ON WEAK POSITIVITY THEOREMS AND IITAKA'S CONJECTURE IN POSITIVE CHARACTERISTIC

SHO EJIRI

1. Introduction

In this article, we would like to explain several results on weak positivity theorems and Iitaka's conjecture in positive characteristic. We fix an algebraically closed field k of characteristic $p > 0$. For simplicity, we only deal with a surjective morphism between smooth projective varieties whose geometric generic fiber is integral and smooth. In this setting, we give proofs of weak positivity theorems (Section 4) and explain the ideas of proofs of results on Iitaka's conjectures (Section 5). For this purpose, we explain trace maps of absolute and relative Frobenius morphisms (Section 2) and an invariant which measures the positivity of coherent sheaves (Section 3).

2. Trace maps

In this section, we consider trace maps of Frobenius and relative Frobenius morphisms.

2.1. **Absolute Frobenius morphisms.** In Sections 4 and 5, we consider the trace map of Frobenius morphisms of the geometric generic fiber $X_{\overline{\eta}}$ of a surjective morphism between smooth projective varieties $f : X \to Z$. In general, $X_{\overline{\eta}}$ is Gorenstein but not smooth. For this reason, we introduce trace maps of Frobenius morphisms of Gorenstein varieties.

Let X be a Gorenstein variety. Let e be a positive integer. We denote by $F_X^e : X \to X$ the e-times iterated absolute Frobenius morphism of X. We denote X by X^e when we regard X as the source space of F_X^e.

Let ω_X be the dualizing sheaf of X. We have $\mathcal{H}om(F_{X*}^e\mathcal{O}_X, \mathcal{O}_X) \cong F_{X*}^e\omega_X^{1-p^e}$ by the Grothendieck duality. Applying the functor $\mathcal{H}om(\_, \mathcal{O}_X)$ to $\mathcal{O}_X \to F_{X*}^e\mathcal{O}_{X^e}$, we obtain the homomorphism of $\mathcal{O}_X$-module

$$\phi_X^{(e)} : F_{X*}^e\omega_X^{1-p^e} \to \mathcal{O}_X.$$

X is said to be *F-pure* if $\phi_X^{(e)}$ is surjective. Let $\mathcal{L}$ be a line bundle on X. We have

$$(F_{X*}^e\omega_X^{1-p^e}) \otimes \mathcal{L} \cong F_{X*}^e(\omega_X^{1-p^e} \otimes F_X^{e*}\mathcal{L}) \cong F_{X*}^e(\omega_X^{1-p^e} \otimes \mathcal{L}^{p^e})$$

by the projection formula. Therefore the tensor product induces the homomorphism

$$\phi_X^{(e)} \otimes \mathcal{L} : F_{X*}^e(\omega_X^{1-p^e} \otimes \mathcal{L}^{p^e}) \to \mathcal{L}$$

of $\mathcal{O}_X$-modules. We consider the morphism between global sections

$$H^0(X, F_{X*}^e(\omega_X^{1-p^e} \otimes \mathcal{L}^{p^e})) \xrightarrow{H^0(X, \phi_X^{(e)} \otimes \mathcal{L})} H^0(X, \mathcal{L})$$

induced by $\phi_X^{(e)} \otimes \mathcal{L}$. Following [17, §3], we put

$$S^0(X,\mathcal{L}) := \bigcap_{e>0} \mathrm{Image}(H^0(X, \phi_X^{(e)} \otimes \mathcal{L})) \subseteq H^0(X,\mathcal{L}).$$

Assume that X is projective and F-pure. Let $\mathcal{L}$ be an ample line bundle on X. Then there exists an m_0 such that $S^0(X,\mathcal{L}^m) = H^0(X,\mathcal{L}^m)$ for every $m \geq m_0$ [7, §3].

2.2. **Relative Frobenius morphisms.** Let $f : X \to Z$ be a surjective morphisms from a Gorenstein projective variety X to a smooth projective variety Z. We denote $f : X \to Z$ by $f^{(e)} : X^e \to Z^e$ when we regard X as X^e and Z as Z^e. We define the e-th relative Frobenius morphism $F_{X/Z}^{(e)}$ of f to be the morphism $f^{(e)} \times F_{X/Z}^{(e)} : X^e \to X \times_Z Z^e =: X_{Z^e}$. When $Z = \mathrm{Spec}\, k$, we sometimes identify $F_{X/Z}^{(e)}$ with F_X^e. Let $\pi^{(e)} : X_{Z^e} =: X \times_Z Z^e \to X$ be the first projection. Now we have the following commutative diagram.

$$\begin{tikzcd}
X^e \arrow[d, "F^{(e)}_{X/Z}"'] \arrow[dr, "F_X^e"] \arrow[dd, bend right=60, "f^{(e)}"'] & \\
X_{Z^e} \arrow[r, "\pi^{(e)}"'] \arrow[d, "f_{Z^e}"'] & X \arrow[d, "f"] \\
Z^e \arrow[r, "F_Z^e"'] & Z
\end{tikzcd}$$

As shown by Kunz [11], the smoothness of Z is equivalent to the flatness of Z. Hence F_Z^e is a Gorenstein morphism, and so is $\pi^{(e)}$, which implies that X_{Z^e} is Gorenstein. Set $\omega_{X/Y} := \omega_X \otimes f^*\omega_Y^{-1}$. Then we have $\mathcal{H}om(F_{X/Z}^{(e)}{}_*\mathcal{O}_{X^e}, \mathcal{O}_{X_{Z^e}}) \cong F_{X/Z}^{(e)}{}_*\omega_{X/Z}^{1-p^e}$ by the Grothendieck duality. Applying the functor $\mathcal{H}om(\_, \mathcal{O}_{X_{Z^e}})$ to the natural morphism $\mathcal{O}_{X_{Z^e}} \to F_{X/Z}^{(e)}{}_*\mathcal{O}_{X^e}$, we obtain the homomorphism of $\mathcal{O}_{X_{Z^e}}$-module

$$\phi_{X/Z}^{(e)} : F_{X/Z}^{(e)}{}_*\omega_{X^e/Z^e}^{1-p^e} \to \mathcal{O}_{X_{Z^e}}.$$

Let $\mathcal{L}$ be a line bundle on X, and set $\mathcal{L}_{Z^e} := \pi^{(e)*}\mathcal{L}$. We have

$$\begin{aligned}
(F_{X/Z}^{(e)}{}_*\omega_{X^e/Z^e}^{1-p^e}) \otimes \mathcal{L}_{Z^e} &\cong (F_{X/Z}^{(e)}{}_*(\omega_{X^e/Z^e}^{1-p^e} \otimes F_{X/Z}^{(e)}{}^*\mathcal{L}_{Z^e})) \\
&\cong (F_{X/Z}^{(e)}{}_*(\omega_{X^e/Z^e}^{1-p^e} \otimes F_X^{e*}\mathcal{L})) \cong F_{X/Z}^{(e)}{}_*(\omega_{X^e/Z^e}^{1-p^e} \otimes \mathcal{L}^{p^e})
\end{aligned}$$

by the projection formula. From this the tensor product induces the homomorphism

$$\phi_{X/Z}^{(e)} \otimes \mathcal{L}_{Z^e} : F_{X/Z}^{(e)}{}_*(\omega_{X^e/Z^e}^{1-p^e} \otimes \mathcal{L}^{p^e}) \to \mathcal{L}_{Z^e}$$

of $\mathcal{O}_{X_{Z^e}}$-modules. Taking the direct image by f_{Z^e}, we obtain the homomorphism of $\mathcal{O}_{Z^e}$-modules

$$f_{Z^e*}(\phi_{X/Z}^{(e)} \otimes \mathcal{L}_{Z^e}) : f^{(e)}{}_*(\omega_{X^e/Z^e}^{1-p^e} \otimes \mathcal{L}^{p^e}) \to f_{Z^e*}\mathcal{L}_{Z^e} \cong F_Z^{e*}f_*\mathcal{L}.$$

Here the isomorphism follows from the flatness of F_Z^e. One of the important property of relative Frobenius morphisms and their trace maps is compatibility with base change. See [16, §2] for more details. Let W be a regular scheme and $h : W \to Z$

be a morphism. Assume that $V := X \times_Z W$ is Gorenstein. Let $g : V \to W$ be the second projection. Then we have the diagram

$$\begin{tikzcd} V^e \arrow[r] \arrow[d, "F^{(e)}_{V/W}"'] & X^e \arrow[d, "F^{(e)}_{X/Z}"] \\ V_{W^e} \arrow[r] \arrow[d, "g_{W^e}"'] & X_{Z^e} \arrow[d, "f_{Z^e}"] \\ W^e \arrow[r, "h^{(e)}"'] & Z^e \end{tikzcd}$$

whose squares are Cartesian. Let $\mathcal{L}$ be a line bundle on X. Set $\mathcal{L}_W := (V \to X)^*\mathcal{L}$ and $\mathcal{L}_{W^e} := (V_{W^e} \to V)^*\mathcal{L}_W$. We have the following commutative diagram.

$$\begin{tikzcd} (F^{(e)}_{X/Z*}(\omega^{1-p^e}_{X/Z} \otimes \mathcal{L}^{p^e}))|_{V_{W^e}} \arrow[rr, "(\phi^{(e)}_{X/Z} \otimes \mathcal{L}_{Z^e})|_{V_{W^e}}"] \arrow[d, "\cong"'] & & \mathcal{L}_{Z^e}|_{V_{W^e}} \arrow[d, equal] \\ F^{(e)}_{V/W*}(\omega^{1-p^e}_{V/W} \otimes \mathcal{L}^{p^e}_W) \arrow[rr, "\phi^{(e)}_{V/W} \otimes \mathcal{L}_{W^e}"'] & & \mathcal{L}_{W^e} \end{tikzcd}$$

Let $\overline{\eta}$ be the geometric generic point of Z. Let $h : \mathrm{Spec}\ k(\overline{\eta}) \to Z$ be the natural morphism and $V := X \times_Z \mathrm{Spec}\ k(\overline{\eta})$ be the geometric generic fiber of f. Then by the flatness of h, we have

$$(f_*\mathcal{L}) \otimes k(\overline{\eta}) \cong H^0(V, \mathcal{L}_{\mathrm{Spec}\ k(\overline{\eta})}).$$

Furthermore, by the above argument, we see that

$$(\mathrm{Image}(f_{Z^e *}(\phi^{(e)}_{X/Z} \otimes \mathcal{L}_{Z^e}))) \otimes k(\overline{\eta}^e) \cong S^0(V, \mathcal{L}_{\mathrm{Spec}\ k(\overline{\eta})}).$$

In particular, if $S^0(V, \mathcal{L}_{\mathrm{Spec}\ k(\overline{\eta})}) = H^0(V, \mathcal{L}_{\mathrm{Spec}\ k(\overline{\eta})})$, then $f_{Z^e *}(\phi^{(e)}_{X/Z} \otimes \mathcal{L}_{Z^e})$ is generically surjective.

3. Weak positivity and numerical invariant

In this section, we introduce an invariant used to measure positivity of coherent sheaves. This plays an important role in the proof of the main theorem.

Throughout this section, we let $\mathcal{G}$ be a coherent sheaf on a smooth projective variety Z over an algebraically closed field k of positive characteristic. We first recall some definitions.

Definition 3.1. $\mathcal{G}$ is said to be *generically globally generated* if the natural morphism $H^0(Z, \mathcal{G}) \otimes_k \mathcal{O}_Z \to \mathcal{G}$ is surjective on the generic point of Z.

Viehweg introduced the notion of weak positivity as a generalization of nefness of vector bundles.

Definition 3.2 ([12, Notation, (vii)])**.** $\mathcal{G}$ is said to be *weakly positive* if for an ample divisor H and for each $\alpha > 0$ there exist some $\beta > 0$ such that $(S^{\alpha\beta}\mathcal{G})^{**} \otimes \mathcal{O}_Z(\beta H)$ is generically globally generated. Here $(S^{\alpha\beta}\mathcal{G})^{**}$ is the double dual of the $\alpha\beta$-th symmetric product of $\mathcal{G}$.

Remark 3.3. (1) The above definition is independent of the choice of ample divisor H (e.g.,[19, Lemma 2.14]).
(2) Assume that $\mathcal{G}$ is a line bundle. Then $\mathcal{G}$ is weakly positive if and only if it is pseudo-effective.
(3) Assume that $\mathcal{G}$ is a vector bundle on a smooth projective curve. Then $\mathcal{G}$ is weakly positive if and only if it is nef.

Definition 3.4. Let D be a $\mathbb{Q}$-Cartier divisor on Z. Then we define

$$T(\mathcal{G}, D) := \left\{ \varepsilon \in \mathbb{Q} \;\middle|\; \begin{array}{c} \text{there exists an integer } e > 0 \text{ such that} \\ p^e\varepsilon \in \mathbb{Z} \text{ and } (F_Z^{e*}\mathcal{G})(-p^e\varepsilon D) \text{ is} \\ \text{generically globally generated.} \end{array} \right\}, \quad \text{and}$$

$$t(\mathcal{G}, D) := \sup T(\mathcal{G}, D) \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

Next we list some properties of $t(\mathcal{G}, D)$.

Lemma 3.5. *With notation as Definition 3.4, let $\mathcal{F}$ be a coherent sheaf on Z.*

(1) *If there exists a generically surjective morphism $\mathcal{F} \to \mathcal{G}$, then $t(\mathcal{F}, D) \le t(\mathcal{G}, D)$.*
(2) *If $\{t(\mathcal{F}, D), t(\mathcal{G}, D)\} \ne \{-\infty, +\infty\}$, then $t(\mathcal{F}, D) + t(\mathcal{G}, D) \le t(\mathcal{F} \otimes \mathcal{G}, D)$.*
(3) *For each $e > 0$, $t(F_Z^{e*}\mathcal{G}, D) = p^e t(\mathcal{G}, D)$.*
(4) *If $t(\mathcal{G}, D) = +\infty$ and $\mathcal{G}$ is of positive rank, then $-D$ is pseudo-effective.*
(5) *Let H be an ample divisor on Z. Then $t(\mathcal{G}, H) \ne -\infty$. Furthermore, if $t(\mathcal{G}, H) \ge 0$, then $\mathcal{G}$ is weakly positive.*

Proof. (1)-(3) are follows directly from the definition. For (4) (resp. (5)), we refer [8, §3] (resp. [7, §4]). □

4. Weak positivity theorems

In this section, we prove two kind of weak positivity theorems in a simple situation. For more general statements, please see [7] and [8].

We first prove a weak positivity theorem under some global conditions on geometric generic fibers.

Theorem 4.1 ([7, Theorem 5.1])**.** *Let $f : X \to Z$ be a separable surjective morphism between smooth projective varieties with connected fibers and $\overline{\eta}$ be the geometric generic point of Z. Assume that*

(i) *the graded ring $\bigoplus_{m \ge 0} H^0(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}^m)$ is a finitely generated k-algebra, and*
(ii) *there exists an integer $m_0 > 0$ such that for each $m \ge m_0$,*

$$S^0(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}^m) = H^0(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}^m).$$

Then $f_*\omega_{X/Y}^m$ is a weakly positive sheaf for every $m \ge m_0$.

Remark 4.2. The assumptions of the theorem are satisfied if one of the followings is satisfied:

(0) $X_{\overline{\eta}}$ is a reduced zero-dimensional $k(\overline{\eta})$-scheme.
(1) $X_{\overline{\eta}}$ is a smooth projective curve of genus at least two and $m_0 \ge 2$.
(2) $X_{\overline{\eta}}$ is a projective surface of general type with at most rational double point singularities, $m_0 \gg 0$ and $p \ge 7$.

(3) $X_{\overline{\eta}}$ is a projective variety with at most F-pure singularities, $K_{X_{\overline{\eta}}}$ is ample and $m_0 \gg 0$.

Remark 4.3. There exists a counter-example to Theorem 4.1 if we do not assume the assumption (ii).

Proof of Theorem 4.1. Let H be an ample divisor on Z. For simplicity we set $t(n) := t(f_*\omega^n_{X/Z}, H)$ for every integer n.
Step 1. We show that there exist integers $l, n_0 > m_0$ such that $t(l)+t(n) \le t(l+n)$ for each $n \ge n_0$. By the assumption (i), we see that for every $l > m_0$ having enough divisors there exists an $n_0 > m_0$ such that the natural morphism

$$H^0(X_{\overline{\eta}}, \omega^l_{X_{\overline{\eta}}}) \otimes H^0(X_{\overline{\eta}}, \omega^n_{X_{\overline{\eta}}}) \to H^0(X_{\overline{\eta}}, \omega^{l+n}_{X_{\overline{\eta}}})$$

is surjective. This implies that the natural morphism

$$f_*\omega^l_{X/Z} \otimes f_*\omega^n_{X/Z} \to f_*\omega^{l+n}_{X/Z}$$

is generically surjective, and hence we have $t(l) + t(n) \le t(l + n)$ by Lemma 3.5.
Step 2. We show that $t((m-1)p^e + 1) \le p^e t(m)$ for each $e > 0$ and for each $m \ge m_0$. We use the notation in Subsection 2.2. Set $\mathcal{L} := \omega^m_{X/Z}$. As shown in Subsection 2.2, we obtain the homomorphism

$$f^{(e)}{}_*\omega^{(m-1)p^e+1}_{X^e/Y^e} \xrightarrow{f_{Z^e*}(\phi^{(e)}_{X/Z}\otimes\omega^m_{X_{Z^e}/Z^e})} F^{e*}_Z f_*\omega^m_{X/Z}$$

of $\mathcal{O}_Z$-modules, and this is generically surjective by the assumption (ii). Hence by Lemma 3.5, we have

$$t((m-1)p^e + 1) \le t((F^{e*}_Z f_*\omega^m_{X/Z}), H) = p^e t(m).$$

Step 3. We prove the theorem. Set $m \ge m_0$. If $m = 1$, then $t(1) \le pt(1)$ by Step 2, which gives $t(1) \ge 0$. Thus we may assume $m_0 \ge 2$. For $e \gg 0$, we let $q_{m,e}$ and $r_{m,e}$ be the quotient and the remainder of $(m-1)p^e + 1 - n_0$ by l respectively. We note that $q_{m,e} > 0$ since $m \ge m_0 \ge 1$, and that $p^e - q_{l,e} \xrightarrow{e\to\infty} \infty$. Then

$$q_{m,e}t(l) + t(r_{m,e} + n_0) \overset{Step1}{\le} t((m-1)p^e + 1) \overset{Step2}{\le} p^e t(m),$$

and so $c := \min\{t(r + n_0) | 0 \le r < l\} \le p^e t(m) - q_{m,e}t(l)$. Substituting l for m, we obtain that $c \le (p^e - q_{l,e})t(l)$ for each $e \gg 0$, which implies that $t(l) \ge 0$. Therefore $c \le p^e t(m)$ for each $e \gg 0$, and hence $t(m) \ge 0$. This completes the proof. $\square$

The following theorem can be viewed as a kind of weak positivity theorem. The theorem requires a strong condition on K_X, but does not require any global condition on $X_{\overline{\eta}}$.

Theorem 4.4 ([8, Theorem 1.4]). *Let $f : X \to Z$ be a separable surjective morphism with connected fibers. Assume that*

(i) *$X_{\overline{\eta}}$ is F-pure.*
(ii) *$K_X \sim_{\mathbb{Q}} f^*(K_Z + L)$ for a $\mathbb{Q}$-Cartier divisor L on Z.*

Then L is pseudo-effective.

Proof. As seen in Subsection 2.1, there exist an ample line bundle $\mathcal{L}$ on X such that
$$S^0(X_{\overline{\eta}}, \mathcal{L}|_{X_{\overline{\eta}}}) = H^0(X_{\overline{\eta}}, \mathcal{L}|_{X_{\overline{\eta}}})$$
and that $f_*\mathcal{L}$ is globally generated. Then we see that
$$f_*(\phi^{(e)}_{X/Z} \otimes \mathcal{L}_{Z^e}) : {f^{(e)}}_*(\omega_{X/Z}^{1-p^e} \otimes \mathcal{L}^{p^e}) \to F_Z^{e*} f_*\mathcal{L}$$
is generically surjective as shown in Subsection 2.2. Let $l > 0$ be an integer such that lD is Cartier and $lK_{X/Z} \sim lf^*D$. For $e \gg 0$, we let q_e and r_e be the quotient and the remainder of $p^e - 1$ by l respectively. Then by the projection formula, we have
$$f_*(\omega_{X/Z}^{1-p^e} \otimes \mathcal{L}^{p^e}) \cong \mathcal{O}_Z(-q_eD) \otimes f_*(\omega_{X/Z}^{-r_e} \otimes \mathcal{L}^{p^e}).$$
Let m_0 be an integer such that for each $0 \le r < l$, $f_*(\omega_{X/Z}^{-r} \otimes \mathcal{L}^{m_0})$ is globally generated and $\omega_{X_{\overline{\eta}}}^{-r} \otimes \mathcal{L}|_{X_{\overline{\eta}}}^{m_0}$ is 0-regular with respect to $\mathcal{L}|_{X_{\overline{\eta}}}$. Then the morphism
$$H^0(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}^{-r} \otimes \mathcal{L}|_{X_{\overline{\eta}}}^{m}) \otimes H^0(X_{\overline{\eta}}, \mathcal{L}|_{X_{\overline{\eta}}}) \to H^0(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}^{-r} \otimes \mathcal{L}|_{X_{\overline{\eta}}}^{m+1})$$
induced by the multiplication is surjective for every $m \ge m_0$. This implies that the natural morphism
$$f_*(\omega_{X/Y}^{-r} \otimes \mathcal{L}^m) \otimes f_*\mathcal{L} \to f_*(\omega_{X/Y}^{-r} \otimes \mathcal{L}^{m+1})$$
is generically surjective for every $m \ge m_0$. By the above argument, for every $e \gg 0$, we have generically surjective morphisms
$$\mathcal{O}_Z(-q_eD) \otimes (f_*\omega_{X/Z}^{-r} \otimes \mathcal{L}^{m_0}) \otimes (\bigotimes^{p^e - m_0} f_*\mathcal{L}) \to f_*(\omega_{X/Z}^{1-p^e} \otimes \mathcal{L}^{p^e}) \to F_Z^{e*} f_*\mathcal{L}.$$
Assume that D is not pseudo-effective. Then by Lemma 3.5 (4), $t(\mathcal{G}, D) \ne +\infty$ for any coherent sheaf $\mathcal{G}$ of positive rank. Applying Lemma 3.5 (2) to the above homomorphism, we obtain
$$t(\mathcal{O}_Z(-q_eD), -D) + t(f_*(\omega_{X/Z}^{-r} \otimes \mathcal{L}^{m_0}), -D) + (p^e - m_0)t(f_*\mathcal{L}, -D) \le p^e t(f_*\mathcal{L}, -D).$$
It is easily seen that $t(\mathcal{O}_Z(-q_eD), -D) = q_e$. Set $c := \min\{t(f_*(\omega_{X/Z}^{-r} \otimes \mathcal{L}^{m_0}, -D) | 0 \le r < l\}$. Note that $c \ne -\infty$ by the choice of m_0. Since $t(f_*\mathcal{L}, -D) \ne -\infty$ by the choice of $\mathcal{L}$, we have $t(f_*\mathcal{L}, -D) \in \mathbb{R}$. Therefore we get
$$q_e + c \le m_0 t(f_*\mathcal{L}, -D)$$
for every $e \gg 0$, which is a contradiction. Hence we conclude that D is pseudo-effective. □

5. Iitaka's $C_{n,m}$ conjecture

In this section, we survey some results on the following Iitaka's conjecture in positive characteristic, and explain briefly the ideas of the proofs of two of them proved in [7] and [9].

Conjecture ($C_{n,m}$)**.** *Let X and Z be smooth projective varieties of dimension n and m respectively over an algebraically closed field and $f : X \to Z$ be a surjective morphism whose geometric generic fiber $X_{\overline{\eta}}$ is integral and smooth. Then*
$$\kappa(X) \ge \kappa(X_{\overline{\eta}}) + \kappa(Z).$$

The *Kodaira dimension* $\kappa(V)$ of a smooth projective variety V is defined as the minimum element $\kappa \in \{-\infty, 0, 1, 2, \ldots\}$ such that there exists $c > 0$ satisfying $\dim_k H^0(X, \omega_X^m) \leq cm^\kappa$ for each $m > 0$. The Kodaira dimension is a birational invariant which plays an important role in birational geometry, and tells us some hints about geometric features of varieties.

In characteristic zero, many results related to Conjecture $C_{n,m}$ are known. (For a list of references, please see [9, §1].) In positive characteristic, Conjecture $C_{n,m}$ has been proved in some cases recently. Chen and Zhang showed $C_{n,n-1}$ [6, Theorem 1.2]. Patakfalvi proved $C_{n,m}$ when Z is of general type and the Hasse-Witt matrix of $X_{\overline{\eta}}$ is not nilpotent [15, Theorem 1.1].

Recall that for a smooth projective variety V we have $\kappa(V) \leq \dim V$ and that V is said to be *of general type* if $\kappa(V) = \dim V$. As an application of Theorem 4.1, we show that Conjecture $C_{n,m}$ holds in some situations.

Theorem 5.1 ([7, §7]). *Let $f : X \to Z$ be a surjective morphism between smooth projective varieties whose geometric generic fiber $X_{\overline{\eta}}$ is integral. Assume that*

(i) $\bigoplus_{m\geq 0} H^0(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}^m)$ *forms a finitely generated k-algebra, and*

(ii) *there exists an integer $m_0 > 0$ such that for every $m \geq m_0$,*

$$S^0(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}^m) = H^0(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}^m).$$

If Z is of general type, or if Z is a curve, then Conjecture $C_{n,m}$ holds.

Idea of the proof. In the situation of the theorem, $f_*\omega_{X/Z}^m$ is weakly positive for every $m \geq m_0$ as shown by Theorem 4.1. From this, the theorem follows by a standard argument if Z is of general type. For more details, please see the proof of [7, Theorem 7.2]. If Z is an elliptic curve, we need to show that $\pi^* f_*\omega_{X/Z}^m$ is globally generated for a finite morphism $\pi : Z' \to Z$ from an elliptic curve Z'. To this end, we use the classification of vector bundles on elliptic curves due to Atiyah [1] and Oda [13]. For more details, please see the proof of [7, Theorem 7.6]. □

The following is a direct corollary of the theorem.

Corollary 5.2. *Let $f : X \to Z$ be a surjective morphism from a smooth projective 3-fold to a smooth projective curve whose geometric generic fiber $X_{\overline{\eta}}$ is a smooth projective surface of general type. Then Conjecture $C_{n,m}$ holds true.*

Next we consider Conjecture $C_{n,m}$ in the case when $\dim X = 3$.

Theorem 5.3 ([9, Theorem 1.2]). *Conjecture $C_{3,m}$ holds when $p > 5$.*

Under the assumption that $p > 5$, Conjecture $C_{3,1}$ was proved when $k = \overline{\mathbb{F}_p}$ by Birkar, Chen and Zhang [4, Theorem 1.2], and when the genus of Z is at least two by Zhang [20, Corollary 1.9].

Idea of the proof of Theorem 5.3. Since Conjecture $C_{n,n-1}$ is settled [6, Theorem 1.2], we only need to consider $C_{3,1}$. We divide the proof into three cases. When $\kappa(X_{\overline{\eta}}) = 2$, $C_{3,1}$ is proved by Corollary 5.2. When $\kappa(X_{\overline{\eta}}) = 1$, by the minimal model program for 3-folds provided by [2, 3, 5, 10], we may assume that K_X is nef over Z. Note that we never obtain a Mori fiber space by the assumption of $\kappa(X_{\overline{\eta}}) = 1$. In this situation, we can show that for every $m \gg 0$, $f_*\omega_{X/Y}^m$ contain a nef subbundle

$V^{(m)}$ of rank large enough. Using this, we prove the assertion by a standard argument if Z is of general type. If Z is an elliptic curve, we need to show that $\pi^*V^{(m)}$ is globally generated for a finite morphism $\pi : Z' \to Z$ from an elliptic curve Z'. To this end, we apply the abundance theorem on two-dimensional log canonical pairs [18] for a log canonical center of a pair which dominates Z. For more details, please see [9, §4]. When $\kappa(X_{\overline{\eta}}) = 0$, by the minimal model program again, we may assume that $K_X \sim_{\mathbb{Q}} f^*(K_Z + L)$ for a Cartier divisor L on Z. Then by Theorem 4.4, we see that L is pseudo-effective. Using this, we can show the assertion by a standard argument if Z is of general type. If Z is an elliptic curve, the assertion follows from Theorem 5.4 below. □

Theorem 5.4 ([9, Theorem 3.2]). *Let $f : X \to Z$ be a surjective morphism from a smooth projective variety X to an elliptic curve Z whose geometric generic fiber is integral and smooth. Assume that $K_X \sim_{\mathbb{Q}} f^*(K_Z + L)$ for a $\mathbb{Q}$-Cartier divisor on L on Z. Then L is semi-ample.*

Idea of the proof. As shown by Theorem 4.4, we have $\deg L \geq 0$. We may assume that $\deg L = 0$. We need to show that L is a torsion line bundle. To this end, we use the trace map of relative Frobenius and the classification of vector bundles on elliptic curves due to Atiyah [1] and Oda [13]. For more details, please see the proof of [9, Theorem 3.2]. □

Acknowledgments. The author would like to thank the organizers of the conference for giving him the opportunity to talk.

References

[1] M. F. Atiyah, Vector bundles over an elliptic curve, Proc. London Math. Soc, **7** (1957), 414–452.

[2] C. Birkar, Existence of flips and minimal models for 3-folds in char p, To appear in Ann. Sci. Ecole Norm. S. (2013).

[3] C. Birkar and J. Waldron, Existence of Mori fibre spaces for 3-folds in char p, arXiv:1410.4511.

[4] C. Birkar, Y. Chen and C. Zhang, Iitaka's $C_{n,m}$ conjecture for 3-folds over finite fields, to appear in Nagoya Math. J. (2016).

[5] P. Cascini, H. Tanaka and C. Xu, On base point freeness in positive characteristic, Ann. Sci. Ecole Norm. S. **48** (2015), 1239–1272.

[6] Y. Chen and L. Zhang, The subadditivity of the Kodaira dimension for fibrations of relative dimension one in positive characteristics, to appear in Math. Res. Lett. (2013).

[7] S. Ejiri, Weak positivity theorem and Frobenius stable canonical rings of geometric generic fibers, arXiv:1508.00484 (2015).

[8] S. Ejiri, Positivity of anti-canonical divisors and F-purity of fibers, arXiv:1604.02022.

[9] S. Ejiri and L. Zhang, Iitaka's $C_{n,m}$ conjecture in positive characteristic,arXiv:1604.01856.

[10] C.D. Hacon and C. Xu, On the three dimensional minimal model program in positive characteristic, J. Amer. Math. Soc. **28** (2015), 711–744

[11] E. Kunz, On Noetherian rings of characteristic p, Amer. J. Math. 98 (1976), 999—1013.

[12] J. Kollár, Subadditivity of the Kodaira dimension: fibers of general type, Algebraic geometry, Sendai, 1985, 361–398, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam, (1987).
[13] T. Oda, Vector bundles on an elliptic curve, Nagoya Math. J. **43** (1971), 41–72.
[14] Z. Patakfalvi, Semi-positivity in positive characteristics, Ann. Sci. Ecole Norm. S. **47** (2014), no. 5, 991–1025.
[15] Z. Patakfalvi, On subadditivity of Kodaira dimension in positive characteristic, http://arxiv.org/abs/1308.5371 (2013).
[16] Z. Parakfalvi, K. Schwede, W. Zhang, F-singularities in families, arXiv:1305.1646.
[17] K. Schwede, A canoical linear system associated to adjoint divisors in characteristic $p > 0$, J. Reine Angew. Math., **696**, (2014), 69-87.
[18] H. Tanaka, Minimal models and abundance for positive characteristic log surfaces, Nagoya Math. J. **216** (2014), 1–70.
[19] E. Viehweg, Quasi-projective moduli for polarized manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **30** Springer-Verlag, Berlin, (1995).
[20] L. Zhang, Subadditivity of Kodaira dimensions for fibrations of three-folds in positive characteristics, http://arxiv.org/abs/1601.06907 (2016).

Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.

E-mail address: `ejiri@ms.u-tokyo.ac.jp`