Grade Restriction Rule and Equivalences of Categories

Kentaro Hori

Kavli Institute for the Physics and Mathematics of the Universe (WPI)
The University of Tokyo, Kashiwa, Chiba 277-8583, Japan

Abstract

This is a talk presented at 2016 Kinosaki Symposium in Algebraic Geometry, October 17-27, 2016. It is based mainly on a work [1] with M. Herbst and D. Page, a work [2] with M. Romo, and a work in progress [3] with R. Eager, J. Knapp and M. Romo.

1 2d (2,2) supersymmetry and categories of branes

For each quantum field theory, one can extract several mathematical structures out of it. If there is some relationship between two quantum field theories, then, it may induce some relationship between the associated mathematical structures. If the two theories are equivalent (we say that they are *dual* to each other), then the mathematical structures are of course equivalent. Such a duality usually appears as a very non-trivial conjecture which is hard to prove.[1] Therefore, if we can establish the equivalence of the associated mathematical structures, that can be regarded as a strong test of the duality. Conversely, if some equivalence of mathematical structures is found, that can be used as a hint to discover a new duality in quantum field theory. In the past two decades, there has been such interaction between physics and mathematics in the arena of two-dimensional (2,2) supersymmetric quantum field theories.

To each 2d $(2,2)$ supersymmetric quantum field theory $\mathcal{T}$, two categories are associated — the category of A-branes $\mathcal{C}_A(\mathcal{T})$ and the category of B-branes $\mathcal{C}_B(\mathcal{T})$. For example, for the non-linear sigma model $\sigma(X)$ associated to a Kähler manifold X, the category of A-branes is the Fukaya category, $\mathcal{C}_A(\sigma(X)) = \mathrm{Fuk}(X)$, and the category of B-branes is the derived category, $\mathcal{C}_B(\sigma(X)) = \mathrm{D}^b_{Coh}(X)$. For the Landau-Ginzburg model

[1]In fact the theories themselves are not even defined (yet), and the words like "conjecture" and "proof" are not in the mathematical sense!

LG(W) associated to $W \in \mathbb{C}[x_1, \ldots, x_n]$, the category of A-branes is the Fukaya category, $\mathcal{C}_A(\mathrm{LG}(W)) = \mathrm{Fuk}(W)$, and the category of B-branes is the category of matrix factorizations, $\mathcal{C}_B(\mathrm{LG}(W)) = \mathrm{MF}(W)$.

If 2d (2,2) supersymmetric quantum field theories $\mathcal{T}_1$ and $\mathcal{T}_2$ are dual to each other, the categories of branes are equivalent, $\mathcal{C}_A(\mathcal{T}_1) \cong \mathcal{C}_A(\mathcal{T}_2)$ and $\mathcal{C}_B(\mathcal{T}_1) \cong \mathcal{C}_B(\mathcal{T}_2)$. There is also a twisted version of duality called *mirror symmetry*, under which the supersymmetry is transformed via a certain automorphism. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are mirror to each other, then $\mathcal{C}_A(\mathcal{T}_1) \cong \mathcal{C}_B(\mathcal{T}_2)$ and $\mathcal{C}_B(\mathcal{T}_1) \cong \mathcal{C}_A(\mathcal{T}_2)$. If a given theory $\mathcal{T}_1$ flows under the renormalization group to another theory $\mathcal{T}_2$, then, we have $\mathcal{C}_A(\mathcal{T}_1) \cong \mathcal{C}_A(\mathcal{T}_2)$ and $\mathcal{C}_B(\mathcal{T}_1) \cong \mathcal{C}_B(\mathcal{T}_2)$. Therefore, if the two theories are dual or mirror at low energies, we have the equivalences of categories as stated above. We often use the term "dual" and "mirror" in this sense. For example, if $\sigma(X)$ and LG(W) are mirror at low eneries, then, we have equivalences $\mathrm{Fuk}(X) \cong \mathrm{MF}(W)$ and $\mathrm{D}^b_{Coh}(X) \cong \mathrm{Fuk}(W)$ (homological mirror symmetry).

Continuous deformations may also result in some mathematical consequences. 2d $(2,2)$ supersymmetric quantum field theories have two distinguished classes of deformations, *A-chiral* deformations and *B-chiral* deformations. The categories $\mathcal{C}_B$ and $\mathcal{C}_A$ are invariant under A- and B-chiral deformations respectively. Let $\mathfrak{M}_A$ and $\mathfrak{M}_B$ be the space of A-chiral and B-chiral parameters. If $\mathcal{T}_1$ and $\mathcal{T}_2$ correspond to the same point in $\mathfrak{M}_B$ and different points in $\mathfrak{M}_A$ connected by a path, then, $\mathcal{C}_B(\mathcal{T}_1)$ and $\mathcal{C}_B(\mathcal{T}_2)$ are equivalent, and the equivalence depends on the homotopy class of the path. To a non-contractible loop in $\mathfrak{M}_A$ associated an autoequivalence of $\mathcal{C}_B(\mathcal{T})$ for the theory $\mathcal{T}$ at each point on the loop. The space $\mathfrak{M}_A$ can have limiting regions that correspond to sigma models and/or orbifold of Landau-Ginzburg models. If a sigma model $\sigma(X)$ and Landau-Ginzburg orbifold LG(W, Γ) appear at different limits of a common space $\mathfrak{M}_A$, then, we have an equivalence $\mathrm{D}^b_{Coh}(X) \cong \mathrm{MF}_\Gamma(W)$ (homological Calabi-Yau/Landau-Ginzburg correspondence). If the sigma models on different target spaces, X and Y, appear at different limits of a common space $\mathfrak{M}_A$, then, we have a derived equivalence $\mathrm{D}^b_{Coh}(X) \cong \mathrm{D}^b_{Coh}(Y)$. Of course these equivalences depend on the homotopy classes of the paths in $\mathfrak{M}_A$ that connects the limiting regions.

In this talk, we shall present two examples of such equivalences of categories which can be seen from a family of 2d (2,2) supersymmetric quantum field theories called the gauged linear sigma models. The main problem we would like to solve is to determine the equivalence for each homotopy class of paths.

2 Gauged Linear Sigma Models

2.1 Basic Data and the Moduli Spaces

A gauged linear sigma model (GLSM) [4] is specified by a choice of a compact Lie group G (gauge group), a finite dimensional faithful representation V of G over the field $\mathbb{C}$ of complex numbers (matter representation), a G invariant polynomial function $W : V \to \mathbb{C}$ (superpotential), and a G invariant polynomial function $\widetilde{W} : \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ (twisted superpotential). We require that the model has vector and axial $U(1)$ R-symmetries with charge integrality, that is, there is an $R \in \mathrm{End}(V)^G$ (vector R-charge) such that $W(\lambda^R\phi) = \lambda^2 W(\phi)$ and $\mathrm{e}^{\pi i R} = J$ for some $J \in G$, the twisted superpotential $\widetilde{W}$ is linear, and $G \subset SL(V)$. The linear twisted superpotential is written as

$$\widetilde{W}(\sigma) = -\langle t, \sigma\rangle, \tag{2.1}$$

for $t \in (\mathfrak{g}^*_{\mathbb{C}})^G$. To be precise, t is subject to a discrete identification, $t \equiv t + 2\pi i n$ for $n \in \mathrm{Im}(\mathrm{Hom}(G, U(1)) \to i(\mathfrak{g}^*)^G) =: \Lambda_G$.

Under the R-symmetry assumptions, the A-chiral parameters are the FI-theta parameters t and the B-chiral parameters are the couplings in the superpotential W. We require that the theory is *regular* so that the energy spectrum is discrete when formulated on $\mathbb{R} \times S^1$. Then, the spaces of parameters of regular theories are (Zariski) open subsets of these spaces

$$\mathfrak{M}_A = (\mathfrak{g}^*_{\mathbb{C}})^G/2\pi i\Lambda_G - \Delta_t, \qquad \mathfrak{M}_B \subset ((\mathrm{Sym}(V^*)^G)^2 - \Delta_W)/\simeq . \tag{2.2}$$

In the latter, the superscript 2 selects the component of vector R-charge 2, $W(\lambda^R\phi) = \lambda^2 W(\phi)$, and $/\simeq$ is the identification by reparametrizations of the argument ϕ. The discriminant loci Δ_t and Δ_W are where the theory becomes irregular.

2.2 Examples

Quintic model

The first model, introduced in [4], is

$$\begin{aligned} G &= U(1), \\ V &= \mathbb{C}(-5) \oplus \mathbb{C}(1)^{\oplus 5} \ni (p, x_1, \dots, x_5) \\ W &= pf(x_1, \dots, x_5), \\ \widetilde{W} &= -t \cdot \sigma. \end{aligned}$$

$f(x_1,\ldots,x_5)$ is a polynomial of degree 5 and $t=\zeta-i\theta\in\mathbb{C}/2\pi i\mathbb{Z}$. The vector R-charge is unique up to gauge $R=(2-5\epsilon,\epsilon,\ldots,\epsilon)$. The theory is regular when the quintic polynomial f is generic in that $\mathrm{d}f=0\Rightarrow x=0$ and when t is away from the discriminant point $t\equiv 5\log(-5)$, i.e.,

$$\zeta=5\log 5,\quad \theta\equiv\pi. \tag{2.3}$$

When ζ is large positive, the theory reduces at low energies to the sigma model $\sigma(X_f)$ whose target space is the quintic threefold $X_f=(f=0)\subset\mathbb{P}^4$. The Kähler class and the B-field is approximately given by ζH and $(\theta+\pi)H$ respectively, where H is the hyperplane class of $\mathbb{P}^4$ restricted to X_f. In the limit $\zeta\to-\infty$, the theory reduces at low energies to the Landau-Ginzburg orbifold $\mathrm{LG}(f(x_1,\ldots,x_5),\mathbb{Z}_5)$ where $\omega\in\mathbb{Z}_5$ acts as $(x_1,\ldots,x_5)\to(\omega x_1,\ldots,\omega x_5)$.

Rødland model

The second model, introduced in [5], is

$$\begin{aligned} G &= U(2),\\ V &= (\det{}^{-1}S)^{\oplus 7}\oplus S^{\oplus 7}\ni(p^1,\ldots,p^7,x_1,\ldots,x_7)\\ W &= \sum_{i,j,k=1}^{7}A^{ij}_k p^k[x_ix_j],\\ \widetilde{W} &= -t\cdot\mathrm{tr}_S\sigma. \end{aligned}$$

$S\cong\mathbb{C}^2$ is the fundamental representation of $U(2)$, $[x_ix_j]=x_i^1x_j^2-x_i^2x_j^1$ is a symplectic pairing of x_i and x_j in S, A^{ij}_k are complex numbers which are antisymmetric in the upper indices, and $t=\zeta-i\theta\in\mathbb{C}/2\pi i\mathbb{Z}$. We may also write $W=\sum_{ij}A^{ij}(p)[x_ix_j]=\sum_k p^kA_k(x)$. The vector R-charge is unique up to gauge; $R=2-2\epsilon$ on p's and $R=\epsilon$ on x's. The theory is regular when A^{ij}_k are generic in a certain sense (i.e. $\mathrm{rank}(x_i^a)=2$ and $A_1(x)=\cdots=A_7(x)=0\Rightarrow \mathrm{d}A_1(x)\wedge\cdots\wedge\mathrm{d}A_7(x)\neq 0$) and when t is away from the discriminant locus $\{t_1,t_2,t_3\}$, where

$$\zeta_a=7\log\left(2\cos\left(\frac{\pi a}{7}\right)\right),\qquad \theta_a\equiv\pi a,\qquad a=1,2,3. \tag{2.4}$$

When ζ is large positive, the theory reduces at low energies to the sigma model $\sigma(X_A)$ whose target space is the complete interesection $X_A=\cap_{k=1}^7(A_k(x)=0)\subset G(2,7)$. When ζ is large negative, the theory reduces at low energies to the sigma model $\sigma(Y_A)$ whose target space is the Pfaffian variety $Y_A=\{\mathrm{rank}A(p)\leq 4\}\subset\mathbb{P}^6$. Both X_A and Y_A are Calabi-Yau threefolds with Hodge numbers $(h^{1,1},h^{2,1})=(1,50)$.

2.3 Expectations

From the general principles described in Section 1, we expect to have the following equivalences of categories. In the quintic model, for each homotopy class of paths in the space of $t \in \mathbb{C}/2\pi i\mathbb{Z}$ that goes from $\zeta \gg 0$ to $\zeta \ll 0$ avoiding the discriminant point (2.3), we have an equivalence

$$\mathrm{D}^b_{Coh}(X_f) \cong \mathrm{MF}_{\mathbb{Z}_5}(f). \tag{2.5}$$

In Rødland model, for each homotopy class of paths in the space of $t \in \mathbb{C}/2\pi i\mathbb{Z}$ that goes from $\zeta \gg 0$ to $\zeta \ll 0$ avoiding the three discriminant points (2.4), we have an equivalence

$$\mathrm{D}^b_{Coh}(X_A) \cong \mathrm{D}^b_{Coh}(Y_A). \tag{2.6}$$

Indeed, the equivalences (2.5) and (2.6) had been proven in [6] and in [7, 8] respectively. The main problem we would like to solve is to identify *which* equivalence is associated to each homotopy class of paths.

3 B-branes in Gauged Linear Sigma Models

3.1 B-brane Data

A B-brane in the GLSM (G, V, W, t) is specified classically by a choice of

- a G-equivariant matrix factorization of W;
  - $\mathrm{M} = \mathrm{M}^{\mathrm{ev}} \oplus \mathrm{M}^{\mathrm{od}}$, a $\mathbb{Z}_2$-graded finite dimensional representation of G over $\mathbb{C}$,
  - $Q : V \to \mathrm{End}^{\mathrm{od}}(\mathrm{M})$, a G-equivariant polynomial function such that

$$Q(\phi)^2 = W(\phi) \cdot \mathrm{id}_{\mathrm{M}}, \tag{3.1}$$

- $\gamma \subset \mathfrak{t}_{\mathbb{C}}$, a Weyl invariant Lagrangian submanifold.

Here $\mathfrak{t}$ is the Lie algebra of a maximal torus T of the gauge group G, which is equipped with a Weyl invariant inner product that induces a symplectic structure on $\mathfrak{t}_{\mathbb{C}}$. We require that the vector $U(1)$ R-symmetry with charge integrality is preserved by the brane: there is an $r \in \mathrm{End}^{\mathrm{ev}}(\mathrm{M})^G$ such that $\lambda^r Q(\lambda^R \phi)\lambda^{-r} = \lambda Q(\phi)$ and $\mathrm{e}^{\pi i r} J = \pm 1$ on even/odd elements of M. Note that G-equivariant matrix factorizations of W satisfying this condition form a differential $\mathbb{Z}$-graded category which we denote by $\mathrm{MF}_G(W)$.

Given the above data, we have a B-brane at the classical level. We note that the discrete identification of the parameter t involves a twist of the data: For $\chi \in \mathrm{Hom}(G, U(1))$ with infinitesimal version $n_\chi \in \Lambda_G$,

$$t \to t - 2\pi i n_\chi \quad \text{is equivalent to} \quad \mathrm{M} \to \mathrm{M}(\chi). \tag{3.2}$$

The main question is whether the data defines a B-brane in the quantum theory. To get some hint, we consider the partition function on the hemisphere [2].

3.2 Hemisphere Partition function and the Grade Restriction Rule

Let

$$V = \bigoplus_{\mathrm{i}} \mathbb{C}(R_{\mathrm{i}}, Q_{\mathrm{i}}) \tag{3.3}$$

be the weight decomposition of the matter representation — R_{i} and Q_{i} are the vector R-charge and the T-weight of the i-th component. Then, the hemisphere partition function with the B-brane (M, Q, γ) at the boundary is given by $[2, 9, 10]$[1]

$$Z_{D^2}(M, Q, \gamma) = \int_\gamma \mathrm{d}^\ell \sigma \prod_{\alpha > 0} \langle \alpha, \sigma \rangle \sinh(\pi \langle \alpha, \sigma \rangle) \prod_{\mathrm{i}} \Gamma\left(i\langle Q_{\mathrm{i}}, \sigma\rangle + \tfrac{R_{\mathrm{i}}}{2}\right) \mathrm{e}^{i\langle t, \sigma\rangle} \operatorname{tr}_{\mathrm{M}}\left(\mathrm{e}^{\pi i r}\, \mathrm{e}^{2\pi\sigma}\right). \tag{3.4}$$

Here $\mathrm{d}^\ell\sigma$ is a flat holomorphic volume form on $\mathfrak{t}_{\mathbb{C}}$ (ℓ is the rank of G), the first product is over positive roots of G, and $\Gamma(x)$ in the second product is Euler's Gamma function. Note that the integrand has poles at the hyperplanes

$$\langle Q_{\mathrm{i}}, \sigma \rangle = i\left(n_{\mathrm{i}} + \frac{R_{\mathrm{i}}}{2}\right), \qquad n_{\mathrm{i}} = 0, 1, 2, 3, \ldots. \tag{3.5}$$

Note also that the convergence of the integral is not trivial when γ is non-compact. The asymptotic behaviour of the integrand can be seen from Stirling's formula.

The formula (3.4) is derived first for the case where we take $\gamma = i\mathfrak{t}$ (the real locus) under the assumption that all the vector R-charges are brought in the band $0 < R_{\mathrm{i}} < 2$ by using the gauge ambiguity if necessary. Note that the pole hyperplanes (3.5) do not meet the real locus $i\mathfrak{t}$ under the assumtion.

The requirement for the brane (M, Q, γ) is that $\gamma \subset \mathfrak{t}_{\mathbb{C}}$ is homotopic to the real locus in the complement of the poles,

$$\gamma \simeq i\mathfrak{t} \quad \text{in } \mathfrak{t}_{\mathbb{C}} - (3.5), \tag{3.6}$$

[1] To be precise, the partition function depends on the radius L of the hemisphere. In the model with the two $U(1)$ R-symmetries, the dependence is an overall power factor $L^{\widehat{c}/2}$ where $\widehat{c} = \operatorname{tr}_V(1-R) - \dim G$. For simplicity, we suppress the dependence from the expression.

on which the integral (3.4) is absolutely convergent. Existence of such γ imposes a severe constraint on the representations on G that can enter into M. This is the *grade restriction rule.*

In what follows, we shall workout the rule in the two models.

4 Quintic

In the quintic model where the gauge group G is $U(1)$, we choose $0 < \epsilon < 2/5$ in order for the bound $0 < R_{\rm i} < 2$ to be satisfied. The integrand of (3.4) has fifth order poles at $\sigma = i(n_x+\epsilon/2)$ on the positive imaginary axis and simple poles at $\sigma = i(-(n_p+1)/5+\epsilon/2)$ on the negative imaginary axis. For the term corresponding to the charge q representation $\mathbb{C}(q)$ in M, the integrand behaves at large $|\sigma|$ as

$$|\text{integrand}_q| \;\sim\; \exp\Big(-(\zeta - 5\log 5)\mathrm{Im}\,\sigma + (\theta+2\pi q)\mathrm{Re}\,\sigma - 5\pi|\mathrm{Re}\,\sigma| \Big), \tag{4.1}$$

where $\exp\left(5\log 5\mathrm{Im}\,\sigma - 5\pi|\mathrm{Re}\,\sigma|\right)$ comes from the Gamma function factors.

When $\zeta \gg 0$ (*resp.* $\zeta \ll 0$), (4.1) decays exponentially at infinity of γ_+ (*resp.* γ_-) for an arbitrary q where

$$\gamma_\pm = \{\mathrm{Im}\,\sigma = \pm(\mathrm{Re}\,\sigma)^2\}. \tag{4.2}$$

In particular, the integral on γ_+ (*resp.* γ_-) is absolutely convergent for any representation $\mathbb{C}(q)$. There is no condition on the representations to be included in M for $\zeta \gg 0$ and for $\zeta \ll 0$. I.e., there is no non-trivial garde restriction rule there.

When t moves along a path from $\zeta \gg 0$ to $\zeta \ll 0$ avoiding the discriminant locus $5\log(-5) + 2\pi i\mathbb{Z}$, then, it must go through the window $\mathbf{w}_n$:

$$\zeta = 5\log 5, \qquad \theta \in ((2n-1)\pi, (2n+1)\pi), \tag{4.3}$$

for some $n \in \mathbb{Z}$. On this window, (4.1) grows exponentially in either $\mathrm{Re}\,\sigma \to +\infty$ or $\mathrm{Re}\,\sigma \to -\infty$ unless $|\theta + 2\pi q| < 5\pi$ for the θ in the window $\mathbf{w}_n$, that is,

$$q \in \{-n-2, -n-1, -n, -n+1, -n+2\}. \tag{4.4}$$

We can find a family of contours γ along the path, starting from γ_+ at $\zeta \gg 0$ and ending with γ_- at $\zeta \ll 0$, on which the integral is absolutely convergent all the way, if and only if all the q's in M are in the range (4.4). This is the grade restriction rule for the paths through the window $\mathbf{w}_n$ given by (4.3).

Let $\mathrm{GR}_{\mathbf{w}_n} \subset \mathrm{MF}_{\mathbb{C}^*}(p \cdot f)$ be the subcategory of grade restricted matrix factorizations with respect to the window $\mathbf{w}_n$, i.e., (M, Q) where M is a direct sums of copies of the $\mathbb{C}(q)$'s for q in the range (4.4). Then, we have a diagram

$$\begin{tikzcd}
 & \mathrm{MF}_{\mathbb{C}^*}(p \cdot f) \arrow[ddl] \arrow[ddr] & \\
 & \mathrm{GR}_{\mathbf{w}_n} \arrow[u, phantom, "\bigcup"] \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\mathrm{MF}_{\mathbb{Z}_5}(f) & & \mathrm{D}^b_{Coh}(X_f)
\end{tikzcd} \tag{4.5}$$

The downward arrows to $\mathrm{D}^b_{Coh}(X_f)$ and $\mathrm{MF}_{\mathbb{Z}_5}(f)$ are functors that represent the reduction to the sigma model at $\zeta \gg 0$ and to the Landau-Ginzburg orbifold at $\zeta \ll 0$ respectively. The arrows from $\mathrm{MF}_{U(1)}(p \cdot f)$ are very far from equivalences — many different objects are sent to the same object. However, when restricted to the grade restricted subcategory $\mathrm{GR}_{\mathbf{w}_n}$, they are equivalences. In particular, we obtain an equivalence from $\mathrm{D}^b_{Coh}(X_f)$ to $\mathrm{MF}_{\mathbb{Z}_5}(f)$ via $\mathrm{GR}_{\mathbf{w}_n}$. This is the equivalence associated to the paths through $\mathbf{w}_n$.

If we change the window, we get a different equivalence. This effect can be used to find the monodromy along the loop around a discriminant point, say, $t = 5\log 5 - 5\pi i$. The window to the right (*resp.* left) of this point is $\mathbf{w}_{-2}$ (*resp.* $\mathbf{w}_{-3}$), for which the set (4.4) is $\{0, 1, 2, 3, 4\}$ (*resp.* $\{1, 2, 3, 4, 5\}$). Let us consider a loop that starts from $\zeta \gg 0$, goes to $\zeta \ll 0$ through the window $\mathbf{w}_{-2}$ and comes back to $\zeta \gg 0$ through the window $\mathbf{w}_{-3}$. Then, the monodromy is the functor from $\mathrm{D}^b_{Coh}(X_f)$ to $\mathrm{MF}_{\mathbb{Z}_5}(f)$ through $\mathrm{GR}_{\mathbf{w}_{-2}}$ followed by the functor backward through $\mathrm{GR}_{\mathbf{w}_{-3}}$. Note that the representation $\mathbb{C}$ is grade restricted with respect to the first window $\mathbf{w}_{-2}$ but not with respect to the second window $\mathbf{w}_{-3}$. Instead, $\mathbb{C}(5)$ is grade restricted with respect to $\mathbf{w}_{-3}$. Let us see what the monodromy does on an object $E \in \mathrm{D}^b_{Coh}(X_f)$. Let (M, Q) be its lift to $\mathrm{MF}_{\mathbb{C}^*}(p \cdot f)$ that is grade restricted with respect to $\mathbf{w}_{-2}$. This can be transported from $\zeta \gg 0$ to $\zeta \ll 0$ along a path through the window $\mathbf{w}_{-2}$. While at $\zeta \ll 0$, we would like to find another matrix factorization which is isomorphic to (M, Q) when reduced to $\mathrm{MF}_{\mathbb{Z}_5}(f)$ and is grade restricted with respect to $\mathbf{w}_{-3}$. To do so, we must replace each $\mathbb{C}$ component in M by something else made of $\mathbb{C}(1), \ldots, \mathbb{C}(5)$. This can be done by using the matrix factorization (M_-, Q_-)

$$\mathrm{M}_- = \mathbb{C}(0)[0] \oplus \mathbb{C}(5)[1 - 5\epsilon], \tag{4.6}$$

$$Q_- = \begin{pmatrix} 0 & p \\ f(x) & 0 \end{pmatrix}, \tag{4.7}$$

and its shifts $(\mathrm{M}_-[i], Q_-)$ for $i \in \mathbb{Z}$. Here $\mathbb{C}(q)[j]$ stands for the charge q representation $\mathbb{C}(q)$ of $G = U(1)$ at $r = j$. When reduced to $\mathrm{MF}_{\mathbb{Z}_5}(f)$, these are empty, and hence binding them to a given matrix factorization does not change the image in $\mathrm{MF}_{\mathbb{Z}_5}(f)$. Once all the $\mathbb{C}$ components of (M, Q) are replaced by $\mathbb{C}(5)$ in this way, the resulting matrix factorization (M', Q') is grade restricted with respect to the window $\mathbf{w}_{-3}$ and can be transported from $\zeta \ll 0$ back to $\zeta \gg 0$ along a path through $\mathbf{w}_{-3}$. The reduction of this (M', Q') to $\mathrm{D}^b_{Coh}(X_f)$ is the monodromy image of the object E. When reduced to $\mathrm{D}^b_{Coh}(X_f)$, the matrix factorization (M_-, Q_-) and its shifts become the structure sheaf $\mathcal{O}_{X_f}$ and its shifts. In effect, the monodromy is the Seidel-Thomas twist

$$\mathrm{ST}_{\mathfrak{V}} : E \longmapsto \mathrm{Cone}\left(E \to \bigoplus_i \mathrm{Hom}(E, \mathfrak{V}[i]) \otimes \mathfrak{V}[i] \right) [-1], \tag{4.8}$$

by $\mathfrak{V} = \mathcal{O}_{X_f}$.

5 Rødland Model

Let us next describe the grade restriction rule and the monodromy action in Rødland model. We choose $0 < \epsilon < 1$ for the bound $0 < R_{\mathrm{i}} < 2$. Note that a finite dimensional irreducible representations of the gauge group $G = U(2)$ is one of

$$S^l(i) := \mathrm{Sym}^l S \otimes (\det S)^{\otimes i}, \qquad l = 0, 1, 2, \ldots; \ i \in \mathbb{Z}. \tag{5.1}$$

We shall write $\mathbb{C}(i) = S^0(i)$, $S(i) = S^1(i)$ and $S^l = S^l(0)$. As a maximal torus T of G, we choose the diagonal matrices and write $\sigma \in \mathfrak{t}_{\mathbb{C}}$ as $\sigma = \mathrm{diag}(\sigma_1, \sigma_2)$.

When $\zeta \gg 0$, the integral (3.4) is absolutely convergent on

$$\gamma_+ = \left\{ \ \mathrm{Im}\,\sigma_1 = (\mathrm{Re}\,\sigma_1)^2, \quad \mathrm{Im}\,\sigma_2 = (\mathrm{Re}\,\sigma_2)^2 \ \right\}, \tag{5.2}$$

for any (M, Q). There is no non-trivial grade restriction rule.

When $\zeta \ll 0$, the integral (3.4) is absolutely convergent on

$$\gamma_- = \left\{ \ \mathrm{Im}\,\sigma_1 = \mathrm{Im}\,\sigma_2 = -(\mathrm{Re}\,\sigma_1 + \mathrm{Re}\,\sigma_2)^2 \ \right\}, \tag{5.3}$$

if M is a direct sum of copies of $S^l(i)$ with $l = 0, 1, 2$ and $i \in \mathbb{Z}$. Also, it is divergent on any $\gamma \simeq i\mathfrak{t}$ if M includes a component $S^l(i)$ with $l \geq 3$. Thus, we have the grade restriction rule:

$$\left\{ \ \mathbb{C}(i), \ \ S(i), \ \ S^2(i) \ \right\}_{i \in \mathbb{Z}}. \tag{5.4}$$

We shall say that a matrix factorization (M, Q) is grade restricted in $\zeta \ll 0$ when M is a direct sum of copies of representations in (5.4).

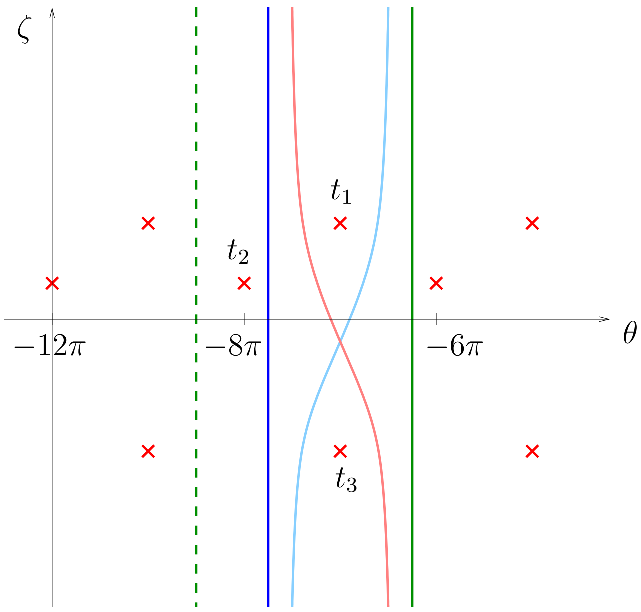

Figure 1: Different types of paths in the t-space

Now, let us describe the grade restriction rule for the paths from $\zeta \gg 0$ to $\zeta \ll 0$ avoiding the discriminant locus $t_a + 2\pi i\mathbb{Z}$ $(a = 1, 2, 3)$, for $t_a \equiv \zeta_a - i\theta_a$ given in (2.4). This time, since there are three arrays of discriminant points, there are several types of paths, in addition to the variety coming from the shift $\theta \to \theta + 2\pi$. See Fig. 1. The grade restriction rule for the four types of paths in Fig. 1 is shown in Fig. 2. To each homotopy

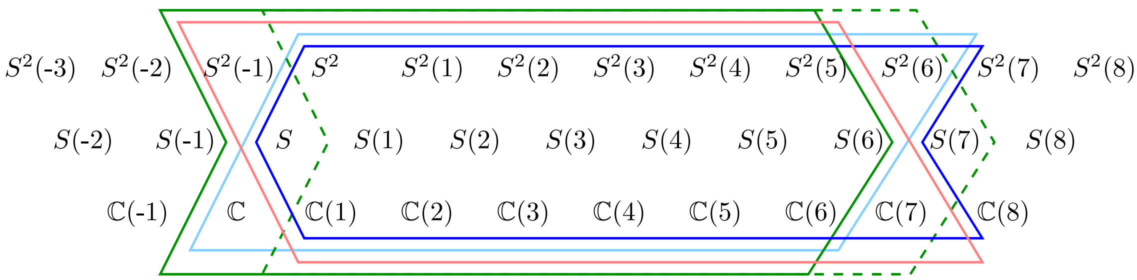

Figure 2: The grade restriction rule in Rødland model

class of paths, we associate a set of twenty one representations of $G = U(2)$ encircled by a line of the corresponding color, which we shall call the *grade restricted subset*. For example, the grade restricted subset for the light blue path is $\{\mathbb{C}(i), S(i), S^2(i)\}_{i=0}^{6}$. If we shift the path by $\theta \to \theta - 2\pi$, the set is shifted by $\otimes\mathbb{C}(1)$, as shown for the green paths in the Figures, in accord with (3.2). The main statement is: along each path, there exists a family of γ's, starting from γ_+ at $\zeta \gg 0$ and ending with γ_- at $\zeta \ll 0$, on which the integral (3.4) is absolutely convergent all the way, if and only if (M, Q) is grade restricted, i.e. M is a direct sum of copies of the representations from the grade restricted subset.

Let GR_{-} and $\mathrm{GR}_{\mathbf{w}}$ be the subcategories of $\mathrm{MF}_{\mathrm{GL}_2(\mathbb{C})}(W)$ consisting of matrix factorizations which are grade restricted in $\zeta \ll 0$ and with respect to the homotopy class $\mathbf{w}$ of paths respectively. Then, we have a diagram

$$
\begin{tikzcd}
 & \mathrm{MF}_{\mathrm{GL}_2(\mathbb{C})}(W) \arrow[dddrr] & & \\
 & \mathrm{GR}_{-} \arrow[u, hook] \arrow[ddl] & & \\
 & & \mathrm{GR}_{\mathbf{w}} \arrow[ul, hook] \arrow[dll, "\cong"'] \arrow[dr, "\cong"] & \\
\mathrm{D}^b_{Coh}(Y_A) & & & \mathrm{D}^b_{Cof}(X_A)
\end{tikzcd}
\tag{5.5}
$$

The downward arrows to $\mathrm{D}^b_{Coh}(X_A)$ and $\mathrm{D}^b_{Coh}(Y_A)$ are functors that represent the reduction to the sigma models at $\zeta \gg 0$ and $\zeta \ll 0$ respectively. The arrows from $\mathrm{MF}_{\mathrm{GL}_2(\mathbb{C})}(W)$ and from GR_{-} are very far from equivalences but the arrows from $\mathrm{GR}_{\mathbf{w}}$ are. In particular, we obtain an equivalence from $\mathrm{D}^b_{Coh}(X_A)$ to $\mathrm{D}^b_{Coh}(Y_A)$ via $\mathrm{GR}_{\mathbf{w}}$. This is the equivalence associated to the homotopy class $\mathbf{w}$ of paths.

We can also find the monodromy along loops around the discriminant points. Let us look at a loop around t_1 in Fig. 1 with a base point at $\zeta \gg 0$. It can be represented as the concatenation of the light blue path and the blue path. The monodromy is the functor from $\mathrm{D}^b_{Coh}(X_A)$ to $\mathrm{D}^b_{Coh}(Y_A)$ through $\mathrm{GR}_{\text{light blue}}$ followed by the functor backward through $\mathrm{GR}_{\text{blue}}$. To see what that is, let us note that the difference between the light blue set and the blue set in Fig. 2 is that $\mathbb{C}$ is in light blue but not in blue while $\mathbb{C}(7)$ is in blue but not in light blue. If we start from a matrix factorization at $\zeta \gg 0$ which is grade restricted with respect to the light blue path, then, while at $\zeta \ll 0$ we must replace each $\mathbb{C}$ component by something else made of the blue set. This can be done using the matrix factorization (M_{-}, Q_{-}):

$$
\mathrm{M}_{-} \;=\; \bigwedge \mathbb{C}(1)^{\oplus 7}[1-2\epsilon] = \bigoplus_{j=0}^{7} \mathbb{C}(j)^{\oplus \binom{7}{j}}[j-2j\epsilon], \tag{5.6}
$$

$$
Q_{-} \;=\; \sum_{k=1}^{7} \left(p^k \eta_k + A_k(x)\overline{\eta}^k \right), \tag{5.7}
$$

and its shifts $(\mathrm{M}_{-}[i], Q_{-})$ for $i \in \mathbb{Z}$. In (5.6)-(5.7), we regard M_{-} as a module over the Clifford algebra generated by η_k and $\overline{\eta}^k$ ($k = 1, \ldots, 7$) obeying the relations $\{\eta_k, \overline{\eta}^l\} = \delta_k^l$, $\{\eta_k, \eta_l\} = \{\overline{\eta}^k, \overline{\eta}^l\} = 0$ such that the $j = 0$ component $\mathbb{C}(0)[0] \subset \mathrm{M}_{-}$ is annihilated by all η_k's. The brane (M_-, Q_-) and its shifts are empty when reduced to $\mathrm{D}^b_{Coh}(Y_A)$ but descend to the structure sheaf $\mathcal{O}_{X_A}$ and its shifts when reduced to $\mathrm{D}^b_{Coh}(X_A)$. Therefore, the monodromy is the Seidel-Thomas twist by $\mathcal{O}_{X_A}$. Similarly, the monodromy along a loop around t_2 is $\mathrm{D}^b_{Coh}(X_A) \to \mathrm{GR}_{\text{blue}} \to \mathrm{D}^b_{Coh}(Y_A)$ followed by $\mathrm{D}^b_{Coh}(Y_A) \to \mathrm{GR}_{\text{dashed green}} \to \mathrm{D}^b_{Coh}(X_A)$. The relevant matrix factorizations for the replacement at $\zeta \ll 0$ are $(\mathrm{M}_- \otimes S, Q_-)$ and its shifts, which descend to the tautological vector bundle S_{X_A} and its shifts when reduced to $\mathrm{D}^b_{Coh}(X_A)$. Therefore, the monodromy is the Seidel-Thomas twist by S_{X_A}. The monodromy along a loop around t_3 is $\mathrm{D}^b_{Coh}(X_A) \to \mathrm{GR}_{\text{pink}} \to \mathrm{D}^b_{Coh}(Y_A)$ followed by $\mathrm{D}^b_{Coh}(Y_A) \to \mathrm{GR}_{\text{blue}} \to \mathrm{D}^b_{Coh}(X_A)$. The relevant matrix factorizations for the replacement at $\zeta \ll 0$ are $(\mathrm{M}_- \otimes S^2(-1), Q_-)$ and its shifts, which descend to $\mathrm{Sym}^2 S_{X_A}(-1)$ and its shifts when reduced to $\mathrm{D}^b_{Coh}(X_A)$. Therefore, the monodromy is the Seidel-Thomas twist by $\mathrm{Sym}^2 S_{X_A}(-1)$. To summarize, the monodromies around t_1, t_2 and t_3 are respectively

$$\mathrm{ST}_{\mathcal{O}_{X_A}}, \quad \mathrm{ST}_{S_{X_A}} \quad \text{and} \quad \mathrm{ST}_{\mathrm{Sym}^2 S_{X_A}(-1)}. \tag{5.8}$$

6 Remarks

The grade restriction rule was first found in [1] for GLSMs with Abelian gauge groups by analyzing the effective potential localized near the boundary. Later in [2] it was shown to be reproduced by looking at the condition of convergence of the hemisphere partition function, as described in this talk.

We would like to make some remarks on related mathematical works. The grade restriction rule, or its purely categorical extraction to be precise, had been completed in [11] and is extended in [12, 13] to a general variation of GIT quotients in the case when the quotients are good. In these works, the term "window" is used for the subcategory $\mathrm{GR}_{\mathbf{w}}$ instead of the actual window $\mathbf{w}$ that determines it, which is understandable as the space $\mathfrak{M}_A$ is out of scope in their current formulation. (However, a mathematical incarnation of $\mathfrak{M}_A$ is discussed in a recent work [14].) These works do not apply to the case with bad GIT quotients such as Rødland model. Nevertheless, GLSM-like proof of the Pfaffian-Grassmannian equivalence (2.6) was given in [15]. In fact, the diagram (5.5) was first obtained in this work for the case $\mathrm{GR}_{\mathbf{w}} = \mathrm{GR}_{\text{light blue}}$, and that was a huge encouragement for the work in progress [3] presented here. Later, the proof is revisited in [16] based on the categorical extraction of the duality found in [17].

Acknowledgement

The speaker would like to thank Richard Eager, Manfred Herbst, Johanna Knapp, Mauricio Romo, David Page and David Tong for the collaborations [5, 1–3] on which this talk is based.

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