COMPLEX HYPERBOLICITY PROBLEMS RELATED TO ABELIAN VARIETIES

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1. Hyperbolicity problems in algebraic setting

Let X be a smooth complex projective variety.

Conjecture 1. *Assume that X is of general type. Let L be an ample line bundle on X. Then there exists a positive number $\alpha > 0$ and a proper Zariski closed subset $Z \subsetneqq X$ such that for every holomorphic map $g : C \to X$ from a compact Riemann surface C with $g(C) \not\subset Z$, we have*

$$\deg_C g^*(L) \leq \alpha \times \{\text{genus}(C) - 1\}$$

We introduce the following special set after S. Lang:

Definition 1 (Special set)**.** *Consider all non-constant rational maps $\varphi : A \dashrightarrow X$ from abelian varieties A and define the special set of X by*

$$\mathrm{Sp}(X) = \overline{\bigcup_{\varphi: A \dashrightarrow X} \varphi(A)}^{\textit{Zariski closure}}$$

By the definition, $\mathrm{Sp}(X)$ is a Zariski closed set of X. Note that the special set $\mathrm{Sp}(X)$ contains all rational and elliptic curves in X. This is trivial for elliptic curves in X by the definition. For the rational curves $f : \mathbb{P}^1 \to X$, we take the composite with a double covering $\varphi : E \to \mathbb{P}^1$ from an elliptic curve E to get a morphism $f \circ \varphi : E \to X$ from the elliptic curve. So $f(\mathbb{P}^1) = f \circ \varphi(E) \subset \mathrm{Sp}(X)$.

Conjecture 2 (Lang [7])**.** *Assume X is of general type. Then the special set is a proper Zariski closed subset $\mathrm{Sp}(X) \subsetneqq X$.*

Conjecture 1 implies Conjecture 2. Indeed, let $\varphi : A \dashrightarrow X$ be a non-constant rational map. We want to show that $\varphi(A) \subset Z$, where $Z \subsetneqq X$ is the Zariski closed set in Conjecture 1. So suppose contrary that $\varphi(A) \not\subset Z$. Let $E \subset A$ be the indeterminacy set of φ. Then $\mathrm{codim} E \geq 2$. Let $n_A : A \to A$ be the n-times map. Let $C \subset A$ be a smooth curve in A such that $n_A(C) \not\subset E \cup \varphi^{-1}(Z)$ for all n. For an ample line bundle L on X, we have $n_A^* c_1(\varphi^* L) = n^2 c_1(\varphi^* L) \in H^2(A, \mathbb{Z})$, where $c_1(\varphi^* L) \in H^2(A, \mathbb{Z})$ is the characteristic class of $\varphi^* L$. Now we consider the holomorphic map $\varphi \circ n_A|_C : C \to X$. Then

$$\deg_C(\varphi \circ n_A|_C)^* L = n^2 \deg_C(\varphi|_C)^* L \to \infty$$

as $n \to \infty$, but $\varphi \circ n_A|_C(C) \not\subset Z$. This contradicts to Conjecture 1. Thus $\varphi(A) \subset Z$. □

These conjectures are open even for the case of $\dim X = 2$. Some known cases are the following:

- Surfaces of general type with $c_1(X)^2 > c_2(X)$. Conjecture 1 is valid ([1]).
- Subvarieties of general type on abelian varieties. Conjecture 2 is valid ([5]).

- Very generic hypersurfaces of degree $d \geq 2n+2$ in $\mathbb{P}^{n+1}$. Conjecture 1 is valid with stronger conclusion that $Z = \emptyset$ ([2]).

We say that X is of *maximal albanese dimension* if the dimension of the image $a(X)$ of the albanese map $a : X \to \mathrm{Alb}(X)$ is equal to the dimension of X.

- Varieties of maximal albanese dimension and of general type. Conjecture 1 is valid ([8]).

2. Hyperbolicity problems in complex analytic setting

Let M be a complex manifold. We first introduce Kobayashi-Royden pseudo-metric $F_M : TM \to \mathbb{R}_{\geq 0}$ as follows. Set $\Delta = \{z \in \mathbb{C};\ |z| < 1\}$.

Definition 2 (Kobayashi-Royden pseudo-metric)**.** *For $v \in T_x(M)$,*

$$F_M(v) = \inf\left\{\frac{1}{r};\ \exists f : \Delta \to M \text{ holomorphic s.t. } f(0) = x,\ \ f'(0) = rv\right\}.$$

Proposition 1. *Let $\varphi : X \to Y$ be a holomorphic map between complex manifolds, let $\varphi_* : TX \to TY$ be the induced map. Then for all $v \in TX$,*

$$F_X(v) \geq F_Y(\varphi_*(v)).$$

In particular, every holomorphic automorphism is an isometry of F_X.

Proof. This follows directly from the definition. We prove, for arbitrary $v \in TX$,

$$F_X(v) \geq F_Y(\varphi_*(v)).$$

Take an arbitrary $1/r > F_X(v)$. It is enough to show $1/r > F_Y(\varphi_*(v))$. There exists $f : \Delta \to X$ such that $f'(0) = rv$. Hence $\varphi \circ f : \Delta \to Y$ satisfies $(\varphi \circ f)'(0) = r\varphi_*(v)$. Hence $F_Y(\varphi_*(v)) < 1/r$, so $F_X(v) \geq F_Y(\varphi_*(v))$. □

Definition 3 (Kobayashi pseudo-distance)**.** *For $x, y \in M$, we define*

$$d_M(x,y) = \inf_{\gamma} \int_{\gamma} F_M(\gamma'(t))dt$$

where the infimum is taken over all piecewise smooth curves γ joining x and y.

Corollary 1. *For all $x, y \in X$, we have $d_X(x,y) \geq d_Y(\varphi(x), \varphi(y))$.*

Definition 4 (S. Kobayashi [6])**.** *A complex manifold M is Kobayashi hyperbolic if d_M is a true distance, i.e., $d_M(p,q) > 0$ whenever $p \neq q$.*

Examples.

(1) If $M = \Delta$, then $F_\Delta = \dfrac{|dz|}{1-|z|^2}$. Hence d_Δ is the Poincaré distance of the unit disc Δ.

Proof. Both F_Δ and $|dz|/(1-|z|^2)$ are isometric under $\mathrm{Aut}(\Delta)$. Hence it is enough to compare them over $0 \in \Delta$, and show $F_\Delta(\partial/\partial z) = 1$. The existence of $\mathrm{id}_\Delta : \Delta \to \Delta$ yields $F_\Delta(\partial/\partial z) \leq 1$. By the Schwarz lemma, if $f : \Delta \to \Delta$ is holomorphic with $f(0) = 0$ and $f'(0) = r\frac{\partial}{\partial z}$, then $1 \leq 1/r$. This shows $1 \leq F_\Delta(\partial/\partial z)$. Hence $F_\Delta(\partial/\partial z) = 1$. □

(1)' If X is a compact Riemann surface of genus$(X) \geq 2$, then d_X is equal to the Poincaré distance on X induced from the universal covering map $\Delta \to X$. In particular, X is Kobayashi hyperbolic.

(2) If $M = \mathbb{C}$, then $F_{\mathbb{C}} \equiv 0$ and $d_{\mathbb{C}} \equiv 0$.

Proof. For arbitrary $R > 0$, let $f_R : \Delta \to \mathbb{C}$ be defined by $f_R(z) = Rz$. Then $f_R(0) = 0$ and $f'_R(0) = R\frac{\partial}{\partial z}$. Hence $F_{\mathbb{C}}(\partial/\partial z) \leq 1/R$. Since $R > 0$ is arbitrary, $F_{\mathbb{C}}(\partial/\partial z) = 0$. Using automorphism of $\mathbb{C}$, $F_{\mathbb{C}} \equiv 0$. $\square$

(3) If X is a projective space or an Abelian variety, then $d_X \equiv 0$.

Proof. For every $x, y \in X$, there exists a holomorphic map $f : \mathbb{C} \to X$ such that $f(0) = x$ and $f(1) = y$. So by the distance decreasing property, we have $d_X(x, y) \leq d_{\mathbb{C}}(0, 1) = 0$ $\square$

Thus if X is a smooth projective variety with $\dim X = 1$, the followings are equivalent:
(1) genus$(X) \geq 2$.
(2) X is of general type.
(3) X is Kobayashi hyperbolic.

Indeed, if genus$(X) = 0$ or 1, then X is $\mathbb{P}^1$ or elliptic curve, so $d_X \equiv 0$. Hence (3) implies (1).

For $\dim X \geq 2$, a projective variety X of general type need not be Kobayashi hyperbolic. Indeed X may contain rational or elliptic curves even if X is of general type, and such X is not Kobayashi hyperbolic. We introduce weaker notion.

Definition 5. *Let M be a complex manifold and let $S \subset M$ be a closed subset. M is Kobayashi hyperbolic modulo S, if d_M is a true distance outside S.*

Conjecture 3 ([6])**.** *If X is a projective variety of general type, then X is pseudo-Kobayashi hyperbolic, i.e., there exists a proper Zariski closed subset $S \subsetneqq X$ such that X is Kobayashi hyperbolic modulo S.*

Conjecture 3 implies Conjecture 2. We want to show that the image of every non-constant rational map $A \dashrightarrow X$ from abelian variety A should be contained in S, where $S \subsetneqq X$ is the proper Zariski closed subset appeared in Conjecture 3. This follows from the distance decreasing property of Kobayashi pseudo-distance and $d_A \equiv 0$. Hence $\mathrm{Sp}(X) \subset S \subsetneqq X$. $\square$

It is not clear whether or not Conjecture 3 implies Conjecture 1. However the following implication is known by Demailly.

Theorem 1 ([3])**.** *Let X be a Kobayashi hyperbolic projective manifold. Let L be an ample line bundle on X. Then there exists a positive number $\alpha > 0$ such that for every holomorphic map $g : C \to X$ from a compact Riemann surface C, we have*

$$\deg_C g^*(L) \leq \alpha \times \{\mathrm{genus}(C) - 1\}.$$

Proof. Let ω_L be a curvature form of L, which is a positive (1,1)-form on X. For $v \in TX$, we denote by $|v|_{\omega_L}$ the norm associated to ω_L. There exists a positive constant $\delta > 0$ such that $F_X(v) \geq \delta |v|_{\omega_L}$ for all $v \in TX$; For otherwise, Brody's reparametrization argument yields a non-constant holomorphic map $\mathbb{C} \to X$, which contradicts to the assumption that X is Kobayashi hyperbolic. By the distance decreasing property, we have $F_C(v) \geq F_X(g_* v)$ for all $v \in TC$. Hence we have $F_C(v) \geq \delta |v|_{g^* \omega_L}$ for all $v \in TC$. Since F_C coincides with Poincaré metric on C, we have

$$\omega_C \geq \delta g^* \omega_L,$$

where ω_C is the (1,1)-form on C associated to the Poincaré metric. By integrating both sides of this estimate, we conclude the proof. □

Recently, Conjecture 3 is verified for subvarieties of general type on abelian varieties ([9]).

Theorem 2. *Let X be a subvariety of general type on an abelian variety. Then X is Kobayashi hyperbolic modulo* $\mathrm{Sp}(X)$. □

Recall that $\mathrm{Sp}(X)$ is a proper Zariski closed set $\mathrm{Sp}(X) \subsetneqq X$ by [5], where X is a subvariety of general type on an abelian variety A. Moreover, $\mathrm{Sp}(X)$ has more clear description without taking Zariski closure:

$$\mathrm{Sp}(X) = \{x \in X; \exists B \subset A, \text{an abelian variety s.t. } \dim(B) > 0 \text{ and } x + B \subset X\}.$$

Corollary 2. *For subvariety X of an abelian variety A, we have*

$$X \textit{ is of general type} \Longleftrightarrow X \textit{ is pseudo-Kobayashi hyperbolic}$$

Proof. The direction $\Longrightarrow$ is by the theorem. The converse follows by a theorem of Ueno. Indeed, if X is not of general type, then the stabilizer of X is positive dimensional. Hence there exists a positive dimensional abelian subvariety $B \subset A$ such that $x + B \subset X$ for all $x \in X$. Hence, for each point $x \in X$, there exists a non-constant map $B \to X$ passing through x. Hence by the distance decreasing property and $d_B \equiv 0$, we conclude that X is not pseudo-Kobayashi hyperbolic. □

Theorem 2 is a generalization of the following theorems for a subvariety X of general type on an abelian variety.

(1) *Green's theorem* states that if $\mathrm{Sp}(X) = \emptyset$, then X is Kobayashi hyperbolic ([4]). Indeed our theorem implies that X is Kobayashi hyperbolic modulo $\emptyset$, hence X is Kobayashi hyperbolic. Green actually proved that there is no non-constant holomorphic map $f : \mathbb{C} \to X$, if $\mathrm{Sp}(X) = \emptyset$. By Brody's criterion, this implies that X is Kobayashi hyperbolic.

(2) More generally, *Bloch-Ochiai-Kawamata's theorem* states that every non-constant holomorphic map $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset \mathrm{Sp}(X)$ ([5]). This follows from our theorem by the distance decreasing property and $d_{\mathbb{C}} \equiv 0$. However, from this theorem, we can't conclude any information for Kobayashi pseudo-distance d_X. We can't even exclude the possibility $d_X \equiv 0$.

In complex analysis, there is a philosophical principle stated by A. Bloch as follows: *Nothing exists in the infinite plane that has not been previously done in the finite disk.* This principle says that if some property $\mathcal{P}$ reduces analytic maps in $\mathbb{C}$ to a constant, then a family of analytic maps in Δ with the property $\mathcal{P}$ will be normal. The property of 'bounded analytic function' is one such example; Liouville's theorem corresponds to Montel's theorem. The following theorem may be considered as one example of this principle which corresponds to Bloch-Ochiai-Kawamata's theorem above.

We denote by $\mathrm{Hol}(\Delta, X)$ the set of all holomorphic mappings $f : \Delta \to X$.

Theorem 3. *Let X be a subvariety of general type on an abelian variety. Then for each sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathrm{Hol}(\Delta, X)$, we have one of the following:*

(1) *$\{f_n\}_{n=1}^{\infty}$ has a subsequence which converges locally uniformly to some $f \in \mathrm{Hol}(\Delta, X)$, or*

(2) *for each compact subset $K \subset \Delta$ and each compact subset $L \subset X \backslash \mathrm{Sp}(X)$, there exists an integer n_0 such that $f_n(K) \cap L = \emptyset$ for all $n \geq n_0$.*

In other words, X is *taut* modulo $\mathrm{Sp}(X)$ with terminology in [6]. Theorem 2 follows from this theorem. Indeed, Theorem 2 is an integrated version of the following lower estimate of F_X.

Corollary 3 (Lower estimate of F_X)**.** *For each open neighborhood $U \subset X$ of $\mathrm{Sp}(X)$, there exists a positive constant $c > 0$ such that $F_X(v) \geq \frac{1}{c}|v|$ for all $v \in T_x X$ with $x \notin U$.*

Proof. We first show that there exists a positive constant $c > 0$ such that for every $f \in \mathrm{Hol}(\Delta, X)$ with $f(0) \notin U$, we have $|f'(0)| < c$. For otherwise, there exists a sequence $\{f_n\} \subset \mathrm{Hol}(\Delta, X)$ such that $f_n(0) \notin U$ and $|f_n'(0)| \to \infty$. For this sequence, the second conclusion of Theorem 3 does not occur. Hence there exists a subsequence $\{f_{n_k}\}$ which converges locally uniformly to some $f \in \mathrm{Hol}(\Delta, X)$, so $|f_{n_k}'(0)| \to |f'(0)|$. This is a contradiction.

Now to measure $F_X(v)$, we take $f \in \mathrm{Hol}(\Delta, X)$ with $f'(0) = rv$. Then we have $|f'(0)| = r|v| \leq c$. Hence $|v|/c \leq 1/r$. So by the definition of $F_X(v)$, we have $|v|/c \leq F_X(v)$. □

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