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Kyoto University
1. Hyperbolicity problems in algebraic setting

Let $X$ be a smooth complex projective variety.

**Conjecture 1.** Assume that $X$ is of general type. Let $L$ be an ample line bundle on $X$. Then there exists a positive number $\alpha > 0$ and a proper Zariski closed subset $Z \subsetneq X$ such that for every holomorphic map $g : C \to X$ from a compact Riemann surface $C$ with $g(C) \not\subset Z$, we have

$$\deg_C g^* (L) \leq \alpha \times \{\text{genus}(C) - 1\}$$

We introduce the following special set after S. Lang:

**Definition 1 (Special set).** Consider all non-constant rational maps $\varphi : A \dashrightarrow X$ from abelian varieties $A$ and define the special set of $X$ by

$$\text{Sp}(X) = \bigcup_{\varphi : A \dashrightarrow X} \text{Zariski closure} \varphi(A)$$

By the definition, $\text{Sp}(X)$ is a Zariski closed set of $X$. Note that the special set $\text{Sp}(X)$ contains all rational and elliptic curves in $X$. This is trivial for elliptic curves in $X$ by the definition. For the rational curves $f : \mathbb{P}^1 \to X$, we take the composite with a double covering $\varphi : E \to \mathbb{P}^1$ from an elliptic curve $E$ to get a morphism $f \circ \varphi : E \to X$ from the elliptic curve. So $f(\mathbb{P}^1) = f \circ \varphi(E) \subset \text{Sp}(X)$.

**Conjecture 2 (Lang [7]).** Assume $X$ is of general type. Then the special set is a proper Zariski closed subset $\text{Sp}(X) \subsetneq X$.

Conjecture 1 implies Conjecture 2. Indeed, let $\varphi : A \dashrightarrow X$ be a non-constant rational map. We want to show that $\varphi(A) \subset Z$, where $Z \subsetneq X$ is the Zariski closed set in Conjecture 1. So suppose contrary that $\varphi(A) \not\subset Z$. Let $E \subset A$ be the indeterminacy set of $\varphi$. Then $\text{codim} E \geq 2$. Let $n_A : A \to A$ be the $n$-times map. Let $C \subset A$ be a smooth curve in $A$ such that $n_A(C) \not\subset E \cup \varphi^{-1}(Z)$ for all $n$. For an ample line bundle $L$ on $X$, we have $n_A^* c_1(\varphi^* L) = n^2 c_1(\varphi^* L) \in H^2(A, \mathbb{Z})$, where $c_1(\varphi^* L) \in H^2(A, \mathbb{Z})$ is the characteristic class of $\varphi^* L$. Now we consider the holomorphic map $\varphi \circ n_A|_C : C \to X$. Then

$$\deg_C(\varphi \circ n_A|_C)^* L = n^2 \deg_C(\varphi|_C)^* L \to \infty$$

as $n \to \infty$, but $\varphi \circ n_A|_C(C) \not\subset Z$. This contradicts to Conjecture 1. Thus $\varphi(A) \subset Z$.  

These conjectures are open even for the case of $\dim X = 2$. Some known cases are the following:

- Surfaces of general type with $c_1(X)^2 > c_2(X)$. Conjecture 1 is valid ([1]).
- Subvarieties of general type on abelian varieties. Conjecture 2 is valid ([5]).
• Very generic hypersurfaces of degree \(d \geq 2n + 2\) in \(\mathbb{P}^{n+1}\). Conjecture 1 is valid with stronger conclusion that \(Z = \emptyset\) ([2]).

We say that \(X\) is of maximal albanese dimension if the dimension of the image \(a(X)\) of the albanese map \(a : X \rightarrow \text{Alb}(X)\) is equal to the dimension of \(X\).

• Varieties of maximal albanese dimension and of general type. Conjecture 1 is valid ([8]).

2. Hyperbolicity problems in complex analytic setting

Let \(M\) be a complex manifold. We first introduce Kobayashi-Royden pseudo-metric \(F_M : TM \rightarrow \mathbb{R}_{\geq 0}\) as follows. Set \(\Delta = \{z \in \mathbb{C}; |z| < 1\}\).

**Definition 2** (Kobayashi-Royden pseudo-metric). For \(v \in T_x(M)\),

\[
F_M(v) = \inf \left\{ \frac{1}{r}; \exists f : \Delta \rightarrow M \text{ holomorphic s.t. } f(0) = x, \ f'(0) = rv \right\}.
\]

**Proposition 1.** Let \(\varphi : X \rightarrow Y\) be a holomorphic map between complex manifolds, let \(\varphi_* : TX \rightarrow TY\) be the induced map. Then for all \(v \in TX\),

\[
F_X(v) \geq F_Y(\varphi_*(v)).
\]

In particular, every holomorphic automorphism is an isometry of \(F_X\).

**Proof.** This follows directly from the definition. We prove, for arbitrary \(v \in TX\),

\[
F_X(v) = F_Y(\varphi_*(v)).
\]

Take an arbitrary \(1/r > F_X(v)\). It is enough to show \(1/r > F_Y(\varphi_*(v))\). There exists \(f : \Delta \rightarrow X\) such that \(f'(0) = rv\). Hence \(\varphi \circ f : \Delta \rightarrow Y\) satisfies \((\varphi \circ f)'(0) = r\varphi_*(v)\).

Hence \(F_Y(\varphi_*(v)) < 1/r\), so \(F_X(v) \geq F_Y(\varphi_*(v))\). \(\square\)

**Definition 3** (Kobayashi pseudo-distance). For \(x, y \in M\), we define

\[
d_M(x, y) = \inf_\gamma \int_\gamma F_M(\gamma'(t))dt
\]

where the infimum is taken over all piecewise smooth curves \(\gamma\) joining \(x\) and \(y\).

**Corollary 1.** For all \(x, y \in X\), we have \(d_X(x, y) \geq d_Y(\varphi(x), \varphi(y))\).

**Definition 4** (S. Kobayashi [6]). A complex manifold \(M\) is Kobayashi hyperbolic if \(d_M\) is a true distance, i.e., \(d_M(p, q) > 0\) whenever \(p \neq q\).

**Examples.**

(1) If \(M = \Delta\), then \(F_\Delta = \frac{|dz|}{1 - |z|^2}\). Hence \(d_\Delta\) is the Poincaré distance of the unit disc \(\Delta\).

**Proof.** Both \(F_\Delta\) and \(|dz|/(1 - |z|^2)\) are isometric under \(\text{Aut}(\Delta)\). Hence it is enough to compare them over \(0 \in \Delta\), and show \(F_\Delta(\partial/\partial z) = 1\). The existence of \(\text{id}_\Delta : \Delta \rightarrow \Delta\) yields \(F_\Delta(\partial/\partial z) \leq 1\). By the Schwarz lemma, if \(f : \Delta \rightarrow \Delta\) is holomorphic with \(f(0) = 0\) and \(f'(0) = r\frac{a}{\partial z}\), then \(1 \leq 1/r\). This shows \(1 \leq F_\Delta(\partial/\partial z)\). Hence \(F_\Delta(\partial/\partial z) = 1\). \(\square\)
(1) If \( X \) is a compact Riemann surface of genus \( \geq 2 \), then \( d_X \) is equal to the Poincaré distance on \( X \) induced from the universal covering map \( \Delta \to X \). In particular, \( X \) is Kobayashi hyperbolic.

(2) If \( M = \mathbb{C} \), then \( F_{\mathbb{C}} \equiv 0 \) and \( d_{\mathbb{C}} \equiv 0 \).

Proof. For arbitrary \( R > 0 \), let \( f_R : \Delta \to \mathbb{C} \) be defined by \( f_R(z) = Rz \). Then \( f_R(0) = 0 \) and \( f_R'(0) = R \frac{\partial}{\partial z} \). Hence \( F_{\mathbb{C}}(\partial / \partial z) \leq 1/R \). Since \( R > 0 \) is arbitrary, \( F_{\mathbb{C}}(\partial / \partial z) = 0 \). Using automorphism of \( \mathbb{C} \), \( F_{\mathbb{C}} \equiv 0 \). □

(3) If \( X \) is a projective space or an Abelian variety, then \( d_X \equiv 0 \).

Proof. For every \( x, y \in X \), there exists a holomorphic map \( f : \mathbb{C} \to X \) such that \( f(0) = x \) and \( f(1) = y \). So by the distance decreasing property, we have \( d_X(x, y) \leq d_{\mathbb{C}}(0, 1) = 0 \). □

Thus if \( X \) is a smooth projective variety with \( \dim X = 1 \), the followings are equivalent:

1. genus(\( X \)) ≥ 2.
2. \( X \) is of general type.
3. \( X \) is Kobayashi hyperbolic.

Indeed, if genus(\( X \)) = 0 or 1, then \( X \) is \( \mathbb{P}^1 \) or elliptic curve, so \( d_X \equiv 0 \). Hence (3) implies (1).

For \( \dim X \geq 2 \), a projective variety \( X \) of general type need not be Kobayashi hyperbolic. Indeed \( X \) may contain rational or elliptic curves even if \( X \) is of general type, and such \( X \) is not Kobayashi hyperbolic. We introduce weaker notion.

**Definition 5.** Let \( M \) be a complex manifold and let \( S \subset M \) be a closed subset. \( M \) is Kobayashi hyperbolic modulo \( S \), if \( d_M \) is a true distance outside \( S \).

**Conjecture 3 ([6]).** If \( X \) is a projective variety of general type, then \( X \) is pseudo-Kobayashi hyperbolic, i.e., there exists a proper Zariski closed subset \( S \subsetneq X \) such that \( X \) is Kobayashi hyperbolic modulo \( S \).

Conjecture 3 implies Conjecture 2. We want to show that the image of every non-constant rational map \( A \to X \) from abelian variety \( A \) should be contained in \( S \), where \( S \subseteq X \) is the proper Zariski closed subset appeared in Conjecture 3. This follows from the distance decreasing property of Kobayashi pseudo-distance and \( d_A \equiv 0 \). Hence \( \text{Sp}(X) \subset S \subsetneq X \). □

It is not clear whether or not Conjecture 3 implies Conjecture 1. However the following implication is known by Demailly.

**Theorem 1 ([3]).** Let \( X \) be a Kobayashi hyperbolic projective manifold. Let \( L \) be an ample line bundle on \( X \). Then there exists a positive number \( \alpha > 0 \) such that for every holomorphic map \( g : C \to X \) from a compact Riemann surface \( C \), we have

\[
\deg_C g^*(L) \leq \alpha \times \{\text{genus}(C) - 1\}.
\]

**Proof.** Let \( \omega_L \) be a curvature form of \( L \), which is a positive (1,1)-form on \( X \). For \( v \in TX \), we denote by \( |v|_{\omega_L} \) the norm associated to \( \omega_L \). There exists a positive constant \( \delta > 0 \) such that \( F_X(v) \geq \delta |v|_{\omega_L} \) for all \( v \in TX \); for otherwise, Brody’s reparametrization argument yields a non-constant holomorphic map \( C \to X \), which contradicts to the assumption that \( X \) is Kobayashi hyperbolic. By the distance decreasing property, we
have \( F_C(v) \geq F_X(g vX) \) for all \( v \in TC \). Hence we have \( F_C(v) \geq \delta|v|g^*\omega_L \) for all \( v \in TC \). Since \( F_C \) coincides with Poincaré metric on \( C \), we have

\[
\omega_C \geq \delta g^*\omega_L,
\]

where \( \omega_C \) is the \((1,1)\)-form on \( C \) associated to the Poincaré metric. By integrating both sides of this estimate, we conclude the proof. \( \square \)

Recently, Conjecture 3 is verified for subvarieties of general type on abelian varieties ([9]).

**Theorem 2.** Let \( X \) be a subvariety of general type on an abelian variety. Then \( X \) is Kobayashi hyperbolic modulo \( \text{Sp}(X) \). \( \square \)

Recall that \( \text{Sp}(X) \) is a proper Zariski closed set \( \text{Sp}(X) \subsetneq X \) by [5], where \( X \) is a subvariety of general type on an abelian variety \( A \). Moreover, \( \text{Sp}(X) \) has more clear description without taking Zariski closure:

\[
\text{Sp}(X) = \{ x \in X; \exists B \subset A, \text{an abelian varietys.t. } \dim(B) > 0 \text{ and } x + B \subset X \}.
\]

**Corollary 2.** For subvariety \( X \) of an abelian variety \( A \), we have

\[
X \text{ is of general type } \iff X \text{ is pseudo-Kobayashi hyperbolic}
\]

**Proof.** The direction \( \Rightarrow \) is by the theorem. The converse follows by a theorem of Ueno. Indeed, if \( X \) is not of general type, then the stabilizer of \( X \) is positive dimensional. Hence there exists a positive dimensional abelian subvariety \( B \subset A \) such that \( x + B \subset X \) for all \( x \in X \). Hence, for each point \( x \in X \), there exists a non-constant map \( B \to X \) passing through \( x \). Hence by the distance decreasing property and \( d_B \equiv 0 \), we conclude that \( X \) is not pseudo-Kobayashi hyperbolic. \( \square \)

Theorem 2 is a generalization of the following theorems for a subvariety \( X \) of general type on an abelian variety.

1. **Green’s theorem** states that if \( \text{Sp}(X) = \emptyset \), then \( X \) is Kobayashi hyperbolic ([4]). Indeed our theorem implies that \( X \) is Kobayashi hyperbolic modulo \( \emptyset \), hence \( X \) is Kobayashi hyperbolic. Green actually proved that there is no non-constant holomorphic map \( f : \C \to X \), if \( \text{Sp}(X) = \emptyset \). By Brody’s criterion, this implies that \( X \) is Kobayashi hyperbolic.

2. More generally, **Bloch-Ochiai-Kawamata’s theorem** states that every non-constant holomorphic map \( f : \C \to X \) satisfies \( f(\C) \subset \text{Sp}(X) \) ([5]). This follows from our theorem by the distance decreasing property and \( d_C \equiv 0 \). However, from this theorem, we can’t conclude any information for Kobayashi pseudo-distance \( d_X \). We can’t even exclude the possibility \( d_X \equiv 0 \).

In complex analysis, there is a philosophical principle stated by A. Bloch as follows:

**Nothing exists in the infinite plane that has not been previously done in the finite disk.**

This principle says that if some property \( \mathcal{P} \) reduces analytic maps in \( \C \) to a constant, then a family of analytic maps in \( \Delta \) with the property \( \mathcal{P} \) will be normal. The property of ‘bounded analytic function’ is one such example; Liouville’s theorem corresponds to Montel’s theorem. The following theorem may be considered as one example of this principle which corresponds to Bloch-Ochiai-Kawamata’s theorem above.

We denote by \( \text{Hol}(\Delta, X) \) the set of all holomorphic mappings \( f : \Delta \to X \).
Theorem 3. Let $X$ be a subvariety of general type on an abelian variety. Then for each sequence $\{f_n\}_{n=1}^{\infty}$ in $\text{Hol}(\Delta, X)$, we have one of the following:

1. $\{f_n\}_{n=1}^{\infty}$ has a subsequence which converges locally uniformly to some $f \in \text{Hol}(\Delta, X)$, or
2. for each compact subset $K \subset \Delta$ and each compact subset $L \subset X \setminus \text{Sp}(X)$, there exists an integer $n_0$ such that $f_n(K) \cap L = \emptyset$ for all $n \geq n_0$.

In other words, $X$ is taut modulo $\text{Sp}(X)$ with terminology in [6]. Theorem 2 follows from this theorem. Indeed, Theorem 2 is an integrated version of the following lower estimate of $F_X$.

Corollary 3 (Lower estimate of $F_X$). For each open neighborhood $U \subset X$ of $\text{Sp}(X)$, there exists a positive constant $c > 0$ such that $F_X(v) \geq \frac{1}{c} |v|$ for all $v \in T_xX$ with $x \notin U$.

Proof. We first show that there exists a positive constant $c > 0$ such that for every $f \in \text{Hol}(\Delta, X)$ with $f(0) \notin U$, we have $|f'(0)| < c$. For otherwise, there exists a sequence $\{f_n\} \subset \text{Hol}(\Delta, X)$ such that $f_n(0) \notin U$ and $|f_n'(0)| \to \infty$. For this sequence, the second conclusion of Theorem 3 does not occur. Hence there exists a subsequence $\{f_{n_k}\}$ which converges locally uniformly to some $f \in \text{Hol}(\Delta, X)$, so $|f_{n_k}'(0)| \to |f'(0)|$. This is a contradiction.

Now to measure $F_X(v)$, we take $f \in \text{Hol}(\Delta, X)$ with $f'(0) = rv$. Then we have $|f'(0)| = r|v| \leq c$. Hence $|v|/c \leq 1/r$. So by the definition of $F_X(v)$, we have $|v|/c \leq F_X(v)$. □

References


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