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1. Hyperbolicity problems in algebraic setting

Let $X$ be a smooth complex projective variety.

**Conjecture 1.** Assume that $X$ is of general type. Let $L$ be an ample line bundle on $X$. Then there exists a positive number $\alpha > 0$ and a proper Zariski closed subset $Z \subseteq X$ such that for every holomorphic map $g : C \to X$ from a compact Riemann surface $C$ with $g(C) \not\subseteq Z$, we have

$$\deg_C g^*(L) \leq \alpha \times \{\text{genus}(C) - 1\}$$

We introduce the following special set after S. Lang:

**Definition 1** (Special set). Consider all non-constant rational maps $\varphi : A \dashrightarrow X$ from abelian varieties $A$ and define the special set of $X$ by

$$\text{Sp}(X) = \bigcup_{\varphi : A \dashrightarrow X} \varphi(A)$$

By the definition, $\text{Sp}(X)$ is a Zariski closed set of $X$. Note that the special set $\text{Sp}(X)$ contains all rational and elliptic curves in $X$. This is trivial for elliptic curves in $X$ by the definition. For the rational curves $f : P^1 \to X$, we take the composite with a double covering $\varphi : E \to P^1$ from an elliptic curve $E$ to get a morphism $f \circ \varphi : E \to X$ from the elliptic curve. So $f(P^1) = f \circ \varphi(E) \subseteq \text{Sp}(X)$.

**Conjecture 2** (Lang [7]). Assume $X$ is of general type. Then the special set is a proper Zariski closed subset $\text{Sp}(X) \subseteq X$.

Conjecture 1 implies Conjecture 2. Indeed, let $\varphi : A \dashrightarrow X$ be a non-constant rational map. We want to show that $\varphi(A) \subseteq Z$, where $Z \subseteq X$ is the Zariski closed set in Conjecture 1. So suppose contrary that $\varphi(A) \not\subseteq Z$. Let $E \subseteq A$ be the indeterminacy set of $\varphi$. Then $\text{codim} E \geq 2$. Let $n_A : A \to A$ be the $n$-times map. Let $C \subseteq A$ be a smooth curve in $A$ such that $n_A(C) \not\subseteq E \cup \varphi^{-1}(Z)$ for all $n$. For an ample line bundle $L$ on $X$, we have $n_A^* c_1(\varphi^* L) = n^2 c_1(\varphi^* L) \in H^2(A, Z)$, where $c_1(\varphi^* L) \in H^2(A, Z)$ is the characteristic class of $\varphi^* L$. Now we consider the holomorphic map $\varphi \circ n_A|_C : C \to X$. Then

$$\deg_C (\varphi \circ n_A|_C)^* L = n^2 \deg_C (\varphi|_C)^* L \to \infty$$

as $n \to \infty$, but $\varphi \circ n_A|_C(C) \not\subseteq Z$. This contradicts to Conjecture 1. Thus $\varphi(A) \subseteq Z$. □

These conjectures are open even for the case of $\dim X = 2$. Some known cases are the following:

- Surfaces of general type with $c_1(X)^2 > c_2(X)$. Conjecture 1 is valid ([1]).
- Subvarieties of general type on abelian varieties. Conjecture 2 is valid ([5]).
• Very generic hypersurfaces of degree \( d \geq 2n + 2 \) in \( \mathbb{P}^{n+1} \). Conjecture 1 is valid with stronger conclusion that \( Z = \emptyset \) ([2]).

We say that \( X \) is of maximal albanese dimension if the dimension of the image \( a(X) \) of the albanese map \( a : X \to \text{Alb}(X) \) is equal to the dimension of \( X \).

• Varieties of maximal albanese dimension and of general type. Conjecture 1 is valid ([8]).

2. Hyperbolicity problems in complex analytic setting

Let \( M \) be a complex manifold. We first introduce Kobayashi-Royden pseudo-metric \( F_M : TM \to \mathbb{R}_{\geq 0} \) as follows. Set \( \Delta = \{ z \in \mathbb{C}; |z| < 1 \} \).

**Definition 2** (Kobayashi-Royden pseudo-metric). For \( v \in T_x(M) \),

\[
F_M(v) = \inf \left\{ \frac{1}{r}; \exists f : \Delta \to M \text{ holomorphic s.t. } f(0) = x, \ f'(0) = rv \right\}.
\]

**Proposition 1.** Let \( \varphi : X \to Y \) be a holomorphic map between complex manifolds, let \( \varphi_* : TX \to TY \) be the induced map. Then for all \( v \in TX \),

\[
F_X(v) \geq F_Y(\varphi_*(v)).
\]

In particular, every holomorphic automorphism is an isometry of \( F_X \).

**Proof.** This follows directly from the definition. We prove, for arbitrary \( v \in TX \),

\[
F_X(v) \geq F_Y(\varphi_*(v)).
\]

Take an arbitrary \( 1/r > F_X(v) \). It is enough to show \( 1/r > F_Y(\varphi_*(v)) \). There exists \( f : \Delta \to X \) such that \( f'(0) = rv \). Hence \( \varphi \circ f : \Delta \to Y \) satisfies \( (\varphi \circ f)'(0) = r\varphi_*(v) \).

Hence \( F_Y(\varphi_*(v)) < 1/r \), so \( F_X(v) \geq F_Y(\varphi_*(v)) \). \( \square \)

**Definition 3** (Kobayashi pseudo-distance). For \( x, y \in M \), we define

\[
d_M(x, y) = \inf_{\gamma} \int_{\gamma} F_M(\gamma'(t))dt
\]

where the infimum is taken over all piecewise smooth curves \( \gamma \) joining \( x \) and \( y \).

**Corollary 1.** For all \( x, y \in X \), we have \( d_X(x, y) \geq d_Y(\varphi(x), \varphi(y)) \).

**Definition 4** (S. Kobayashi [6]). A complex manifold \( M \) is Kobayashi hyperbolic if \( d_M \) is a true distance, i.e., \( d_M(p, q) > 0 \) whenever \( p \neq q \).

**Examples.**

1. If \( M = \Delta \), then \( F_\Delta = \frac{|dz|}{1-|z|^2} \). Hence \( d_\Delta \) is the Poincaré distance of the unit disc \( \Delta \).

**Proof.** Both \( F_\Delta \) and \( |dz|/(1-|z|^2) \) are isometric under \( \text{Aut}(\Delta) \). Hence it is enough to compare them over \( 0 \in \Delta \), and show \( F_\Delta(\partial/\partial z) = 1 \). The existence of \( \text{id}_\Delta : \Delta \to \Delta \) yields \( F_\Delta(\partial/\partial z) \leq 1 \). By the Schwarz lemma, if \( f : \Delta \to \Delta \) is holomorphic with \( f(0) = 0 \) and \( f'(0) = r \frac{\partial}{\partial z} \), then \( 1 \leq 1/r \). This shows \( 1 \leq F_\Delta(\partial/\partial z) \). Hence \( F_\Delta(\partial/\partial z) = 1 \). \( \square \)
(1)' If \( X \) is a compact Riemann surface of genus(\( X \)) \( \geq 2 \), then \( d_X \) is equal to the Poincaré distance on \( X \) induced from the universal covering map \( \Delta \rightarrow X \). In particular, \( X \) is Kobayashi hyperbolic.

(2) If \( M = \mathbb{C} \), then \( F_C \equiv 0 \) and \( d_C \equiv 0 \).

\[ \text{Proof.} \quad \text{For arbitrary } R > 0, \text{ let } f_R : \Delta \rightarrow \mathbb{C} \text{ be defined by } f_R(z) = Rz. \text{ Then } f_R(0) = 0 \text{ and } f_R'(0) = R \frac{\partial}{\partial z}. \text{ Hence } F_C(\partial / \partial z) \leq 1 / R. \text{ Since } R > 0 \text{ is arbitrary, } F_C(\partial / \partial z) = 0. \text{ Using automorphism of } \mathbb{C}, F_C \equiv 0. \quad \square \]

(3) If \( X \) is a projective space or an Abelian variety, then \( d_X \equiv 0 \).

\[ \text{Proof.} \quad \text{For every } x, y \in X, \text{ there exists a holomorphic map } f : \mathbb{C} \rightarrow X \text{ such that } f(0) = x \text{ and } f(1) = y. \text{ So by the distance decreasing property, we have } d_X(x, y) \leq d_C(0, 1) = 0 \quad \square \]

Thus if \( X \) is a smooth projective variety with \( \dim X = 1 \), the followings are equivalent:

(1) genus(\( X \)) \( \geq 2 \).

(2) \( X \) is of general type.

(3) \( X \) is Kobayashi hyperbolic.

Indeed, if genus(\( X \)) = 0 or 1, then \( X \) is \( \mathbb{P}^1 \) or elliptic curve, so \( d_X \equiv 0 \). Hence (3) implies (1).

For \( \dim X \geq 2 \), a projective variety \( X \) of general type need not be Kobayashi hyperbolic. Indeed \( X \) may contain rational or elliptic curves even if \( X \) is of general type, and such \( X \) is not Kobayashi hyperbolic. We introduce weaker notion.

\textbf{Definition 5.} Let \( M \) be a complex manifold and let \( S \subset M \) be a closed subset. \( M \) is Kobayashi hyperbolic modulo \( S \), if \( d_M \) is a true distance outside \( S \).

\textbf{Conjecture 3} ([6]). If \( X \) is a projective variety of general type, then \( X \) is pseudo-Kobayashi hyperbolic, i.e., there exists a proper Zariski closed subset \( S \subsetneq X \) such that \( X \) is Kobayashi hyperbolic modulo \( S \).

\textit{Conjecture 3 implies Conjecture 2.} We want to show that the image of every non-constant rational map \( A \rightarrow X \) from abelian variety \( A \) should be contained in \( S \), where \( S \subsetneq X \) is the proper Zariski closed subset appeared in Conjecture 3. This follows from the distance decreasing property of Kobayashi pseudo-distance and \( d_A \equiv 0 \). Hence \( \text{Sp}(X) \subset S \subsetneq X \). \( \square \)

It is not clear whether or not Conjecture 3 implies Conjecture 1. However the following implication is known by Demailly.

\textbf{Theorem 1} ([3]). Let \( X \) be a Kobayashi hyperbolic projective manifold. Let \( L \) be an ample line bundle on \( X \). Then there exists a positive number \( \alpha > 0 \) such that for every holomorphic map \( g : C \rightarrow X \) from a compact Riemann surface \( C \), we have

\[ \deg_C g^*(L) \leq \alpha \times \{ \text{genus}(C) - 1 \}. \]

\[ \text{Proof.} \quad \text{Let } \omega_L \text{ be a curvature form of } L, \text{ which is a positive (1,1)-form on } X. \text{ For } v \in TX, \text{ we denote by } |v|_{\omega_L} \text{ the norm associated to } \omega_L. \text{ There exists a positive constant } \delta > 0 \text{ such that } F_X(v) \geq \delta |v|_{\omega_L} \text{ for all } v \in TX; \text{ For otherwise, Brody’s reparametrization argument yields a non-constant holomorphic map } C \rightarrow X, \text{ which contradicts to the assumption that } X \text{ is Kobayashi hyperbolic. By the distance decreasing property, we} \]
have $F_C(v) \geq F_X(g,v)$ for all $v \in TC$. Hence we have $F_C(v) \geq \delta|v|g^\omega_L$ for all $v \in TC$. Since $F_C$ coincides with Poincaré metric on $C$, we have

$$\omega_C \geq \delta g^*\omega_L,$$

where $\omega_C$ is the (1,1)-form on $C$ associated to the Poincaré metric. By integrating both sides of this estimate, we conclude the proof. □

Recently, Conjecture 3 is verified for subvarieties of general type on abelian varieties ([9]).

**Theorem 2.** Let $X$ be a subvariety of general type on an abelian variety. Then $X$ is Kobayashi hyperbolic modulo $\text{Sp}(X)$. □

Recall that $\text{Sp}(X)$ is a proper Zariski closed set $\text{Sp}(X) \subset X$ by [5], where $X$ is a subvariety of general type on an abelian variety $A$. Moreover, $\text{Sp}(X)$ has more clear description without taking Zariski closure:

$$\text{Sp}(X) = \{x \in X; \exists B \subset A, \text{ an abelian variety s.t. } \dim(B) > 0 \text{ and } x + B \subset X\}.$$

**Corollary 2.** For subvariety $X$ of an abelian variety $A$, we have

$$X \text{ is of general type } \iff X \text{ is pseudo-Kobayashi hyperbolic}$$

**Proof.** The direction $\implies$ is by the theorem. The converse follows by a theorem of Ueno. Indeed, if $X$ is not of general type, then the stabilizer of $X$ is positive dimensional. Hence there exists a positive dimensional abelian subvariety $B \subset A$ such that $x + B \subset X$ for all $x \in X$. Hence, for each point $x \in X$, there exists a non-constant map $B \to X$ passing through $x$. Hence by the distance decreasing property and $d_B \equiv 0$, we conclude that $X$ is not pseudo-Kobayashi hyperbolic. □

Theorem 2 is a generalization of the following theorems for a subvariety $X$ of general type on an abelian variety.

1. **Green’s theorem** states that if $\text{Sp}(X) = \emptyset$, then $X$ is Kobayashi hyperbolic ([4]). Indeed our theorem implies that $X$ is Kobayashi hyperbolic modulo $\emptyset$, hence $X$ is Kobayashi hyperbolic. Green actually proved that there is no non-constant holomorphic map $f : \mathbb{C} \to X$, if $\text{Sp}(X) = \emptyset$. By Brody’s criterion, this implies that $X$ is Kobayashi hyperbolic.

2. More generally, **Bloch-Ochiai-Kawamata’s theorem** states that every non-constant holomorphic map $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset \text{Sp}(X)$ ([5]). This follows from our theorem by the distance decreasing property and $d_C \equiv 0$. However, from this theorem, we can’t conclude any information for Kobayashi pseudo-distance $d_X$. We can’t even exclude the possibility $d_X \equiv 0$.

In complex analysis, there is a philosophical principle stated by A. Bloch as follows: **Nothing exists in the infinite plane that has not been previously done in the finite disk.** This principle says that if some property $\mathcal{P}$ reduces analytic maps in $\mathbb{C}$ to a constant, then a family of analytic maps in $\Delta$ with the property $\mathcal{P}$ will be normal. The property of ’bounded analytic function’ is one such example; Liouville’s theorem corresponds to Montel’s theorem. The following theorem may be considered as one example of this principle which corresponds to Bloch-Ochiai-Kawamata’s theorem above.

We denote by $\text{Hol}(\Delta, X)$ the set of all holomorphic mappings $f : \Delta \to X$. 


Theorem 3. Let $X$ be a subvariety of general type on an abelian variety. Then for each sequence $\{f_n\}_{n=1}^\infty$ in $\text{Hol}(\Delta, X)$, we have one of the following:

1. $\{f_n\}_{n=1}^\infty$ has a subsequence which converges locally uniformly to some $f \in \text{Hol}(\Delta, X)$, or
2. for each compact subset $K \subset \Delta$ and each compact subset $L \subset X \setminus \text{Sp}(X)$, there exists an integer $n_0$ such that $f_n(K) \cap L = \emptyset$ for all $n \geq n_0$.

In other words, $X$ is taut modulo $\text{Sp}(X)$ with terminology in [6]. Theorem 2 follows from this theorem. Indeed, Theorem 2 is an integrated version of the following lower estimate of $F_X$.

Corollary 3 (Lower estimate of $F_X$). For each open neighborhood $U \subset X$ of $\text{Sp}(X)$, there exists a positive constant $c > 0$ such that $F_X(v) \geq \frac{1}{c}|v|$ for all $v \in T_xX$ with $x \not\in U$.

Proof. We first show that there exists a positive constant $c > 0$ such that for every $f \in \text{Hol}(\Delta, X)$ with $f(0) \not\in U$, we have $|f'(0)| < c$. For otherwise, there exists a sequence $\{f_n\} \subset \text{Hol}(\Delta, X)$ such that $f_n(0) \not\in U$ and $|f'_n(0)| \to \infty$. For this sequence, the second conclusion of Theorem 3 does not occur. Hence there exists a subsequence $\{f_{n_k}\}$ which converges locally uniformly to some $f \in \text{Hol}(\Delta, X)$, so $|f_{n_k}(0)| \to |f'(0)|$. This is a contradiction.

Now to measure $F_X(v)$, we take $f \in \text{Hol}(\Delta, X)$ with $f'(0) = rv$. Then we have $|f'(0)| = r|v| \leq c$. Hence $|v|/c \leq 1/r$. So by the definition of $F_X(v)$, we have $|v|/c \leq F_X(v)$.

References


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