STABLE RATIONALITY OF ORBIFOLD FANO 3-FOLD HYPERSURFACES

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1. Stable rationality

The aim of this report is to explain the main result of [20] concerning stable non-rationality of Fano 3-folds, and an idea of its proof. We refer readers to the above mentioned paper for details.

A projective variety X of dimension n is *rational* if X is birational to $\mathbb{P}^n$. We say that X is *stably rational* if $X \times \mathbb{P}^m$ is rational for some $m \geq 0$. We have the implications

$$\text{Rational} \overset{(1)}{\Longrightarrow} \text{Stably rational} \overset{(2)}{\Longrightarrow} \text{Unirational} \overset{(3)}{\Longrightarrow} \text{Rationally connected.}$$

In dimensions 1 and 2, the above implications are equivalent. In dimension at least 3, neither (1) nor (2) is equivalent: see [4] and [1] for the existence of stably rational non-rational varieties and unirational stably non-rational varieties. However, it is unknown that the implication (3) is an equivalence or not.

Rationality (resp. stable rationality) questions, which asks whether a given variety is rational or not (resp. stably rational or not), is a fundamental question in algebraic geometry. These questions are birational in nature and the objects to be considered are necessarily rationally connected. Thus, in view of the Minimal Model Program, the questions for 3-folds split into those for Fano 3-folds, del Pezzo fibrations over $\mathbb{P}^1$ and conic bundles over a rational surface. By contributions of many mathematicians, rationality questions for (general) smooth Fano 3-folds have been settled. The following result settles stable rationality questions for (very general) smooth Fano 3-folds except for cubic 3-folds. Note that it is a well known fact that a smooth cubic 3-fold is not rational ([9]) while its stable (non-)rationality remains unknown.

Theorem 1.1 ([15, Theorem 1])**.** *Let X be a very general smooth non-rational Fano 3-fold over $\mathbb{C}$. Assume that X is not birational to a cubic 3-fold. Then X is not stably rational.*

As a next step it is natural to consider singular Fano 3-folds (of Picard number one and with only terminal singularities). We consider stable rationality questions for Fano 3-folds with only terminal quotient singularities embedded in a weighted projective space as a hypersurface. By the notation $X_d \subset \mathbb{P}(a_0, \dots, a_4)$, we mean that X_d is a weighted hypersurface of degree d in $\mathbb{P}(a_0, \dots, a_4)$. The number $I = \sum a_i - d$ is called the *index* of the Fano variety X for which we have $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(I)$. These Fano 3-folds are classified by [16], [5] and [6]: there are 95 families of index 1 Fanos and 35 families of index > 1 Fanos. We list previously known results on (stable) rationality for these objects:

- A Fano 3-fold weighted hypersurface of index 1 is birationally rigid and in particular not rational ([8], [13]).

- Cubic 3-folds $X_3 \subset \mathbb{P}^4$ are non-rational ([9]).
- A very general $X_{10} \subset \mathbb{P}(1,1,2,3,5)$ and $X_{15} \subset \mathbb{P}(1,2,3,5,7)$ are not rational ([18]).
- A very general $X_4 \subset \mathbb{P}^4$ ([11]), $X_6 \subset \mathbb{P}(1,1,1,1,3)$ ([3]), $X_4 \subset \mathbb{P}(1,1,1,1,2)$ ([23]) and $X_6 \subset \mathbb{P}(1,1,1,2,3)$ ([15]) are not stably rational.

We state the main theorem of this report.

Theorem 1.2 ([20, Theorem 1.2]). *Let $X_d \subset \mathbb{P}(a_0, \dots, a_4)$, $a_0 \le \cdots \le a_4$, be a very general Fano 3-fold weighted hypersurface of degree d. Then the following are equivalent:*

(1) *Either $d < 2a_4$ or $d = 2a_4 = 2a_3$.*
(2) *X is rational.*

Moreover if X is not a cubic 3-folds, then the above conditions are equivalent to the following:

(3) *X is stably rational.*

The implication (1) $\Rightarrow$ (2) can be proved easily (see Section 4) and the implication (2) $\Rightarrow$ (3) follows from the definition. The main part is to prove the implication (3) $\Rightarrow$ (1), or in other words, to prove stable non-rationality of a Fano weighted hypersurface X_d other than a cubic 3-fold which fails to satisfy the condition in (1).

As a consequence of Theorem 1.2, we have the following:

- A very general Fano 3-fold weighted hypersurface of index 1 is not stably rational.
- A very general Fano 3-fold weighted hypersurface of index 2 is not stably rational except possibly for cubic 3-folds.
- Among the 27 families of Fano 3-fold weighted hypersurfaces of index > 2, 20 families consist of rational varieties and a very general ember of he remaining 7 families is not stably rational.

See Table 1 for the families of Fano 3-fold weighted hypersurfaces of index > 1 and their (stable) rationality. As far as very general members are concerned, this settles stable rationality questions for Fano 3-fold weighted hypersurfaces except for cubic 3-folds.

In Section 2, we explain the arguments, specialization of universal CH_0-triviality, proving stable non-rationalities of varieties which are introduced by Voisin and amplified by Colliot-Thélìne, Pirutka and Totaro. In Section 3 we explain the construction of (global) differential forms on cyclic covers in positive characteristic which is due to Kollár. Finally in Section 4 we explain the sketch of proof of Theorem 1.2.

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2. Specialization of universal CH_0-triviality

We explain that the property "universal CH_0-triviality" can be used to detect stable rationality and it can be specialized in some sense. We refer readers to [23] and [11] for details and to [2] and [21] for surveys on this subject.

In this section we assume for simplicity that the base field k is an algebraically closed field unless otherwise specified.

TABLE 1. (Stable) Rationality of Fano 3-folds of index > 1: In the column "Rat", the signs $+$, $-$ and $--$ mean that a very general member is rational, not rational and not stably rational, respectively. The column "Ind" indicates the index of members of the family.

No.	$X_d \subset \mathbb{P}(a_0, \dots, a_4)$	Rat	Ind	No.	$X_d \subset \mathbb{P}(a_0, \dots, a_4)$	Rat	Ind
96	$X_3 \subset \mathbb{P}(1,1,1,1,1)$	$-$	2	113	$X_4 \subset \mathbb{P}(1,1,2,2,3)$	$+$	5
97	$X_4 \subset \mathbb{P}(1,1,1,1,2)$	$--$	2	114	$X_6 \subset \mathbb{P}(1,1,2,3,4)$	$+$	5
98	$X_6 \subset \mathbb{P}(1,1,1,2,3)$	$--$	2	115	$X_6 \subset \mathbb{P}(1,2,2,3,3)$	$+$	5
99	$X_{10} \subset \mathbb{P}(1,1,2,3,5)$	$--$	2	116	$X_{10} \subset \mathbb{P}(1,2,3,4,5)$	$--$	5
100	$X_{18} \subset \mathbb{P}(1,2,3,5,9)$	$--$	2	117	$X_{15} \subset \mathbb{P}(1,3,4,5,7)$	$--$	5
101	$X_{22} \subset \mathbb{P}(1,2,3,7,11)$	$--$	2	118	$X_6 \subset \mathbb{P}(1,1,2,3,5)$	$+$	6
102	$X_{26} \subset \mathbb{P}(1,2,5,7,13)$	$--$	2	119	$X_6 \subset \mathbb{P}(1,2,2,3,5)$	$+$	7
103	$X_{38} \subset \mathbb{P}(2,3,5,11,19)$	$--$	2	120	$X_6 \subset \mathbb{P}(1,2,3,3,4)$	$+$	7
104	$X_2 \subset \mathbb{P}(1,1,1,1,1)$	$+$	3	121	$X_8 \subset \mathbb{P}(1,2,3,4,5)$	$+$	7
105	$X_3 \subset \mathbb{P}(1,1,1,1,2)$	$+$	3	122	$X_{14} \subset \mathbb{P}(2,3,4,5,7)$	$--$	7
106	$X_4 \subset \mathbb{P}(1,1,1,2,2)$	$+$	3	123	$X_6 \subset \mathbb{P}(1,2,3,3,5)$	$+$	8
107	$X_6 \subset \mathbb{P}(1,1,2,2,3)$	$--$	3	124	$X_{10} \subset \mathbb{P}(1,2,3,5,7)$	$+$	8
108	$X_{12} \subset \mathbb{P}(1,2,3,4,5)$	$--$	3	125	$X_{12} \subset \mathbb{P}(1,3,4,5,7)$	$+$	8
109	$X_{15} \subset \mathbb{P}(1,2,3,5,7)$	$--$	3	126	$X_6 \subset \mathbb{P}(1,2,3,4,5)$	$+$	9
110	$X_{21} \subset \mathbb{P}(1,3,5,7,8)$	$--$	3	127	$X_{12} \subset \mathbb{P}(2,3,4,5,7)$	$+$	9
111	$X_4 \subset \mathbb{P}(1,1,1,2,3)$	$+$	4	128	$X_{12} \subset \mathbb{P}(1,4,5,6,7)$	$+$	11
112	$X_6 \subset \mathbb{P}(1,1,2,3,3)$	$+$	4	129	$X_{10} \subset \mathbb{P}(2,3,4,5,7)$	$+$	11
				130	$X_{12} \subset \mathbb{P}(3,4,5,6,7)$	$+$	13

Definition 2.1. For a variety Y defined over a field F, we denote by $\mathrm{CH}_0(Y) = Z_0(Y)/\sim_{\mathrm{rat}}$ the *Chow group of* 0*-cycles* on Y, which is the free $\mathbb{Z}$-module generated by 0-dimensional integral subschemes of Y modulo rational equivalence.

For a projective variety X over k, we have the degree map $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$, which is nothing but the push-forward of 0-cycles via $X \to \operatorname{Spec} k$. If X is a nonsingular projective curve (over k), then the kernel of the degree map is the Jacobian variety $J(X)$. Although this is a trivial observation, we see that a smooth projective curve X is rational if and only if the degree map is an isomorphism. This is no more true in higher dimensions: on one hand, it is known that, for a rationally connected smooth projective variety over k, the degree map $\mathrm{CH}_0(X) \to \mathbb{Z}$ is an isomorphism, and, on the other hand, there are a lot of non-rational rationally connected varieties in dimension at least 3. However the implication

$$X \text{ is rational} \implies \text{the degree map is an isomorphism}$$

remains true in arbitrary dimension (below we will discuss a more general statement). To capture this phenomenon more precisely, we give a few more definitions.

Definition 2.2. Let X be a projective variety over k.

(1) X is *universally* CH_0*-trivial* if for any field F containing k the degree map $\mathrm{CH}_0(X_F) \to \mathbb{Z}$ is an isomorphism.
(2) A projective morphism $\varphi\colon Y \to X$ defined over k is *universally* CH_0*-trivial* if for any field extension $F \supset k$ the push-forward map $\varphi_*\colon \mathrm{CH}_0(Y_F) \to \mathrm{CH}_0(X_F)$ is an isomorphism.

The following shows that the property "universal CH_0-triviality" is a stable birational invariant.

Lemma 2.3 ([10, Proposition 6.3], [14, Example 16.1.12]). *Let X and Y be smooth projective varieties over k. Assume that $X \times \mathbb{P}^m_k$ is birational to $Y \times \mathbb{P}^n_k$ for some $m, n \geq 0$. Then X is universally CH_0-trivial if and only if Y is universally CH_0-trivial.*

Note that in the above lemma the case when $Y = \operatorname{Spec} k$ is allowed. Note also that $\mathbb{P}^n_k$ is clearly universally CH_0-trivial. Combining these, we obtain:

Lemma 2.4. *Let X be a smooth projective variety over k. If X is stably rational, then it is universally CH_0-trivial.*

Thus in order to show that a given projective variety X (over k) is not stably rational, it is enough to show that X admits a resolution $\tilde{X} \to X$ such that $\tilde{X}$ is not universally CH_0-trivial. However it is quite difficult to conclude that a given variety is not universally CH_0-trivial.

Voisin brought a breakthrough idea in [23] and we explain her argument briefly: Let X be a very general quartic double solid defined over $\mathbb{C}$, i.e. a double cover of $\mathbb{P}^3_{\mathbb{C}}$ branched along a very general divisor of degree 4. Then consider a degeneration of X to a so-called Artin-Mumford nodal double solid Y, i.e. a double cover Y of $\mathbb{P}^3_{\mathbb{C}}$ branched along a divisor of degree 4 possessing 10 nodes in special position. It is proved in [1] that the blowup $\tilde{Y} \to Y$ at the 10 nodes gives a resolution of singularities and the torsion part of $H^3(X, \mathbb{Z})$ is non-zero (this shows that $\tilde{Y}$ is not stably rational). Voisin then shows that (i) the non-vanishing of the torsion part of $H^3(X, \mathbb{Z})$ implies that $\tilde{Y}$ is not universally CH_0-trivial, and that (ii) non-universal CH_0-triviality of $\tilde{Y}$ implies non-universal CH_0-triviality of X. These show that X is not universally CH_0-trivial, hence X is not stably rational. The argument (ii) is given in a more general setting (see [23, Theorem 2.1]) and it is frequently referred to as "specialization of universal CH_0-triviality".

We apply a version of specialization argument due to Colliot-Thélène and Pirutka:

Theorem 2.5 ([11, Théorème 1.14]). *Let A be a disctrete valuation ring with fraction field K and residue field k, with k algebraically closed. Let $\mathcal{X}$ be a flat proper scheme over A with geometrically integral fibers. Let X be the generic fiber $\mathcal{X} \times_A K$ and Y the special fiber fiber $\mathcal{X} \times_A k$. Assume that Y admits a universally CH_0-trivial resolution $\tilde{Y} \to Y$ of singularities. Let $\overline{K}$ be an algebraic closure of K and assume that the geometric generic fiber $X_{\overline{K}}$ admits a resolution $\tilde{X} \to X_{\overline{K}}$. If $\tilde{X}$ is universally CH_0-trivial, then so is $\tilde{Y}$.*

This in particular enables us to consider reduction of a variety defined over $\mathbb{C}$ into a positive characteristic. The reduction modulo p specialization in combination with Kollár's arguments on cyclic coverings which will be explained in the next subsection, was firstly applied by Totaro [22] to hypersurfaces. The following result is crucial in concluding universal CH_0-non-triviality.

Lemma 2.6 ([22, Lemma 2.2]). *Let X be a smooth projective variety over k. If $H^0(X, \Omega^i_X) \neq 0$ for some $i > 0$, then X is not universally CH_0-trivial.*

Recall that the characteristic of k is not assumed to be 0. It should be stressed that, for a rationally connected smooth projective variety X over $\mathbb{C}$, $H^0(X, \Omega^i_X) = 0$

for any $i > 0$, hence considering reduction into a positive characteristic is an essential feature of the proof.

3. Differential forms in positive characteristic

The aim of this section is to explain Kollár's arguments [17, Section V.5] on cyclic coverings in positive characteristic. We work over an algebraically closed field $\Bbbk$ of characteristic $p > 0$.

Let Z be a smooth (quasi-projective) variety of dimension n over $\Bbbk$, $\mathcal{L}$ an invertible sheaf on Z, m a positive integer and $s \in H^0(Z, \mathcal{L}^m)$. Let

$$W = \operatorname{Spec}\left(\oplus_{i \geq 0} \mathcal{L}^{-i}\right) \to Z$$

be the total space of $\mathcal{L}$ and $y \in H^0(W, \pi^* \mathcal{L})$ the canonical section. Then y^m and s (or more precisely $\pi^* s$) can be viewed as sections of $\pi^* \mathcal{L}^m$ and we denote by X the closed subscheme defined by $y^m - s = 0$ in W. In the following we assume that the branched divisor $(s = 0) \subset Z$ is reduced.

By restricting (to X) the exact sequence on Ω induced by the projection $W \to Z$, we obtain an exact sequence

$$0 \to \pi^* \Omega^1_X \to \Omega^1_W|_X \to \pi^* \mathcal{L}^{-1} \to 0. \tag{1}$$

In view of $\mathcal{O}_W(-X) \cong \pi^* \mathcal{L}^{-m}$, the closed immersion $X \hookrightarrow W$ induces an exact sequence

$$\pi^* \mathcal{L}^{-m} \xrightarrow{d_X} \Omega^1_W|_X \to \Omega^1_X \to 0. \tag{2}$$

Lemma 3.1 (cf. [17, Lemma V.5.3])**.** *If $p \mid m$, then the image of $d_X \colon \pi^* \mathcal{L}^{-m} \to \Omega^1_W|_X$ is contained in $\pi^* \Omega^1_X$.*

Proof. We may assume that $Z = \operatorname{Spec} A$ is affine. Then, by regarding s as an element of A, we have $X = \operatorname{Spec} A[y]/(y^m - s) \hookrightarrow W = \operatorname{Spec} A[y]$. In this case the image of d_X is generated by

$$d(y^m - s) = my^{m-1} dy - ds = -ds \in \pi^* \Omega^1_Z$$

since $p \mid m$. $\square$

In the following we assume that $p \mid m$. By Lemma 3.1, we have a homomorphism $\delta_X \colon \pi^* \mathcal{L}^{-m} \to \pi^* \Omega^1_X$ such that the composition of δ_X and $\pi^* \Omega^1_X \to \Omega^1_W|_X$ coincides with d_X. Combining the exact sequences (1) and (2), we obtain a diagram

$$\begin{tikzcd}
 & \pi^* \mathcal{L}^{-m} \arrow[r, "d_X"] \arrow[d, "\delta_X"'] & \Omega^1_W|_X \arrow[r] \arrow[d, equal] & \Omega^1_X \arrow[r] & 0 \\
0 \arrow[r] & \pi^* \Omega^1_X \arrow[r] & \Omega^1_W|_X \arrow[r] & \pi^* \mathcal{L}^{-m} \arrow[r] & 0
\end{tikzcd}$$

which induces an exact sequence

$$0 \to \operatorname{Coker}(\delta_X) \to \Omega^1_X \to \pi^* \mathcal{L}^{-m} \to 0. \tag{3}$$

Note that the sheaf $\operatorname{Coker}(\delta_X)$ has generic rank $n - 1$, where we recall that $n = \dim Z = \dim X$.

Definition 3.2. We define $\mathcal{M}$ to be the double dual of the sheaf

$$\bigwedge^{n-1} \operatorname{Coker}(\delta_X).$$

By the exact sequence (3), we obtain an inclusion $\mathcal{M} \hookrightarrow (\Omega^1_X)^{\vee\vee}$.

Lemma 3.3 (cf. [17, Lemma V.5.9])**.** *Under the above setting, the sheaf $\mathcal{M}$ is invertible and it is isomorphic to $\pi^*(\omega_Z \otimes \mathcal{L}^m)$.*

The variety X is almost always singular. In the following we explain that X admits a universally CH_0-trivial resolution $\varphi\colon \tilde{X} \to X$ of singularities of X such that $\varphi^*\mathcal{M} \hookrightarrow \Omega^{n-1}_{\tilde{X}}$ when the singularities of X are mild. In order to understand the singularities of X, we give the definition of critical points (of s).

Definition 3.4. Let $\mathsf{q} \in Z$ be a point and $x_1, \dots, x_n$ local coordinates of Z at q. Take a local generator μ of $\mathcal{L}$ at q and write $s = f\mu^m$ locally around q, where $f = f(x_1, \dots, x_n)$. We write $f = f_0 + f_1 + f_2 + \cdots$, where f_i is homogeneous of degree i (Here degree is the usual degree, $\deg x_i = 1$).

We say that s has a *critical point* at q if $f_1 = 0$. Suppose that s has a critical point at q. We say that s has an *admissible critical point* at q if, in a suitable choice of local coordinates $x_1, \dots, x_n$,

$$f_2 = \begin{cases} x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n, & \text{if } n \text{ is even,} \\ x_1^2 + x_2x_3 + \cdots + x_{n-1}x_n, & \text{if } n \text{ is odd and either } p \neq 2 \text{ or } p = 2 \text{ and } 4 \mid m, \\ \beta x_1^2 + x_2x_3 + \cdots + x_{n-1}x_n, & \text{if } n \text{ is odd, } p = 2 \text{ and } 4 \nmid m, \end{cases}$$

where $\beta \in \Bbbk$, and in case n is odd, $p = 2$ and $4 \nmid m$ the coefficient of x_1^3 in f_3 is non-zero.

In the above definition, the quadratic part f_2, up to a multiple by a non-zero constant, depends only on the choice of local coordinates and does not depend on the choice of the local generator μ except when $p = 2$ and $4 \nmid m$. See Remark 3.5 for the subtleties in the case $p = 2$ and $4 \nmid m$.

Remark 3.5. Suppose that $p = 2$ and $4 \nmid m$. Let μ and μ' be both local generators of $\mathcal{L}$ and $s = f\mu^m = f'\mu'^m$ be two local descriptions. Then $\mu = u\mu'$ where $u = u(x_1, \dots, x_n)$ does not vanish at q. Write $u = \gamma + h(x_1, \dots, x_n)$, where $0 \neq \gamma \in \Bbbk$ and $h \in (x_1, \dots, x_n)$, and also write $m = 2k$, where k is odd. Then we have

$$f' = u^{2k} f = (\gamma^m + k(2k-1)\gamma^{m-2}h^2 + \cdots +)f,$$

so that, for the quadratic part, we have $f'_2 = \gamma^m f_2 + k(2k-1)\gamma^{m-2} f_0 h_1^2$. Hence the quadratic part can differ by a square of a linear form in $x_1, \dots, x_n$. This does not cause any trouble when n is even: if we can choose coordinates so that $f_2 = x_1x_2 + \cdots + x_{n-1}x_n$, then we can choose another coordinates $x'_1, \dots, x'_n$ so that $f'_2 = x'_1x'_2 + \cdots + x'_{n-1}x'_n$. In the case when n is odd, even if we can choose coordinates so that $f_2 = x_1^2 + x_2x_3 + \cdots + x_{n-1}x_n$, we can always kill the term x_1^2 and have $f'_2 = x_2x_3 + \cdots + x_{n-1}x_n$ by a suitable choice of μ'. This is why we do not (cannot) assume $\beta \neq 0$ in the difinition.

Let $\mathsf{p} \in X$ be a point and $\mathsf{q} = \pi(\mathsf{p}) \in Z$. Let $x_1, \dots, x_n$ be local coordinates of Z at q. Then, in a neighborhood of $\pi^{-1}(\mathsf{q})$, X is defined by the equation $y^m - f(x_1, \dots, x_n) = 0$, where $s = f\mu^m$ be the local description of s. Thus, since $p \mid m$, it is easy to deduce that the singular locus of X is the inverse image of the set of critical points of s.

Definition 3.6. We say that X has *an admissible singular point* at p if s has an *admissible critical point* at $\pi(\mathsf{p})$.

The following result is proved by [22], [12], [7] in some special cases, and by [19] in general.

Proposition 3.7 ([19, Proposition 4.1])**.** *If X admits only admissible singularities, then it admits a universally CH_0-trivial resolution $\varphi \colon \tilde{X} \to X$ of singularities such that $\varphi^* \mathcal{M} \hookrightarrow \Omega^{n-1}_{\tilde{X}}$.*

4. Proof of Theorem 1.2

The implication (1) $\Rightarrow$ (2) of Theorem 1.2 can be proved easily as follows.

Proof of (1) $\Rightarrow$ (2) in Theorem 1.2. Let $X = X_d \subset \mathbb{P} := \mathbb{P}(a_0, \dots, a_4)$, $a_0 \le \cdots \le a_4$, be a Fano 3-fold weighted hypersurface satisfying the condition in (1), i.e. either $d < 2a_4$ or $d = 2a_4 = 2a_3$. Let $x_0, \dots, x_4$ be the homogeneous coordinates of $\mathbb{P}$ of degree $\deg x_i = a_i$ and $F = F(x_0, \dots, x_4)$ the defining polynomial of X. By the assumption, we may assume that $F = x_4 f + g$ for some non-zero $f, g \in \mathbb{C}[x_0, \dots, x_3]$ after replacing coordinates. This is clearly true when $d < 2a_4$. When $d = 2a_4 = 2a_3$, we have a priori $F = \alpha x_4^2 + \beta x_4 x_3 + \gamma x_3^2 + \cdots$ but by replacing coordinates we can eliminate the term x_4^2 and we can assume $F = x_4 f + g$ for some $f, g \in \mathbb{C}[x_0, \dots, x_3]$. Now it is easy to see that the projection $X \dashrightarrow \mathbb{P}(a_0, a_1, a_2, a_3)$ is birational and therefore X is rational. $\square$

For the proof of the implication (3) $\Rightarrow$ (1) of Theorem 1.2, we pick up one family:

Theorem 4.1. *A very general $X_{10} \subset \mathbb{P}(1,2,3,4,5)$ is not stably rational.*

The rest of this section is devoted to the explanation of the proof of Theorem 4.1. The proof for the other families are done in a similar manner although sometimes further involved arguments are required.

Step 0: Setup. The ambient weighted projective space can be defined over an arbitrary field (or more generally a ring) k. To be more specific, we set

$$\mathbb{P}_k(1,2,3,4,5) = \operatorname{Proj} k[x,y,z,t,w],$$

where degrees of x, y, z, t, w are given as $1, 2, 3, 4, 5$, respectively.

We denote by $\Bbbk$ an uncountable algebraically closed field of characteristic 2 and let $g \in \Bbbk[x,y,z,t]$ be a polynomials of degree 10 with the property that the coefficients of the monomials of degree 10 in variables x, y, z, t in g are algebraically independent over the prime field of $\Bbbk$. Then we define

$$Y = (w^2 - g = 0) \subset \mathbb{P}_{\Bbbk}(1,2,3,4,5).$$

In Step 1 below, we work with Y, a variety over a field of characteristic 2, and then in Step 2, we lift Y to a very general member X_{10} defined over $\mathbb{C}$.

Step 1. We explain that Y admits a universally CH_0-trivial resolution $\varphi \colon \tilde{Y} \to Y$ such that $H^0(\tilde{Y}, \Omega^2_{\tilde{Y}}) \ne 0$.

Let $\pi \colon Y \to \mathbb{P}_{\Bbbk}(1,2,3,4)$ be the natural projection, which is a double cover. Let Z be the nonsingular locus of $\mathbb{P}_{\Bbbk}(1,2,3,4)$, or explicitly,

$$Z = \mathbb{P}_{\Bbbk}(1,2,3,4) \setminus ((x = z = 0) \cup \{(0{:}0{:}1{:}0)\}).$$

We set $Y^\circ = \pi^{-1}(Z)$ which is an open subset of Y and we see that the codimension of $Y \setminus Y^\circ$ is at least 2. The restriction $\pi^\circ = \pi|_{Y^\circ} \colon Y^\circ \to Z$ is the double cover explained in Section 3 for $p = m = 2$, $\mathcal{L} = \mathcal{O}_Z(5)$ and $s = g \in H^0(Z, \mathcal{L}^2)$. Let $\mathcal{M}^\circ$

be the invertible sheaf on Y° (defined in Definition 3.2) associated to the double cover π°. By Lemma 3.3 and a straightforward computation, we have

$$\mathcal{M}^\circ = \pi^{\circ *}(\omega_Z \otimes \mathcal{L}^2) \cong \pi^{\circ *}\mathcal{O}_Z \cong \mathcal{O}_{Y^\circ}.$$

We define $\mathcal{M}$ to be the push-forward of $\mathcal{M}^\circ$ via the open immersion $Y^\circ \hookrightarrow Y$, which is an invertible sheaf $\mathcal{M} \cong \mathcal{O}_Y$.

For the singularities of Y, we prove the following.

Lemma 4.2. (1) *Y° has only admissible singular points.*

(2) *Y has only cyclic quotient singular points along $Y \setminus Y^\circ$. More specifically, the number and the type of singularities of Y along $Y \setminus Y^\circ$ are the following:*

$$1 \times \frac{1}{2}(1,1,1),\ 1 \times \frac{1}{3}(1,1,2),\ 1 \times \frac{1}{4}(1,1,3).$$

Our aim is to construct a "good resolution" which is already discussed in Proposition 3.7 for admissible singularities. For cyclic quotient singularities, we can prove a more general result.

Lemma 4.3 (cf. [20, Lemma 3.7])**.** *Let $\mathsf{p} \in V$ be a germ of an isolated toric singularity and $\mathcal{N}$ an invertible sheaf on V such that $\mathcal{N} \hookrightarrow \Omega^i_V$ for some $i > 0$. Let $\varphi \colon \tilde{V} \to V$ be any toric resolution with simple normal crossing exceptional divisor. Then φ is universally CH_0-trivial and $\varphi^*\mathcal{N} \hookrightarrow \Omega^i_{\tilde{V}}$.*

As a summary of Step 1, we conclude that Y admits a universally CH_0-trivial resolution $\varphi \colon \tilde{Y} \to Y$ such that $\mathcal{O}_{\tilde{Y}} \cong \varphi^*\mathcal{M} \hookrightarrow \Omega^2_{\tilde{Y}}$. In particular $H^0(\tilde{Y}, \Omega^2_{\tilde{Y}}) \neq 0$.

Step 2. We lift Y to a very general weighted hypersurface in $\mathbb{P}_{\mathbb{C}}(1,2,3,4,5)$ of degree 10.

Let $F \in \mathbb{C}[x,y,z,t,w]$ be a defining polynomial of X. Since $w^2 \in F$ (otherwise X has a non-cyclic quotient singularity at $(0:0:0:0:1)$), we can assume that $F = w^2 + f(x,y,z,t)$, where $f \in \mathbb{C}[x,y,z,t]$. By a very generality, we require that the coefficients of the degree 10 monomials in f are algebraically independent over $\mathbb{Q}$.

Let $A = W(\Bbbk)$ be the ring of Witt vectors over $\Bbbk$, which is a complete DVR whose residue field is $\Bbbk$ and whose fraction field K is of characteristic 0. Take any lift $g_A \in A[x,y,z,t]$ of g via the surjection $A[x,y,z,t] \to \Bbbk[x,y,z,t]$ and we set

$$\mathcal{X} = (w^2 - g_A = 0) \subset \mathbb{P}_A(1,2,3,4,5),$$

which is a flat prpjective scheme over $\operatorname{Spec} A$. The coefficients of g_A, viewed as elements of K, are algebraically independent over $\mathbb{Q}$. Thus we can choose an embedding $K \hookrightarrow \overline{K} \hookrightarrow \mathbb{C}$, where $\overline{K}$ denote an algebraic closure of K, so that the base change $(\mathcal{X}_{\overline{K}})_{\mathbb{C}}$ of the geometric generic fiber $\mathcal{X}_{\overline{K}}$ via $\overline{K} \hookrightarrow \mathbb{C}$ coincides with X. The variety $\mathcal{X}_{\overline{K}}$ is a very general weighted hypersurface of degree 10 in $\mathbb{P}_{\overline{K}}(1,2,3,4,5)$ so that it admits only (terminal) cyclic quotient singularities (which are the ones described in (2) of Lemma 4.2). It admits a resolution $\tilde{\mathcal{X}}_{\overline{K}} \to \mathcal{X}_{\overline{K}}$ of singularities (which can be obtained as successive weighted blowups) and the base change via $\overline{K} \hookrightarrow \mathbb{C}$ gives a resolution $\tilde{X} \to X$ of X.

As a summary of Step 1 and Step 2, we obtain the diagram:

$$
\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] & \tilde{X}_{\overline{K}} \arrow[d] & & \tilde{Y} \arrow[d, "\varphi"'] \\
X \arrow[r] \arrow[d] & \mathcal{X}_{\overline{K}} \arrow[d] \arrow[dr] & & Y \arrow[dl] \arrow[d] \\
\operatorname{Spec}\mathbb{C} \arrow[r] & \operatorname{Spec}\overline{K} \arrow[dr] & \mathcal{X} \arrow[d] & \operatorname{Spec}\Bbbk \arrow[dl] \\
& & \operatorname{Spec} A &
\end{tikzcd} \tag{4}
$$

Step 3: Conclusion. By Lemma 2.6 and Step 1, the variety $\tilde{Y}$ is not universally CH_0-trivial. We can apply the specialization argument 2.5 to $\mathcal{X} \to \operatorname{Spec} A$ given in Step 2 and conclude that the resolution $\tilde{\mathcal{X}}_{\overline{K}}$ of the geometric generic fiber is not universally CH_0-trivial. This implies that $\tilde{X} = (\tilde{\mathcal{X}}_{\overline{K}})_{\mathbb{C}}$ is not universally CH_0-trivial. Therefore, by Lemma 2.4, $\tilde{X}$, hence X, is not stably rational.

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