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COMPLEX HYPERBOLICITY PROBLEMS RELATED TO ABELIAN VARIETIES

KATSUTOSHI YAMANOI

1. HYPERBOLICITY PROBLEMS IN ALGEBRAIC SETTING

Let $X$ be a smooth complex projective variety.

**Conjecture 1.** Assume that $X$ is of general type. Let $L$ be an ample line bundle on $X$. Then there exists a positive number $\alpha > 0$ and a proper Zariski closed subset $Z \subsetneq X$ such that for every holomorphic map $g : C \to X$ from a compact Riemann surface $C$ with $g(C) \not\subset Z$, we have

$$\deg_C g^*(L) \leq \alpha \times \{\operatorname{genus}(C) - 1\}$$

We introduce the following special set after S. Lang:

**Definition 1** (Special set). Consider all non-constant rational maps $\varphi : A \dashrightarrow X$ from abelian varieties $A$ and define the special set of $X$ by

$$\operatorname{Sp}(X) = \bigcup_{\varphi : A \dashrightarrow X} \overline{\operatorname{Zariski closure}} \varphi(A)$$

By the definition, $\operatorname{Sp}(X)$ is a Zariski closed set of $X$. Note that the special set $\operatorname{Sp}(X)$ contains all rational and elliptic curves in $X$. This is trivial for elliptic curves in $X$ by the definition. For the rational curves $f : P^1 \to X$, we take the composite with a double covering $\varphi : E \to P^1$ from an elliptic curve $E$ to get a morphism $f \circ \varphi : E \to X$ from the elliptic curve. So $f(P^1) = f \circ \varphi(E) \subset \operatorname{Sp}(X)$.

**Conjecture 2** (Lang [7]). Assume $X$ is of general type. Then the special set is a proper Zariski closed subset $\operatorname{Sp}(X) \subsetneq X$.

Conjecture 1 implies Conjecture 2. Indeed, let $\varphi : A \dashrightarrow X$ be a non-constant rational map. We want to show that $\varphi(A) \subset Z$, where $Z \subsetneq X$ is the Zariski closed set in Conjecture 1. So suppose contrary that $\varphi(A) \not\subset Z$. Let $E \subset A$ be the indeterminacy set of $\varphi$. Then $\operatorname{codim} E \geq 2$. Let $n_A : A \to A$ be the $n$-times map. Let $C \subset A$ be a smooth curve in $A$ such that $n_A(C) \not\subset E \cup \varphi^{-1}(Z)$ for all $n$. For an ample line bundle $L$ on $X$, we have $n_A^* c_1(\varphi^* L) = n^2 c_1(\varphi^* L) \in H^2(A, \mathbb{Z})$, where $c_1(\varphi^* L) \in H^2(A, \mathbb{Z})$ is the characteristic class of $\varphi^* L$. Now we consider the holomorphic map $\varphi \circ n_A|_C : C \to X$. Then

$$\deg_C (\varphi \circ n_A|_C)^* L = n^2 \deg_C (\varphi|_C)^* L \to \infty$$

as $n \to \infty$, but $\varphi \circ n_A|_C(C) \not\subset Z$. This contradicts to Conjecture 1. Thus $\varphi(A) \subset Z$. □

These conjectures are open even for the case of $\dim X = 2$. Some known cases are the following:

- Surfaces of general type with $c_1(X)^2 > c_2(X)$. Conjecture 1 is valid ([1]).
- Subvarieties of general type on abelian varieties. Conjecture 2 is valid ([5]).
1. Very generic hypersurfaces of degree \( d \geq 2n + 2 \) in \( \mathbb{P}^{n+1} \). Conjecture 1 is valid with stronger conclusion that \( Z = \emptyset \) ([2]).

We say that \( X \) is of maximal albanese dimension if the dimension of the image \( a(X) \) of the albanese map \( a : X \to \text{Alb}(X) \) is equal to the dimension of \( X \).

2. Hyperbolicity problems in complex analytic setting

Let \( M \) be a complex manifold. We first introduce Kobayashi-Royden pseudo-metric \( F_M : TM \to \mathbb{R}_{\geq 0} \) as follows. Set \( \Delta = \{ z \in \mathbb{C}; |z| < 1 \} \).

**Definition 2** (Kobayashi-Royden pseudo-metric). For \( v \in T_x(M) \),

\[
F_M(v) = \inf \left\{ \frac{1}{r} ; \exists f : \Delta \to M \text{ holomorphic s.t. } f(0) = x, \ f'(0) = rv \right\}.
\]

**Proposition 1.** Let \( \varphi : X \to Y \) be a holomorphic map between complex manifolds, let \( \varphi_* : TX \to TY \) be the induced map. Then for all \( v \in TX \),

\[
F_X(v) \geq F_Y(\varphi_*(v)).
\]

In particular, every holomorphic automorphism is an isometry of \( F_X \).

**Proof.** This follows directly from the definition. We prove, for arbitrary \( v \in TX \),

\[
F_X(v) \geq F_Y(\varphi_*(v)).
\]

Take an arbitrary \( 1/r > F_X(v) \). It is enough to show \( 1/r > F_Y(\varphi_*(v)) \). There exists \( f : \Delta \to X \) such that \( f'(0) = rv \). Hence \( \varphi \circ f : \Delta \to Y \) satisfies \( (\varphi \circ f)'(0) = r\varphi_*(v) \). Hence \( F_Y(\varphi_*(v)) < 1/r \), so \( F_X(v) \geq F_Y(\varphi_*(v)) \). \( \square \)

**Definition 3** (Kobayashi pseudo-distance). For \( x, y \in M \), we define

\[
d_M(x, y) = \inf_{\gamma} \int_{\gamma} F_M(\gamma'(t))dt
\]

where the infimum is taken over all piecewise smooth curves \( \gamma \) joining \( x \) and \( y \).

**Corollary 1.** For all \( x, y \in X \), we have \( d_X(x, y) \geq d_Y(\varphi(x), \varphi(y)) \).

**Definition 4** (S. Kobayashi [6]). A complex manifold \( M \) is Kobayashi hyperbolic if \( d_M \) is a true distance, i.e., \( d_M(p, q) > 0 \) whenever \( p \neq q \).

**Examples.**

1. If \( M = \Delta \), then \( F_\Delta = \frac{|dz|}{1 - |z|^2} \). Hence \( d_\Delta \) is the Poincaré distance of the unit disc \( \Delta \).

**Proof.** Both \( F_\Delta \) and \( |dz|/(1 - |z|^2) \) are isometric under \( \text{Aut}(\Delta) \). Hence it is enough to compare them over \( 0 \in \Delta \), and show \( F_\Delta(\partial/\partial z) = 1 \). The existence of \( \text{id}_\Delta : \Delta \to \Delta \) yields \( F_\Delta(\partial/\partial z) \leq 1 \). By the Schwarz lemma, if \( f : \Delta \to \Delta \) is holomorphic with \( f(0) = 0 \) and \( f'(0) = r \frac{a}{b} \), then \( 1 \leq 1/r \). This shows \( 1 \leq F_\Delta(\partial/\partial z) \). Hence \( F_\Delta(\partial/\partial z) = 1 \). \( \square \)
(1) If $X$ is a compact Riemann surface of genus$(X) \geq 2$, then $d_X$ is equal to the Poincaré distance on $X$ induced from the universal covering map $\Delta \to X$. In particular, $X$ is Kobayashi hyperbolic.

(2) If $M = \mathbb{C}$, then $F_\mathbb{C} \equiv 0$ and $d_\mathbb{C} \equiv 0$.

Proof. For arbitrary $R > 0$, let $f_R : \Delta \to \mathbb{C}$ be defined by $f_R(z) = Rz$. Then $f_R(0) = 0$ and $f'_R(0) = R \frac{\partial}{\partial z}$. Hence $F_\mathbb{C}(\partial/\partial z) \leq 1/R$. Since $R > 0$ is arbitrary, $F_\mathbb{C}(\partial/\partial z) = 0$. Using automorphism of $\mathbb{C}$, $F_\mathbb{C} \equiv 0$. \qed

(3) If $X$ is a projective space or an Abelian variety, then $d_X \equiv 0$.

Proof. For every $x, y \in X$, there exists a holomorphic map $f : \mathbb{C} \to X$ such that $f(0) = x$ and $f(1) = y$. So by the distance decreasing property, we have $d_X(x, y) \leq d_\mathbb{C}(0, 1) = 0$. \qed

Thus if $X$ is a smooth projective variety with dim $X = 1$, the followings are equivalent:

(1) genus$(X) \geq 2$.
(2) $X$ is of general type.
(3) $X$ is Kobayashi hyperbolic.

Indeed, if genus$(X) = 0$ or 1, then $X$ is $\mathbb{P}^1$ or elliptic curve, so $d_X \equiv 0$. Hence (3) implies (1).

For dim $X \geq 2$, a projective variety $X$ of general type need not be Kobayashi hyperbolic. Indeed $X$ may contain rational or elliptic curves even if $X$ is of general type, and such $X$ is not Kobayashi hyperbolic. We introduce weaker notion.

Definition 5. Let $M$ be a complex manifold and let $S \subset M$ be a closed subset. $M$ is Kobayashi hyperbolic modulo $S$, if $d_M$ is a true distance outside $S$.

Conjecture 3 ([6]). If $X$ is a projective variety of general type, then $X$ is pseudo-Kobayashi hyperbolic, i.e., there exists a proper Zariski closed subset $S \subsetneq X$ such that $X$ is Kobayashi hyperbolic modulo $S$.

Conjecture 3 implies Conjecture 2. We want to show that the image of every non-constant rational map $A \dasharrow X$ from abelian variety $A$ should be contained in $S$, where $S \subsetneq X$ is the proper Zariski closed subset appeared in Conjecture 3. This follows from the distance decreasing property of Kobayashi pseudo-distance and $d_A \equiv 0$. Hence $\text{Sp}(X) \subset S \subsetneq X$. \qed

It is not clear whether or not Conjecture 3 implies Conjecture 1. However the following implication is known by Demailly.

Theorem 1 ([3]). Let $X$ be a Kobayashi hyperbolic projective manifold. Let $L$ be an ample line bundle on $X$. Then there exists a positive number $\alpha > 0$ such that for every holomorphic map $g : C \to X$ from a compact Riemann surface $C$, we have

$$\deg_C g^*(L) \leq \alpha \times \{\text{genus}(C) - 1\}.$$ 

Proof. Let $\omega_L$ be a curvature form of $L$, which is a positive (1,1)-form on $X$. For $v \in TX$, we denote by $|v|_{\omega_L}$ the norm associated to $\omega_L$. There exists a positive constant $\delta > 0$ such that $F_X(v) \geq \delta |v|_{\omega_L}$ for all $v \in TX$; For otherwise, Brody’s reparametrization argument yields a non-constant holomorphic map $C \to X$, which contradicts to the assumption that $X$ is Kobayashi hyperbolic. By the distance decreasing property, we
have $F_C(v) \geq F_X(g, v)$ for all $v \in TC$. Hence we have $F_C(v) \geq \delta|v|g_\omega \omega_L$ for all $v \in TC$. Since $F_C$ coincides with Poincaré metric on $C$, we have
\[
\omega_C \geq \delta g^* \omega_L,
\]
where $\omega_C$ is the (1,1)-form on $C$ associated to the Poincaré metric. By integrating both sides of this estimate, we conclude the proof. □

Recently, Conjecture 3 is verified for subvarieties of general type on abelian varieties ([9]).

**Theorem 2.** Let $X$ be a subvariety of general type on an abelian variety. Then $X$ is Kobayashi hyperbolic modulo $\text{Sp}(X)$. □

Recall that $\text{Sp}(X)$ is a proper Zariski closed set $\text{Sp}(X) \subsetneq X$ by [5], where $X$ is a subvariety of general type on an abelian variety $A$. Moreover, $\text{Sp}(X)$ has more clear description without taking Zariski closure:
\[
\text{Sp}(X) = \{x \in X; \exists B \subset A, \text{an abelian variety s.t. dim}(B) > 0 \text{ and } x + B \subset X\}.
\]

**Corollary 2.** For subvariety $X$ of an abelian variety $A$, we have
\[
X \text{ is of general type } \iff X \text{ is pseudo-Kobayashi hyperbolic}
\]

**Proof.** The direction $\Longrightarrow$ is by the theorem. The converse follows by a theorem of Ueno. Indeed, if $X$ is not of general type, then the stabilizer of $X$ is positive dimensional. Hence there exists a positive dimensional abelian subvariety $B \subset A$ such that $x + B \subset X$ for all $x \in X$. Hence, for each point $x \in X$, there exists a non-constant map $B \to X$ passing through $x$. Hence by the distance decreasing property and $d_B \equiv 0$, we conclude that $X$ is not pseudo-Kobayashi hyperbolic. □

Theorem 2 is a generalization of the following theorems for a subvariety $X$ of general type on an abelian variety.

(1) **Green’s theorem** states that if $\text{Sp}(X) = \emptyset$, then $X$ is Kobayashi hyperbolic ([4]). Indeed our theorem implies that $X$ is Kobayashi hyperbolic modulo $\emptyset$, hence $X$ is Kobayashi hyperbolic. Green actually proved that there is no non-constant holomorphic map $f : \mathbb{C} \to X$, if $\text{Sp}(X) = \emptyset$. By Brody’s criterion, this implies that $X$ is Kobayashi hyperbolic.

(2) More generally, **Bloch-Ochiai-Kawamata’s theorem** states that every non-constant holomorphic map $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset \text{Sp}(X)$ ([5]). This follows from our theorem by the distance decreasing property and $d_C \equiv 0$. However, from this theorem, we can’t conclude any information for Kobayashi pseudo-distance $d_X$.

In complex analysis, there is a philosophical principle stated by A. Bloch as follows: *Nothing exists in the infinite plane that has not been previously done in the finite disk.* This principle says that if some property $\mathcal{P}$ reduces analytic maps in $\mathbb{C}$ to a constant, then a family of analytic maps in $\Delta$ with the property $\mathcal{P}$ will be normal. The property of ‘bounded analytic function’ is one such example; Liouville’s theorem corresponds to Montel’s theorem. The following theorem may be considered as one example of this principle which corresponds to Bloch-Ochiai-Kawamata’s theorem above.

We denote by $\text{Hol}(\Delta, X)$ the set of all holomorphic mappings $f : \Delta \to X$.
Theorem 3. Let $X$ be a subvariety of general type on an abelian variety. Then for each sequence $\{f_n\}_{n=1}^{\infty}$ in $\text{Hol}(\Delta, X)$, we have one of the following:

1. $\{f_n\}_{n=1}^{\infty}$ has a subsequence which converges locally uniformly to some $f \in \text{Hol}(\Delta, X)$, or

2. for each compact subset $K \subset \Delta$ and each compact subset $L \subset X \setminus \text{Sp}(X)$, there exists an integer $n_0$ such that $f_n(K) \cap L = \emptyset$ for all $n \geq n_0$.

In other words, $X$ is taut modulo $\text{Sp}(X)$ with terminology in [6]. Theorem 2 follows from this theorem. Indeed, Theorem 2 is an integrated version of the following lower estimate of $F_X$.

Corollary 3 (Lower estimate of $F_X$). For each open neighborhood $U \subset X$ of $\text{Sp}(X)$, there exists a positive constant $c > 0$ such that $F_X(v) \geq \frac{1}{c|v|}$ for all $v \in T_xX$ with $x \notin U$.

Proof. We first show that there exists a positive constant $c > 0$ such that for every $f \in \text{Hol}(\Delta, X)$ with $f(0) \notin U$, we have $|f'(0)| < c$. For otherwise, there exists a sequence $\{f_n\} \subset \text{Hol}(\Delta, X)$ such that $f_n(0) \notin U$ and $|f'_n(0)| \to \infty$. For this sequence, the second conclusion of Theorem 3 does not occur. Hence there exists a subsequence $\{f_{n_k}\}$ which converges locally uniformly to some $f \in \text{Hol}(\Delta, X)$, so $|f'_{n_k}(0)| \to |f'(0)|$. This is a contradiction.

Now to measure $F_X(v)$, we take $f \in \text{Hol}(\Delta, X)$ with $f'(0) = rv$. Then we have $|f'(0)| = r|v| \leq c$. Hence $|v|/c \leq 1/r$. So by the definition of $F_X(v)$, we have $|v|/c \leq F_X(v)$. □

References


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