^2, 3, 4, 5)$  is interesting as there are **two possible constructions**, given by the **Tom and Jerry**. We have one 36-dimensional family of Tom unprojections and one 34-dimensional family of Jerry unprojections which correspond to Sarkisov links  $Y \dashrightarrow \mathbb{P}(1^3, 2)$  ending in a contraction to a curve with singularity  $\Gamma_{(1,3)}$  and  $\Gamma_{(4,0)}$  respectively.



**Figure 1:** Two families of  $Y \subset \mathbb{P}(1^3, 2^2, 3, 4, 5)$   $\mathbb{Q}$ -Fano 3-folds.

They intersect in a 31-dimensional family, where the general member has a Sarkisov link  $Y' \dashrightarrow \mathbb{P}(1^3, 2)$  ending in a contraction to a curve with singularity  $\Gamma_{(3,2)}$  (a **common degeneration** of  $\Gamma_{(1,3)}$  and  $\Gamma_{(4,0)}$ ). However such  $Y'$  has a non-terminal singularity of index 1.

**Further directions.** Construct Sarkisov links with flips, flops and antiflips in the middle, or Sarkisov links which end with a Mori fibre space contractions, or a different type of divisorial contraction.

## References

[1] T. Ducat, Divisorial extractions from singular curves in smooth 3-folds. *Int. J. Math.*, **27**, Issue 01 (2016), 23 pp.  
 [2] T. Ducat, Constructing  $\mathbb{Q}$ -Fano 3-folds à la Prokhorov & Reid, preprint, arxiv:1610.01773, 15 pp.  
 [3] Y. Prokhorov and M. Reid, On  $\mathbb{Q}$ -Fano threefolds of Fano index 2, in *Minimal Models and Extremal Rays (Kyoto 2011)*, Adv. Stud. in Pure Math., **70**, 2016, 397–420