DERIVED FACTORIZATION CATEGORIES AND KNÖRRER PERIODICITY

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1. Introduction

1.1. **Background.** Let $R_n := \mathbb{C}[\![x_1, ..., x_n]\!]$ be the formal power series ring of n-valuables, and let $f \in R_n$ be a non-zero element. A **matrix factorization** of the pair (R_n, f) is a sequence of the following form

$$\left(F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_1\right),$$

where F_i are finitely generated free R_n-modules and φ_i are R_n-linear maps such that $\varphi_0\varphi_1 = f\cdot\mathrm{id}_{F_1}$ and $\varphi_1\varphi_0 = f\cdot\mathrm{id}_{F_0}$. Matrix factorizations are introduced by D. Eisenbud in [Eis]. In [Eis], Eisenbud consider the **homotopy category of matrix factorizations**

$$\mathrm{KMF}(R_n, f)$$

of (R_n, f) whose objects are matrix factorizations, and prove that the category $\mathrm{KMF}(R_n, f)$ is equivalent to the sable category $\underline{\mathrm{CM}}(R_n/\langle f\rangle)$ of maximal Cohen-Macaulay (CM) modules;

$$\mathrm{KMF}(R_n, f) \cong \underline{\mathrm{CM}}(R_n/\langle f\rangle).$$

By the above equivalence, we can apply the theory of matrix factorizations to the representation theory of CM modules over hypersurface singularities. In [Knö], Knörrer showed the following result, which is called **Knörrer periodicity**:

Theorem 1.1 ([Knö]). *We have the following equivalence*

$$\mathrm{KMF}(R_n, f) \cong \mathrm{KMF}(R_{n+2}, f + x_{n+1}^2 + x_{n+2}^2).$$

The hypersurface singularity $R_n/\langle f\rangle$ is called **simple singularities** if f is one of the following polynomials (up to change of variables):

$$\begin{array}{lcll}
(A_k) & : & x_1^{k+1} + x_2^2 + x_3^2 + x_4^2 + \cdots + x_n^2 & (k \geq 1) \\
(D_l) & : & x_1^2x_2 + x_2^{l-1} + x_3^2 + x_4^2 \cdots + x_n^2 & (l \geq 4) \\
(E_6) & : & x_1^3 + x_2^4 + x_3^2 + x_4^2 + \cdots + x_n^2 & \\
(E_7) & : & x_1^3 + x_1x_2^3 + x_3^2 + x_4^2 + \cdots + x_n^2 & \\
(E_8) & : & x_1^3 + x_2^5 + x_3^2 + x_4^2 + \cdots + x_n^2 &
\end{array}$$

Using Knörrer periodicity, we can reduce the representation theory of CM modules over higher dimensional simple singularities to the case of lower dimensional simple singularities, and by this reduction we can show that any dimensional simple singularities is of finite representation type, i.e. the number of irreducible CM modules are finite.

1.2. **Main result.** A data $(X,\chi,W)^G$ is called **gauged Landau-Ginzburg model** (or **gauged LG model**) if X is a scheme over $\mathbb{C}$, G is an algebraic group acting on X, $\chi : G \to \mathbb{G}_m$ is a character of G, and $W : X \to \mathbb{A}^1$ is a χ-semi invariant regular function, i.e. $W(g\cdot x) := \chi(g)W(x)$ for any $g \in G$ and $x \in X$. If $G = 0$, we denote the gauged LG model by (X, W). We can consider gauged LG models are generalizations of the pair (R_n, f) in the previous section. As a generalization of homotopy category of matrix factorizations, we consider the **derived factorization category** $\mathrm{Dcoh}_G(X,\chi,W)$ of $(X,\chi,W)^G$. In fact, we have the following equivalence;

$$\mathrm{Dcoh}(\operatorname{Spec} R_n, f) \cong \mathrm{KMF}(R_n, f).$$

Our main result is a Knörrer periodicity type equivalence of derived factorization categories. To state our main result, we set up notation as follows:

Let X be a smooth quasi-projective variety over $\mathbb{C}$, and let G be a reductive affine algebraic group acting on X. Let $\mathcal{E}$ be a G-equivariant locally free sheaf of finite rank, and choose a G-invariant regular section $s \in \Gamma(X,\mathcal{E}^\vee)^G$. Denote by $Z \subset X$ the zero scheme of s. Let $\chi : G \to \mathbb{G}_m$ be a character of G, and set $\mathcal{E}(\chi) := \mathcal{E}\otimes\mathcal{O}(\chi)$, where $\mathcal{O}(\chi)$ is the G-equivariant invertible sheaf corresponding to χ. Then $\mathcal{E}(\chi)$ induces a vector bundle $\mathrm{V}(\mathcal{E}(\chi))$ over X with a G-action induced by the equivariant structure of $\mathcal{E}(\chi)$. Let $q : \mathrm{V}(\mathcal{E}(\chi)) \to X$ and $p : \mathrm{V}(\mathcal{E}(\chi))|_Z \to Z$ be natural projections, and let $i : \mathrm{V}(\mathcal{E}(\chi))|_Z \to \mathrm{V}(\mathcal{E}(\chi))$ be a natural inclusion. The regular section s induces a χ-semi invariant regular function $Q_s : \mathrm{V}(\mathcal{E}(\chi)) \to \mathbb{A}^1$.

Theorem 1.2 ([H]). *Let $W : X \to \mathbb{A}^1$ be a χ-semi invariant regular function, such that the restricted function $W|_Z : Z \to \mathbb{A}^1$ is flat. Then the functor*

$$i_*p^* : \mathrm{Dcoh}_G(Z,\chi,W|_Z) \xrightarrow{\sim} \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,q^*W+Q_s).$$

is an equivalence.

As a special case of the above theorem, we obtain the following global version of Theorem 1.1;

$$\mathrm{Dcoh}(\operatorname{Spec}\mathbb{C}[x_1,...,x_n],f) \cong \mathrm{Dcoh}(\operatorname{Spec}\mathbb{C}[x_1,....,x_{n+2}],f+x_{n+1}^2+x_{n+2}^2).$$

Theorem 1.2 is also an analogy of the result obtained by Shipman and Isik independently, where they consider the case when $G = \mathbb{G}_m$, the G-action on X is trivial, $\chi = \mathrm{id}_{\mathbb{G}_m}$, and $W = 0$ [Shi], [Isi]. The proof of Theorem 1.1 is quite different from Shipman's and Isik's proofs, and we use relative singularity categories introduced in [EP], which are equivalent to derived factorization categories.

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2. Derived factorization categories

In this section, we give the definition of derived factorization categories of gauged LG models. For the definition of a gauged LG model, see section 1.2 in the introduction.

Definition 2.1. Let $(X,\chi,W)^G$ be a gauged LG model. A **factorization** F of $(X,\chi,W)^G$ is a sequence

$$F = \left(F_1 \xrightarrow{\varphi_1^F} F_0 \xrightarrow{\varphi_0^F} F_1(\chi)\right),$$

where F_i is a G-equivariant quasi-coherent sheaf on X and φ_i^F is a G-invariant homomorphism for $i = 0,1$ such that $\varphi_0^F\circ\varphi_1^F = W\cdot\mathrm{id}_{F_1}$ and $\varphi_1^F(\chi)\circ\varphi_0^F = W\cdot\mathrm{id}_{F_0}$.

Equivariant quasi-coherent sheaves F_0 and F_1 in the above sequence are called **components** of the factorization F.

Definition 2.2. (1) For a gauged LG model $(X,\chi,W)^G$, we define an abelian category

$$\mathrm{Qcoh}_G(X,\chi,W)$$

whose objects are factorizations of $(X,\chi,W)^G$. For $E,F\in \mathrm{Qcoh}_G(X,\chi,W)$, the set of morphisms $\mathrm{Hom}_{\mathrm{Qcoh}_G(X,\chi,W)}(E,F)$ is defined as the set of of pairs (f_1,f_0) such that $f_i\in \mathrm{Hom}_{\mathrm{Qcoh}_G X}(E_i,F_i)$ and that the following diagram is commutative

$$\begin{tikzcd} E_1 \arrow[r, "\varphi_1^E"] \arrow[d, "f_1"'] & E_0 \arrow[r, "\varphi_0^E"] \arrow[d, "f_0"] & E_1(\chi) \arrow[d, "f_1(\chi)"] \\ F_1 \arrow[r, "\varphi_1^F"] & F_0 \arrow[r, "\varphi_0^F"] & F_1(\chi). \end{tikzcd}$$

(2) Two morphisms $f=(f_1,f_0)$ and $g=(g_1,g_0)$ in $\mathrm{Hom}_{\mathrm{Qcoh}_G(X,\chi,W)}(E,F)$ are **homotopy equivalent**, denoted by $f\sim g$, if there exist two G-invariant homomorphisms

$$h_0: E_0\to F_1 \quad\text{and}\quad h_1: E_1(\chi)\to F_0$$

such that $f_0-g_0=\varphi_1^F h_0+h_1\varphi_0^E$ and $f_1(\chi)-g_1(\chi)=\varphi_0^F h_1+h_0(\chi)\varphi_1^E(\chi)$. The **homotopy category of factorizations**

$$\mathrm{KQcoh}_G(X,\chi,W)$$

of $(X,\chi,W)^G$ is defined as the category such that the objects are factorizations of $(X,\chi,W)^G$ and the set of morphisms is defined as the homotopy equivalence classes of the morphisms in $\mathrm{Qcoh}_G(X,\chi,W)$;

$$\mathrm{Hom}_{\mathrm{KQcoh}_G(X,\chi,W)}(E,F):=\mathrm{Hom}_{\mathrm{Qcoh}_G(X,\chi,W)}(E,F)/\sim$$

Similarly, we consider the full subcategories

$$\mathrm{coh}_G(X,\chi,W)$$
$$\mathrm{MF}_G(X,\chi,W)$$

of $\mathrm{Qcoh}_G(X,\chi,W)$ consisting of factorizations whose components are equivariant coherent sheaves and equivariant locally free sheaves of finite ranks respectively, and we define its homotopy categories

$$\mathrm{Kcoh}_G(X,\chi,W)$$
$$\mathrm{KMF}_G(X,\chi,W).$$

We easily see that $\mathrm{coh}_G(X,\chi,W)$ and $\mathrm{MF}(X,\chi,W)$ are exact categories, and if X is Noetherian, $\mathrm{coh}_G(X,\chi,W)$ is an abelian category.

We next define the totalizations of bounded complexes of factorizations.

Definition 2.3. Let $F^\bullet=(\cdots\to F^i\xrightarrow{\delta^i}F^{i+1}\to\cdots)$ be a bounded complex of $\mathrm{Qcoh}_G(X,\chi,W)$. For $l=0,1$, set

$$T_l:=\bigoplus_{i+j=-l} F^i_{\overline{j}}(\chi^{\lceil j/2\rceil}),$$

and let

$$t_l: T_l\to T_{\overline{l+1}}$$

be a G-invariant homomorphism given by

$$t_l|_{F^i_{\overline{j}}(\chi^{\lceil j/2\rceil})}:=\delta^i_{\overline{j}}(\chi^{\lceil j/2\rceil})+(-1)^i\varphi^{F^i}_{\overline{j}}(\chi^{\lceil j/2\rceil}),$$

where $\overline{n}$ is n modulo 2, and $\lceil m \rceil$ is the minimum integer which is greater than or equal to a real number m. We define the **totalization** $\mathrm{Tot}(F^\bullet) \in \mathrm{Qcoh}_G(X, \chi, W)$ of $F^\bullet$ as

$$\mathrm{Tot}(F^\bullet) := \Big(T_1 \xrightarrow{t_1} T_0 \xrightarrow{t_0} T_1(\chi) \Big).$$

Totalizations define a functor

$$\mathrm{Ch}^{\mathrm{b}}(\mathrm{Qcoh}_G(X, \chi, W)) \to \mathrm{Qcoh}_G(X, \chi, W),$$

where $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Qcoh}_G(X, \chi, W))$ is the abelian category of bounded chain complexes in $\mathrm{Qcoh}_G(X, \chi, W)$.

In what follows, we will recall that the category $\mathrm{KQcoh}_G(X, \chi, W)$ has a structure of a triangulated category.

Definition 2.4. We define an automorphism T on $\mathrm{KQcoh}_G(X, \chi, W)$, which is called **shift functor**, as follows. For an object $F \in \mathrm{KQcoh}_G(X, \chi, W)$, we define an object $T(F)$ as

$$T(F) := \Big(F_0 \xrightarrow{-\varphi_0^F} F_1(\chi) \xrightarrow{-\varphi_1^F(\chi)} F_0(\chi) \Big)$$

and for a morphism $f = (f_1, f_0) \in \mathrm{Hom}(E, F)$, we set $T(f) := (f_0, f_1(\chi)) \in \mathrm{Hom}(T(E), T(F))$. For any integer $n \in \mathbb{Z}$, denote by $(-)[n]$ the functor $T^n(-)$.

Definition 2.5. Let $f : E \to F$ be a morphism in $\mathrm{Qcoh}_G(X, \chi, W)$. We define its **mapping cone** $\mathrm{Cone}(f)$ to be the totalization of the complex

$$(\cdots \to 0 \to E \xrightarrow{f} F \to 0 \to \cdots)$$

with F in degree zero.

A **distinguished triangle** is a sequence in $\mathrm{KQcoh}_G(X, \chi, W)$ which is isomorphic to a sequence of the form

$$E \xrightarrow{f} F \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} E[1],$$

where i and p are natural injection and projection respectively.

Proposition 2.6. $\mathrm{KQcoh}_G(X, \chi, W)$ *is a triangulated category with respect to its shift functor and its distinguished triangles defined above. Full subcategories* $\mathrm{Kcoh}_G(X, \chi, W)$ *and* $\mathrm{KMF}_G(X, \chi, W)$ *are full triangulated subcategories.*

Following Positselski ([Pos1], [EP]), we define derived factorization categories.

Definition 2.7. Denote by $\mathrm{Acoh}_G(X, \chi, W)$ the smallest thick subcategory of $\mathrm{Kcoh}_G(X, \chi, W)$ containing all totalizations of short exact sequences in $\mathrm{coh}_G(X, \chi, W)$. We define the **derived factorization category** of $(X, \chi, W)^G$ as the Verdier quotient

$$\mathrm{Dcoh}_G(X, \chi, W) := \mathrm{Kcoh}_G(X, \chi, W)/\mathrm{Acoh}_G(X, \chi, W).$$

Similarly, consider the smallest thick subcategory $\mathrm{AMF}_G(X, \chi, W)$ of $\mathrm{KMF}_G(X, \chi, W)$ containing all totalizations of short exact sequences in $\mathrm{MF}_G(X, \chi, W)$, and denote the Verdier quotient by

$$\mathrm{DMF}_G(X, \chi, W) := \mathrm{KMF}_G(X, \chi, W)/\mathrm{AMF}_G(X, \chi, W).$$

Denote by $\mathrm{A^{co}Qcoh}_G(X, \chi, W)$ the smallest thick subcategory of the triangulated category $\mathrm{KQcoh}_G(X, \chi, W)$ which is closed under taking small direct sums and contains all totalizations of short exact sequences in $\mathrm{Qcoh}_G(X,\chi,W)$. Consider the following Verdier quotient;

$$\mathrm{D^{co}Qcoh}_G(X,\chi,W) := \mathrm{KQcoh}_G(X,\chi,W)/\mathrm{A^{co}Qcoh}_G(X,\chi,W).$$

If G is trivial, we drop G and χ from the above notation, and denote each triangulated categories by $\mathrm{Dcoh}(X,W)$, etc.

The following lemma says that, in affine cases, derived factorization categories are equivalent to homotopy categories.

Lemma 2.8 ([BDFIK, Lemma 2.24]). *If X is an affine scheme, then the localization functor*

$$\mathrm{KMF}_G(X,\chi,W) \to \mathrm{DMF}_G(X,\chi,W)$$

is an equivalence.

Lemma 2.9 ([BFK, Proposition 3.14]). (1) *Assume that X is a smooth variety. Then the natural functor*

$$\mathrm{DMF}_G(X,\chi,W) \to \mathrm{Dcoh}_G(X,\chi,W)$$

is an equivalence. In particular, if X is a smooth affine variety, the natural functor

$$\mathrm{KMF}_G(X,\chi,W) \to \mathrm{Dcoh}_G(X,\chi,W)$$

is an equivalence.

Similarly to the derived categories of coherent sheaves, we have standard functors between derived factorization categories such as direct mage functors, inverse image functors, and tensor product functors. See [H] for the details.

3. Relative singularity categories

In this section, we recall the definition and properties of relative singularity categories. Let X be a quasi-projective scheme, and let G be an affine algebraic group acting on X. Throughout this section, we assume that X *has a G-equivariant ample line bundle*. If X is normal, this condition is satisfied by [Tho, Lemma 2.10]. The equivariant triangulated category of singularities $\mathrm{D}^{\mathrm{sg}}_G(X)$ of X is defined as the Verdier quotient

$$\mathrm{D}^{\mathrm{sg}}_G(X) := \mathrm{D^b}(\mathrm{coh}_G X)/\operatorname{Perf}_G(X)$$

of $\mathrm{D^b}(\mathrm{coh}_G X)$ by the thick subcategory $\operatorname{Perf}_G(X)$ of equivariant perfect complexes

We recall relative singularity categories following [EP]. Let $i : Z \hookrightarrow X$ be a G-equivariant closed immersion of X such that $\mathcal{O}_Z$ *has finite G-flat dimension as an $\mathcal{O}_X$-module* i.e., the G-equivariant sheaf $i_*\mathcal{O}_Z \in \mathrm{coh}_G(X)$ has a finite resolution $F^\bullet \to i_*\mathcal{O}_Z$ of G-equivariant flat sheaves on X. Under this assumption, we have the derived inverse image $\mathbf{L}i^* : \mathrm{D^b}(\mathrm{coh}_G X) \to \mathrm{D^b}(\mathrm{coh}_G Z)$.

Definition 3.1 ([EP] Section 2.1). We consider the following Verdier quotient

$$\mathrm{D}^{\mathrm{sg}}_G(Z/X) := \mathrm{D^b}(\mathrm{coh}_G Z)/\langle \mathrm{Im}(\mathbf{L}i^* : \mathrm{D^b}(\mathrm{coh}_G X) \to \mathrm{D^b}(\mathrm{coh}_G Z))\rangle,$$

where $\langle - \rangle$ denotes the smallest thick subcategory containing objects in $(-)$. The quotient category $\mathrm{D}^{\mathrm{sg}}_G(Z/X)$ is called the **equivariant triangulated category of singularities of Z relative to X**.

Remark 3.2. If X is regular, relative singularity categories are equivalent to the usual singularity categories;

$$\mathrm{D}^{\mathrm{sg}}_G(Z/X) \cong \mathrm{D}^{\mathrm{sg}}_G(Z).$$

In what follows, we recall that, under some assumptions on gauged LG models, derived factorization categories are equivalent to relative singularity categories. Let $\chi : G \to \mathbb{G}_m$ be a character of G, and let $W : X \to \mathbb{A}^1$ be a χ-semi-invariant regular function. We assume that G is reductive and W is flat morphism. Denote by X_0 the zero scheme of W and let $i : X_0 \hookrightarrow X$ be the closed immersion.

We have an exact functor $\tau : \mathrm{coh}_G X_0 \to \mathrm{coh}_G(X, \chi, W)$ defined by

$$\tau(F) := \Big(0 \to i_*(F) \to 0\Big).$$

We define a natural functor

$$\overline{\Upsilon} : \mathrm{D^b}(\mathrm{coh}_G X_0) \to \mathrm{Dcoh}_G(X, \chi, W)$$

as the composition of functors

$$\mathrm{D^b}(\mathrm{coh}_G X_0) \xrightarrow{\tau} \mathrm{D^b}(\mathrm{coh}_G(X, \chi, W)) \xrightarrow{\mathrm{Tot}} \mathrm{Dcoh}_G(X, \chi, W),$$

where the second functor is induced by totalizations defined in Definition 2.3. The functor $\overline{\Upsilon}$ annihilates the thick category $\langle \mathrm{Im}(\mathbf{L}i^*) \rangle \subset \mathrm{D^b}(\mathrm{coh}_G X_0)$, and so, by the universal property of the Verdier quotient, it induces an exact functor

$$\Upsilon : \mathrm{D}^{\mathrm{sg}}_G(X_0/X) \to \mathrm{Dcoh}_G(X, \chi, W).$$

The following result is an equivariant version of [EP, Theorem 2.7], and it follows from the argument in Remark 2.7 in loc. cit.

Theorem 3.3 (cf. [EP] Theorem 2.7, Remark 2.7)**.** *The functor*

$$\Upsilon : \mathrm{D}^{\mathrm{sg}}_G(X_0/X) \to \mathrm{Dcoh}_G(X, \chi, W)$$

is an equivalence.

4. Proof of the main result

In this section, we provide a sketch of the proof of Theorem 1.2, which says that the functor

$$i_*p^* : \mathrm{Dcoh}_G(Z, \chi, W|_Z) \to \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, q^*W + Q_s)$$

is an equivalence (see the introduction for our notation). In the first subsection, we will prove that the functor i_*p^* is fully-faithful, and then we will show that i_*p^* is essentially surjective in the second subsection. See [H] for the detailed argument.

4.1. **Fully-faithfulness.** At first, we will introduce *Koszul factorizations.* Let $(X, \chi, W)^G$ be the gauged LG model as in Theorem 1.2. Let $\mathcal{F}$ be a G-equivariant locally free sheaf on X of rank r, and let

$$f : \mathcal{F} \to \mathcal{O}_X \quad \text{and} \quad g : \mathcal{O}_X \to \mathcal{F}(\chi)$$

be morphisms in $\mathrm{coh}_G X$ such that $g \circ f = W \cdot \mathrm{id}_{\mathcal{F}}$ and $f(\chi) \circ g = W$. Let $Z_f \subset X$ be the zero scheme of the section $f \in \Gamma(X, \mathcal{F}^\vee)^G$. We say that f is **regular** if the codimension of Z_f in X equals to the rank r.

Definition 4.1. We define an object $K(f, g) \in \mathrm{MF}_G(X, \chi, W)$ as

$$K(s,t) := \Big(K_1 \xrightarrow{k_1} K_0 \xrightarrow{k_0} K_1(\chi)\Big)$$

where

$$K_1 := \bigoplus_{n=0}^{\lceil r/2 \rceil - 1} (\bigwedge^{2n+1} \mathcal{F})(\chi^n), \qquad K_0 := \bigoplus_{n=0}^{\lfloor r/2 \rfloor} (\bigwedge^{2n} \mathcal{F})(\chi^n)$$

and

$$k_i := g \wedge (-) \oplus f \vee (-).$$

The following property will be necessary in the proof of the fully-faithfulness.

Lemma 4.2 ([BFK] Lemma 3.21 and Proposition 3.20)**.**
(1) *We have a natural isomorphism*

$$K(f,g)^\vee \cong K(g^\vee, f^\vee).$$

(2) *If f is regular, we have a natural isomorphisms in* $\mathrm{Dcoh}_G(X,\chi,W)$

$$\mathcal{O}_{Z_f} \cong K(f,g) \quad \text{and} \quad \mathcal{O}_{Z_f} \otimes \bigwedge^r \mathcal{F}^\vee(\chi^{-1})[-r] \cong K(f,g)^\vee,$$

where $\mathcal{O}_{Z_f} := \Big(0 \to \mathcal{O}_{Z_f} \to 0\Big)$ *and* $\bigwedge^r \mathcal{F}^\vee(\chi^{-1})[-r]$ *is a complex in* $\mathrm{coh}_G X$*.*

In the following proposition, we don't need to assume the condition that $W|_Z$ is flat, which is assumed in the main result.

Proposition 4.3. *The functor*

$$i_*p^* : \mathrm{Dcoh}_G(Z,\chi,W|_Z) \to \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,q^*W+Q_s)$$

is fully faithful.

Proof. The functor i_*p^* can be extended to the following functor

$$i_*p^* : \mathrm{D^{co}Qcoh}_G(Z,\chi,W|_Z) \to \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,q^*W+Q_s),$$

and the following diagram is commutative;

$$\begin{tikzcd} \mathrm{D^{co}Qcoh}_G(Z,\chi,W|_Z) \arrow[r, "i_*p^*"] & \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,q^*W+Q_s) \\ \mathrm{Dcoh}_G(Z,\chi,W|_Z) \arrow[u] \arrow[r, "i_*p^*"] & \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,q^*W+Q_s) \arrow[u] \end{tikzcd}$$

where the vertical arrows are fully faithful functors. Hence it is enough to show that the extended functor $i_*p^* : \mathrm{D^{co}Qcoh}_G(Z,\chi,W|_Z) \to \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,q^*W+Q_s)$ is fully faithful.

Set

$$\omega_j := \bigwedge^r (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee \quad \text{and} \quad \omega_i := p^*\omega_j,$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z in X. These are G-equivariant invertible sheaves on Z and $\mathrm{V}(\mathcal{E}(\chi))|_Z$ respectively. We define an exact functor

$$i^! : \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,W+Q_s) \to \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi))|_Z,\chi,W)$$

as $i^!(-) := \mathbf{L}i^*(-) \otimes \omega_i[-r]$. Then the above functor $i^!$ is right adjoint to $i_* : \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi))|_Z,\chi,W) \to \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,W+Q_s)$. Hence the composition

$$p_*i^! : \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,q^*W+Q_s) \to \mathrm{D^{co}Qcoh}_G(Z,\chi,W|_Z)$$

is right adjoint to i_*p^*. Let

$$K := K(q^*s,t) \in \mathrm{MF}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,Q_s)$$

be the Koszul factorization of $q^*s \in \Gamma(\mathrm{V}(\mathcal{E}(\chi)),q^*\mathcal{E}^\vee)^G$ and $t \in \Gamma(\mathrm{V}(\mathcal{E}(\chi)),q^*\mathcal{E})^G$, where t is the tautological section. By easy computation, there exists an object $P \in \mathrm{D^{co}Qcoh}_G(Z,\chi,0)$ such that

$$(*) \quad p_*i^!i_*p^*(-) \cong (-) \otimes P.$$

If $W=0$, by Lemma 4.2, we have the following isomorphisms;

$$p_*i^!i_*p^*(\mathcal{O}_Z) \cong p_*i^!(K) \cong p_*\mathbf{L}i^*(K^\vee) \cong p_*\mathbf{L}i^*(\mathcal{O}_{Z_{t^\vee}}) \cong \mathcal{O}_Z,$$

where $\mathcal{O}_Z := \Big(0 \to \mathcal{O}_Z \to 0\Big) \in \mathrm{MF}_G(Z, \chi, 0)$. Substituting $\mathcal{O}_Z$ into the isomorphism $(*)$ when $W = 0$, we obtain an isomorphism $P \cong \mathcal{O}_Z$. Since P doesn't depend on W, we have $p_* i^! i_* p^* \cong \mathrm{id}$. Since $i_* p^* \dashv p_* i^!$, the functor $i_* p^*$ is fully faithful. $\square$

4.2. **Essentially surjectivity.** Denote by Z_0, $V|_{Z_0}$ and V_0 the zero schemes of $W|_Z \in \Gamma(Z, \mathcal{O}(\chi))^G$, $p^*(W|_Z) \in \Gamma(\mathrm{V}(\mathcal{E}(\chi))|_Z, \mathcal{O}(\chi))^G$ and $q^*W + Q_s \in \Gamma(\mathrm{V}(\mathcal{E}(\chi)), \mathcal{O}(\chi))^G$ respectively. Let $p_0 : V|_{Z_0} \to Z_0$ and $i_0 : V|_{Z_0} \to V_0$ be natural projection and injection. Then the composition of functors

$$\mathrm{D^b}(\mathrm{coh}_G Z_0) \xrightarrow{p_0^*} \mathrm{D^b}(\mathrm{coh}_G V|_{Z_0}) \xrightarrow{i_{0*}} \mathrm{D^b}(\mathrm{coh}_G V_0)$$

induces the following functor

$$\Phi : \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{sg}}_G(V_0).$$

Then the following diagram is commutative;

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \arrow[r, "\Phi"] \arrow[d, "\Upsilon"'] & \mathrm{D}^{\mathrm{sg}}_G(V_0) \arrow[d, "\Upsilon"] \\
\mathrm{Dcoh}_G(Z, \chi, W) \arrow[r, "i_* p^*"] & \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, W + Q_s),
\end{tikzcd}$$

where the vertical arrows are equivalences by Theorem 3.3. Hence it is enough to show the following proposition:

Proposition 4.4. *The functor* $\Phi : \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{sg}}_G(V_0)$ *is essentially surjective.*

Proof. To prove the above proposition, we need to compactify the vector bundle $\mathrm{V}(\mathcal{E}(\chi))$. Let

$$P := \mathbb{P}(\mathcal{E}(\chi) \oplus \mathcal{O}_X) = \underline{\mathrm{Proj}}(\mathrm{Sym}(\mathcal{E}(\chi) \oplus \mathcal{O}_{\mathrm{X}})^{\vee})$$

be the projective space bundle over X with a G-action induced by the equivariant structure of $\mathcal{E}(\chi) \oplus \mathcal{O}_X$. Then we have a natural equivariant open immersion

$$l : \mathrm{V}(\mathcal{E}(\chi)) \to P.$$

Denote by $\overline{q} : P \to X$ the natural projection. Let P_0 be the G-invariant subscheme of P defined by the G-invariant section $\overline{s \oplus W} \in \Gamma(P, \mathcal{O}(1)(\chi))^G$ which is corresponding to the composition

$$\mathcal{O}_P \xrightarrow{\overline{q}^*(s \oplus W)} \overline{q}^*(\mathcal{E} \oplus \mathcal{O}(\chi^{-1}))^{\vee} \xrightarrow{\sigma} \mathcal{O}_P(1)(\chi),$$

where σ is the canonical surjection. Since the pull-back of $\overline{s \oplus W}$ by the open immersion l is equal to $W + Q_s$, the open immersion $l_0 : V_0 \hookrightarrow V_0$ induces the following functor

$$l_0{}^* : \mathrm{D}^{\mathrm{sg}}_G(P_0) \to \mathrm{D}^{\mathrm{sg}}_G(V_0).$$

We have the functor

$$\overline{\Phi} : \mathrm{D}^{\mathrm{sg}}_G(Z_0) \to \mathrm{D}^{\mathrm{sg}}_G(P_0),$$

which is defined similarly to Φ, and the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{sg}}_G(Z_0) \arrow[r, "\overline{\Phi}"] \arrow[d, "\pi"'] & \mathrm{D}^{\mathrm{sg}}_G(P_0) \arrow[d, "l_0{}^*"] \\
\mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \arrow[r, "\Phi"] & \mathrm{D}^{\mathrm{sg}}_G(V_0),
\end{tikzcd}$$

where the vertical arrow on the left side is a Verdier localization. Using [Orl, Theorem 2.1], we see that the functor $\overline{\Phi}$ is an equivalence. Since the functor $l_0{}^* : \mathrm{D}^{\mathrm{sg}}_G(P_0) \to \mathrm{D}^{\mathrm{sg}}_G(V_0)$ is also essentially surjective, so is the composition $l_0{}^* \circ \overline{\Phi}$. Hence Φ is also essentially surjective. $\square$

5. Implication of Knörrer periodicity

At the end of this report, we will explain that our main result is a generalization of Knörrer periodicity.

Let S be a smooth quasi-projective variety, and let G be an affine reductive group acting on S. Let $W : S \to \mathbb{A}^1$ be a $\chi := \chi_1 + \chi_2$-semi invariant non-constant regular function for some characters $\chi_i : G \to \mathbb{G}_m$. Let $X := \mathrm{V}(\mathcal{O}(\chi_1)) \cong S \times \mathbb{A}^1_{x_1}$ be the G-vector bundle over S, and let $s \in \Gamma(X, \mathcal{O}(\chi_1))^G$ be the section corresponding to the χ_1-semi invariant function $S \times \mathbb{A}^1_{x_1} \to \mathbb{A}^1$ which is defined as the projection $(s, x_1) \mapsto x_1$. Then S is isomorphic to the zero scheme of s, and the G-vector bundle $\mathrm{V}(\mathcal{O}(-\chi_1)(\chi))$ over X is isomorphic to the G-variety $S \times \mathbb{A}^2_{x_1,x_2}$, where the G-weights of x_i is given by χ_i. By Theorem 1.2, we have the following result.

Corollary 5.1. *We have the following equivalence*

$$\mathrm{Dcoh}_G(S, \chi, W) \simeq \mathrm{Dcoh}_G(S \times \mathbb{A}^2_{x_1,x_2}, \chi, W + x_1x_2).$$

When you consider the case that $G = 0$, since we are working over $\mathbb{C}$, the category $\mathrm{Dcoh}(S \times \mathbb{A}^2_{x_1,x_2}, W + x_1x_2)$ is equivalent to the category $\mathrm{Dcoh}(S \times \mathbb{A}^2_{x_1,x_2}, W + x_1^2 + x_2^2)$. Hence the above corollary can be regarded as a generalization of Theorem 1.1.

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