Filtered de Rham-Witt complexes and wildly ramified higher class field theory over finite fields

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(report on joint work with Shuji Saito and Yigeng Zhao)

Let k be a finite field of characteristic $p > 0$, and let X be a smooth projective variety of dimension d over k. Let $D \subset X$ be a divisor with simple normal crossings, and let $U = X - D$ be the open complement. Then for any prime $\ell \neq p$, each natural number m, and each integer j we have a perfect pairing of finite groups for the étale cohomology groups

$$H^i(U, \mathbb{Z}/\ell^m(j)) \times H_c^{2d+1-i}(U, \mathbb{Z}/\ell^m(d-j)) \longrightarrow H_c^{2d+1}(U, \mathbb{Z}/\ell^m(d)) \cong \mathbb{Z}/\ell^m\,,$$

where $\mathbb{Z}/\ell^m(j)$ denotes the j-th Tate twist of the constant étale sheaf $\mathbb{Z}/\ell^m$, and $H_c^m(U, -)$ is the cohomology with compact support. This can be used to describe the quotient $\pi_1^{ab}(U)/\ell^n$ of the abelianized fundamental group $\pi_1^{ab}(U)$, and, by passing to the inverse limit, the maximal abelian ℓ-adic quotient of $\pi_1^{ab}(U)$. In fact, for $j = 0$ and $i = 1$ we get isomorphisms for all m

$$H_c^{2d}(U, \mathbb{Z}/\ell^m(d)) \cong H^1(U, \mathbb{Z}/\ell^m)^\vee \cong \pi_1^{\mathrm{ab}}(U)/\ell^m\,,$$

and an exact sequence

$$H^{2d-1}(D, \mathbb{Z}/\ell^m(d)) \to H_c^{2d}(U, \mathbb{Z}/\ell^m(d)) \to H^{2d}(X, \mathbb{Z}/\ell^m(d))$$

which provides a certain description of the middle group.

Now we consider p-coefficients. If D is empty, Milne obtained a perfect duality of finite groups

$$H^i(X, \nu_m^r) \times H^{d+1-i}(X, \nu_m^{d-r}) \to H^{d+1}(X, \nu_m^d) \cong \mathbb{Z}/p^r$$

where $\nu_m^r = \nu_{m,X}^r = W_m\Omega_{X,\log}^r \subset W_m\Omega_X^r$ are Illusie's logarithmic de Rham-Witt sheaves inside the components of the de Rham-Witt sheaves, which can be defined as the isomorphic image of the $d\log$ map

(1) $$d\log : \mathcal{K}_{r,X}^M/p^m \xrightarrow{\cong} \nu_m^r \subset W_m\Omega_X^r\,,$$

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on the Milnor K-sheaf of X sending $\{a_1,\dots,a_r\}$ to $d\log[a_1]_m\wedge\ldots\wedge d\log[a_r]_m$, where $[a]_m=(a,0,\dots,0)\in W_m(\mathcal{O}_X)$ is the Teichmüller representative. For $r=0$ one has $\nu^0_m=\mathbb{Z}/p^m$, and Milne's duality induces isomorphisms

$$H^d(X,\nu^d_m)\cong H^1(X,\mathbb{Z}/p^m)^\vee\cong\pi_1^{ab}(X)/p^m$$

If one tries to extend the above duality to the case where D is non-empty, one encounters the problem that there is no obvious analog of cohomology with compact support for de Rham-Witt sheaves or logarithmic de Rham-Witt sheaves. We propose the following approach. Let $\{D_\lambda\}_{\lambda\in\Lambda}$ be the (smooth) irreducible components of D. For $r\geq 1$ and $D=\sum_\lambda n_\lambda D_\lambda$ with $n_\lambda\in\mathbb{N}_0$, not all zero, let

$$\nu^r_{m,X|D}=W_m\Omega^r_{X|D,\log}\subset j_*W_m\Omega^r_{U,\log}$$

be the étale subsheaf generated étale locally by $d\log[x_1]_m\wedge\ldots d\log[x_r]_m$ with $x_\nu\in\mathcal{O}_U^\times$ for all ν and $x_1\in 1+\mathcal{O}_X(-D)$, and let $\nu^0_{m,X|D}=0$.

As in the classical situation we have the following theorem:

Theorem 1. *The map dlog induces an isomorphism*

$$d\log[-]:\mathcal{K}^M_{r,X|D}/p^m\xrightarrow{\cong}W_m\Omega^r_{X|D,\log}\,;\,\{x_1,\dots,x_r\}\mapsto d\log[x_1]_m\wedge\ldots d\log[x_r]_m.$$

Here $\mathcal{K}^M_{r,X|D}$ is the sheaf of relative (to D) Milnor K-groups which has been studied by one of the authors (S. Saito) with K. Rülling in [1].

The following property is immediate.

Lemma 1. *For $D_1\leq D_2$ we have $\nu^r_{m,X|D_2}\subseteq\nu^r_{m,X|D_1}$.*

Moreover we show

Theorem 2. *There is an exact sequence*

$$0\to\nu^r_{m-1,X|[D/p]}\to\nu^r_{m,X|D}\to\nu^r_{1,X|D}\to 0\ ,$$

where $[D/p]=\sum_{\lambda\in\Lambda}[n_\lambda/p]D_\lambda$, with $[n_\lambda/p]=min\{n'\in\mathbb{Z}\mid n'\geq n/p\}$.

By the isomorphism (1) above this is reduced to (difficult) calculations in Milnor K-theory of local rings (see [1]). Moreover, in analogy to Illusie's exact sequence

$$0\to\nu^r_{1,X}\to\Omega^r_X\overset{1-C^{-1}}{\longrightarrow}\Omega^r_X/d\Omega^{r-1}_X\to 0 \tag{2}$$

we prove the following.

Theorem 3. *One has an exact sequence*

$$0 \to \nu^r_{1,X|D} \to \Omega^r_{X|D} \overset{1-C^{-1}}{\longrightarrow} \Omega^r_{X|D}/d\Omega^{r-1}_{X|D} \to 0 \ ,$$

where $\Omega^r_{X|D} = \Omega^r_X(\log D_{red})(-D)$.

An important tool for the duality is the introduction of Γ-filtered rings A for a not necessarily totally ordered abelian group Γ, given by collections of subgroups A^γ with $A^{\gamma'} \subset A^\gamma$ for $\gamma \leq \gamma'$ (descending filtration!) and $A^\gamma \dots A^{\gamma'} \subset A^{\gamma+\gamma'}$.

For a Γ-filtered ring A we consider an associated filtration on the Witt rings which is inspired by a filtration introduced by Kato and Brylinski: We say that a Witt vector $a = (a_0, a_1, a_2, \dots)$ is in $W(A)^\gamma$ if $a_i \in A^{p^i\gamma}$ for all $i \geq 0$.

Moreover, with this we define filtered de Rham-Witt sheaves, by using the universal definition of Hesselholt and Madsen [1] for $p \neq 2$, and Costeanu [3] for $p = 2$.

In our situation we start with the descending filtration on $j_*\mathcal{O}_U$, where for a divisor $D = \sum_\lambda n_\lambda D_\lambda$ with $n_\lambda \in \mathbb{Z}$ we define

$$f^D\mathcal{O}_U := \mathcal{O}_X(-D),$$

and the associated filtered de Rham-Witt complex

$$f^D W_m\Omega^r_U \subset j_* W_m\Omega^r_U.$$

Then we get

Theorem 4. *There is a a perfect pairing between a discrete group and a profinite group*

$$H^i(U, \nu^r_{m,U}) \times \lim_{\underset{D}{\longleftarrow}} H^{d+1-i}(X, \nu^{d-r}_{m,X|D}) \longrightarrow H^{d+1}(X, \nu^d_{m,X}) \cong \mathbb{Z}/p^m\mathbb{Z}$$

where the inverse limit is over the divisors (with multiplicities) D with support in D_{red} *(compare Lemma 1).*

The proof is in two steps. First of all, the pairings give a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \lim\limits_{\underset{D}{\longleftarrow}} H^{d+1-i}(X,\nu^{d-r}_{m-1,X|[D/p]}) & \longrightarrow & \lim\limits_{\underset{D}{\longleftarrow}} H^{d+1-i}(X,\nu^{d-r}_{m,X|D}) & \longrightarrow & \lim\limits_{\underset{D}{\longleftarrow}} H^{d+1-i}(X,\nu^{d-r}_{1,X|D}) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^i(U,\nu^r_{m-1,U})^\vee & \longrightarrow & H^i(U,\nu^r_m,U)^\vee & \longrightarrow & H^i(U,\nu^r_1)^\vee & \longrightarrow \cdots,
\end{array}$$

where the first row is induced by Theorem 1, and its exactness is due to the fact that the inverse limit is exact for projective systems of finite groups; and where the second row comes from the classical exact sequence

$$0 \to \nu^r_{1,U} \to \nu^r_{m,U} \to \nu^r_{m-1,U} \to 0.$$

so that its exactness is clear. Using this commutative diagram and induction on m, we reduce our question to the case $m = 1$. For $m = 1$ we use Theorem 2 to replace $\nu^r_{1,X|D}$ in the derived category by the two-term complex

$$\mathcal{F}^\bullet = [\Omega^r_{X|-D+D_{red}} \xrightarrow{1-C^{-1}} \Omega^r_{X|-D+D_{red}}/d\Omega^{r-1}_{X|-D+D_{red}}],$$

and similarly one can show that, in the derived category, $\nu^{d-r}_{1,U}$ is isomorphic to the direct limit(with respect to D) of two-term complexes

$$\mathcal{G}^\bullet = [Z\Omega^{d-r}_{X|D} \xrightarrow{1-C} \Omega^{d-r}_{X|D}],$$

where $Z\Omega^{d-r}_{X|D} = Z\Omega^{d-r}_X \cap \Omega^{d-r}_{X|D}$.

Finally $\nu^d_{1,X}$ is isomorphic to the two-term complex

$$\mathcal{H}^\bullet = [\Omega^d_X \xrightarrow{1-C} \Omega^d_X].$$

Now we use Milne's method of pairings of two-term complexes to see that one has a non-degenerate pairing

$$\mathcal{F}^\bullet \times \mathcal{G}^\bullet \longrightarrow \mathcal{H}^\bullet$$

which reduces the pairing to a duality in coherent $\mathcal{O}_X$-sheaves, which also works in étale cohomology, and therefore also gives perfect parings in étale cohomology

$$H^{d+1-i}(X, \mathcal{F}^\bullet) \times H^i(X, \mathcal{G}^\bullet) \longrightarrow H^{d+1}(X, \mathcal{H}^\bullet) \cong \mathbb{Z}/p^m\mathbb{Z}.$$

Passing to the inductive limit over D on the first terms and the inverse limit on D on the right term we obtain the wanted pairing in Theorem 3, because the left limit gives $H^i(U, \nu^r_{1,U})$. For $i = 1$ and $r = 0$ Theorem 3 now gives a continuous isomorphism

$$\varprojlim_D H^d(X, \nu^d_{m,X|D}) \longrightarrow H^1(U, \mathbb{Z}/p^m)^\vee \cong \pi_1^{ab}(U)/p^m$$

which gives a canonical ramification filtration of the abelianized fundamental group on the right. A quotient is ramified of order D if it factors through $H^{d+1-i}(X, \nu^{d-r}_{1,X|D})$.

References

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