APPLICATIONS OF NONCOMMUTATIVE DEFORMATIONS

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Abstract. For a general class of contractions of a variety X to a base Y, I discuss recent joint work with M. Wemyss defining a noncommutative enhancement of the locus in Y over which the contraction is not an isomorphism, along with applications to the derived symmetries of X.

Contents

Derived symmetry groups of algebraic varieties extend classical symmetry groups to include contributions from symplectic geometry via homological mirror symmetry, and from birational geometry. In a recent joint paper with M. Wemyss [9], for a general class of birational contractions $f\colon X \to Y$, we construct a sheaf of noncommutative algebras $\mathcal{A}$ on Y which, in an appropriate crepant setting, induces a derived symmetry of X. This short note explains key features of our results.

The sheaf $\mathcal{A}$ is supported on the locus of Y over which f is not an isomorphism. In previous joint work [6, 8] we considered contractions of 3-folds for which this locus is just a point. In this setting we studied an algebra of noncommutative deformations A which allowed new constructions of derived symmetries, and extended and unified known invariants of such contractions. I begin by reviewing this, as $\mathcal{A}$ may be viewed as a sheafy version of the algebra A.

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I also briefly discuss an example in which f is a Springer resolution (§3), and indicate recent work in which deformation algebras are used to recover the geometry of contractions (§4).

Acknowledgements. I am grateful to the organisers and supporters of the Kinosaki symposium for the opportunity to take part in the fine tradition of this meeting. My thanks also go to M. Wemyss, as our joint work forms the subject of this note. The presentation of this work has benefited from comments from many people: in particular, I am grateful for recent conversations with A. Bodzenta, A. Bondal, Z. Hua, Y. Kawamata, T. Logvinenko, D. Piyaratne, and Y. Toda.

Conventions. I work over the ground field $\mathbb{C}$, though this assumption can be weakened. Varieties X are assumed quasi-projective, with bounded derived category of coherent sheaves denoted by $D(X)$. The variety of hyperplanes in a vector space V is denoted by $\mathbb{P}V$.

1. Deformation algebras for 3-folds

The theorem below applies noncommutative deformations to study derived symmetries of 3-folds. Given smooth 3-folds X and X' related by a flop, Bridgeland [3] constructs certain canonical Fourier–Mukai equivalences

$$D(X) \underset{\mathsf{F}'}{\overset{\mathsf{F}}{\underset{\sim}{\rightleftarrows}}} D(X').$$

These equivalences are not mutually inverse: the theorem explains this using deformations of curves on X.

Consider a 3-fold Y with an isolated rational singular point p, and a resolution $f\colon X \to Y$ of this singularity, with one-dimensional exceptional locus. Write C^i for the components of the exceptional locus.

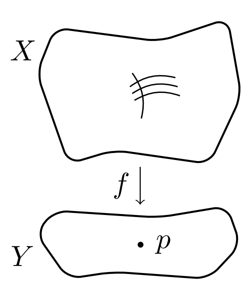

Figure A. Contraction f for Theorem A.

Theorem A. [6, 7, 8] *The subvarieties $C^1, \dots, C^n$ of the resolution X are projective lines, and:*

(1) *there exists a $\mathbb{C}^n$-algebra A which represents the functor of noncommutative deformations of the sheaves $\mathcal{O}_{C^i}(-1)$ on X.*

Write E for the corresponding universal sheaf on X. If the contraction f corresponds to a flop of X, then:

(2) *there is a Fourier–Mukai autoequivalence T_E of $D(X)$, fitting into a distinguished triangle of functors*

$$\mathbb{R}\mathrm{Hom}_X(E, -) \overset{\mathbb{L}}{\underset{A}{\otimes}} E \longrightarrow \mathrm{Id}_{D(X)} \longrightarrow \mathsf{T}_E \longrightarrow;$$

(3) *there is a natural isomorphism of functors*

$$\mathsf{T}_E \cong \left(\mathsf{F}' \circ \mathsf{F}\right)^{-1}.$$

In the simplest flopping situation, where f contracts a $(-1,-1)$-curve, the autoequivalence T_E is a spherical twist in the sense of Seidel–Thomas [18]. For a contraction of a $(-2,0)$-curve, it is a generalized spherical twist as first constructed by Toda [19], who furthermore established the conclusion of Theorem A(3) in this case.

Remark. The noncommutative deformation theory used here relies on work of Laudal [15], Eriksen [11], E. Segal [17], and Efimov–Lunts–Orlov [10].

Remark. The algebra A above, and similar noncommutative deformation algebras, have now been applied in a number of settings including: enumerative geometry of curves on 3-folds by Toda and Hua–Toda [20, 13]; flops of families of curves in higher dimensions by Bodzenta and Bondal [2]; construction of autoequivalences and exceptional objects by Kawamata [14]; and new braid-type groups of derived symmetries of 3-folds by the author and Wemyss [8].

Remark. The full statement of Theorem A does not require X to be smooth: I leave details to the references.

2. General results

The following theorem from [9] gives a sheafy analogue of the deformation algebra A, applicable in higher dimensions. For a birational contraction $f \colon X \to Y$ satisfying the assumption below, we define a sheaf of algebras $\mathcal{A}$ on Y which is supported on the locus over which f is not an isomorphism. We furthermore construct an associated autoequivalence of $D(X)$.

Assumption. *Suppose that* $f\colon X \to Y$ *is a contraction with* $\dim X \geq 2$*, and that either:*

(a) *the variety* X *has an* f*-relative tilting generator with summand* $\mathcal{O}_X$*, where* f *is crepant, and* Y *is Gorenstein;*

or, alternatively,

(b) *the fibres of* f *have dimension at most one.*

Remark. The tilting generator assumption from (a) is satisfied in a range of situations, including symplectic resolutions of quotient singularities as established by Bezrukavnikov and Kaledin [1], and contractions with fibres of dimension at most two under conditions of Toda and Uehara [21].

Write Z for the locus in Y over which f is not an isomorphism.

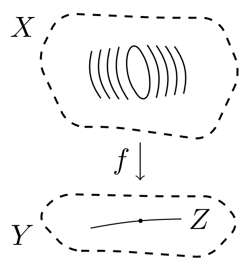

Figure B. Contraction f for Theorem B.

Theorem B. [9] *Under the Assumption above, there is a sheaf of algebras* $\mathcal{A}$ *on* Y*, inducing an object* $\mathcal{E}$ *of* $D(X)$*, such that:*

(1) *the support of* $\mathcal{A}$ *is* Z*;*

and, assuming furthermore that

(i) *the contraction* f *is crepant,*

(ii) *the base* Y *is complete locally a hypersurface at each point of* Z*,*

and that either $\operatorname{codim} Z \geq 3$ *or, alternatively,*

(iii) *the sheaf* $\mathcal{A}$ *is Cohen–Macaulay, and*

(iv) *the object* $\mathcal{E}$ *is perfect,*

then:

(2) *there is a Fourier–Mukai autoequivalence* $\mathsf{T}_{\mathcal{E}}$ *of* $D(X)$*, fitting into a distinguished triangle of functors*

$$f^{-1}\mathbb{R}f_*\mathbb{R}\mathcal{H}om_X(\mathcal{E},-) \overset{\mathbb{L}}{\underset{f^{-1}\mathcal{A}}{\otimes}} \mathcal{E} \longrightarrow \mathrm{Id}_{D(X)} \longrightarrow \mathsf{T}_{\mathcal{E}} \longrightarrow .$$

Remark. When Z is a point, the autoequivalence $\mathsf{T}_{\mathcal{E}}$ reduces to the autoequivalence T_E appearing in Theorem A(2).

I now indicate the construction of the sheaf of algebras $\mathcal{A}$. Under the Assumption above, we have an f-relative tilting generator $\mathcal{O}_X \oplus N$, either by assertion in case (a), or by a theorem of Van den Bergh [22] in case (b). Let $\mathcal{T}$ denote the relative endomorphism algebra of $\mathcal{O}_X \oplus N$, a sheaf of algebras on Y. We then establish that

$$\mathcal{T} = f_* \mathcal{E}nd_X(\mathcal{O}_X \oplus N) \cong \mathcal{E}nd_Y f_*(\mathcal{O}_X \oplus N).$$

This allows us to make the following definition, generalizing a construction of the algebra A from our previous work [6].

Definition. Let $\mathcal{A} = \mathcal{T}/\mathcal{I}$, a sheaf of algebras on Y, where $\mathcal{I}$ is the ideal of sections of $\mathcal{T}$ which factor, at each stalk, through a sum of copies of $\mathcal{O}_Y$.

The object $\mathcal{E}$ of $D(X)$ is then defined as the image of $\mathcal{A}$ under an appropriate tilting equivalence: I refer to [9, Section 3] for a precise statement.

Remark. Although the tilting generator $\mathcal{O}_X \oplus N$, and thence the sheaf of algebras $\mathcal{A}$, is not canonically defined (see for instance the construction of Van den Bergh in [22]) it seems that the autoequivalence $\mathsf{T}_{\mathcal{E}}$ may be canonical, given a choice of contraction f.

Remark. For f a flopping contraction, it would be interesting to establish when $\mathsf{T}_{\mathcal{E}}$ is related to a flop-flop functor, as in Theorem A(3).

Remark. It is tempting to speculate that the 'tilting' condition in the requirement for an f-relative tilting generator in (a) may be relaxed by upgrading $\mathcal{A}$ to an appropriate sheaf of differential graded algebras.

In the case of one-dimensional fibres, the construction above has the following deformation-theoretic interpretation.

Theorem B (continued)**.** *For a point z of Z such that $f^{-1}(z)$ is one-dimensional with components C^i, then:*

(3) *the completion $\mathcal{A}_z$ is an algebra which prorepresents the functor of noncommutative deformations of the sheaves $\mathcal{O}_{C^i}(-1)$ on X, up to Morita equivalence;*

(4) *the restriction of $\mathcal{E}$ to the formal fibre of f over z is a sheaf, namely the universal family corresponding to (3), up to summands of finite sums of sheaves.*

I record the following 3-fold setting where the assumptions of Theorem B may be established.

Theorem C. [9] *With* $\dim X = 3$, *assume that*

(i) *the contraction* f *is crepant,*
(ii) *the base* Y *is complete locally a hypersurface at each point of* Z*,*
(iii) *the exceptional fibres of* f *are irreducible curves.*

Then the assumptions of Theorem B hold, and there exists an associated autoequivalence $\mathsf{T}_{\mathcal{E}}$ *of* $D(X)$*.*

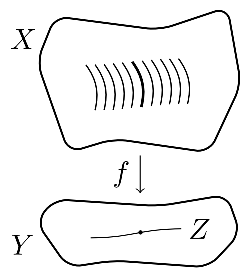

FIGURE C. Contraction f for Theorem C.

3. Springer resolution example

For an example in which the theory of the previous section applies to a contraction with higher-dimensional fibres, consider the Springer resolution of the variety of singular d-by-d matrices. Namely, for a vector space V of dimension d with $d \geq 2$, take the singular cone

$$Y = \{ M \in \operatorname{End} V \mid \det M = 0 \},$$

which is a Gorenstein hypersurface. It has a resolution by

$$X = \{ (M, H) \in \operatorname{End} V \times \mathbb{P}V \mid \operatorname{Im} M \subseteq H \}$$

whose natural projection f to $\operatorname{End} V$ surjects onto Y. This resolution f is crepant. Its exceptional fibres lie over points M in Y with $\operatorname{rk} M < d - 1$, and are projective spaces of dimension $d - 1 - \operatorname{rk} M$.

A tilting generator for X has been constructed by Buchweitz, Leuschke, and Van den Bergh [4], so that we are in the setting of Assumption (a). Conditions (i) and (ii) of Theorem B are noted above, and $\operatorname{codim} Z \geq 3$, so we can apply Theorem B(2) to obtain an autoequivalence $\mathsf{T}_{\mathcal{E}}$ of $D(X)$.

Remark. For $d = 2$, the variety X is just a 3-fold resolving a conifold Y with exceptional fibre a $(-1, -1)$-curve, and we are in the setting of Theorem A.

Remark. In joint work with E. Segal [5], I studied the resolution f, along with more general resolutions where Y is the variety of d-by-d matrices of rank at most r for $0 < r < d$. We constructed certain 'Grassmannian twist' autoequivalences of the corresponding $D(X)$ by quite different methods: it would be interesting to compare these with $\mathsf{T}_{\mathcal{E}}$.

Remark. The sheaf of algebras $\mathcal{A}$ for this example may be computed from the presentation of the endomorphism algebra $\mathcal{T}$ in [4].

4. Conjectures

In the setting of a 3-fold flopping contraction as in Theorem A, we made a conjecture [6, Conjecture 1.4] that the complete local neighbourhood of the 3-fold Y near the singularity p is determined, up to isomorphism, by the deformation algebra A. This conjecture is clear in the following simple cases, namely the two kinds of flopping curve for which A is commutative, but remains open more generally.

(1) Contractions of $(-1, -1)$-curves. In this case $A = \mathbb{C}$, and the completion of Y at p is necessarily a conifold singularity.
(2) Contractions of $(-2, 0)$-curves. Here $A \cong \mathbb{C}[\varepsilon]/\varepsilon^w$ with $w \geq 2$, where w is the width invariant of Reid [16], and the completion of Y at p is determined by w.

Hua and Toda subsequently proposed an A_∞ version of the conjecture [13, Conjecture 5.3] in which A is endowed with the structure of an A_∞-algebra. They established their conjecture for contractions to weighted homogeneous hypersurface singularities [13, Theorem 5.5], and it has now been settled in general by Hua [12]. A key idea in these works is that the deformation algebra A may be viewed as a noncommutative analogue of the Milnor algebra, and that the A_∞ structure on it allows recovery of the Milnor algebra along with enough information to apply a version of the Mather–Yau theorem.

Remark. It would be interesting to extend these results to higher dimensions, and to non-isolated singularities, for instance to establish assumptions under which the complete local neighbourhood of the variety Y near the non-isomorphism locus Z is determined by $(Z, \mathcal{A})$, potentially along with some appropriate A_∞ structure.

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