ON GENERIC VANISHING FOR PLURICANONICAL BUNDLES

TAKAHIRO SHIBATA

1. Introduction

Thoughout this article, we work over the complex number field.

Generic vanishing theory is the study of the family of the cohomologies

$$\{H^i(X, \mathcal{F} \otimes \mathcal{L}) | \mathcal{L} \in \mathrm{Pic}^0(X)\}$$

of a fixed coherent sheaf $\mathcal{F}$. Let $f : X \to A$ be a morphism from a smooth projective variety X to an abelian variety A and $\mathcal{F}$ a coherent sheaf on X. We take a closed subset

$$V^i_k(\mathcal{F}, f) = \{\alpha \in \mathrm{Pic}^0(A) | h^i(X, \mathcal{F} \otimes f^*\alpha) \geq k\}$$

of $\mathrm{Pic}^0(A)$ for integers $i, k \geq 0$. Our aim is to investigate the dimensions and structures of $V^i_k(\mathcal{F}, f)$. If f is the Albanese morphism, $V^i_k(\mathcal{F}, f)$ is denoted by $V^i_k(\mathcal{F})$. Moreover we also denote $V^i_1(\mathcal{F}, f)$ by $V^i(\mathcal{F}, f)$.

For $\mathcal{F} = \omega_X$, we have the following good results.

Theorem 1.1 (The generic vanishing theorem, Green–Lazarsfeld [GrLa]). *Let $f : X \to A$ be a morphism from a smooth projective variety X to an abelian variety A. Then* $\operatorname{codim} V^i(\omega_X, f) \geq i - (\dim X - \dim f(X))$ *for every $i \geq 1$.*

Definition 1.2 (GV-sheaf). Let A be an abelian variety and $\mathcal{F}$ a coherent sheaf on A. $\mathcal{F}$ is called a *generic vanishing sheaf* (*GV-sheaf*, for short) if $\operatorname{codim} V^i(\mathcal{F}) \geq i$ for every $i \geq 1$.

Theorem 1.3 (Hacon [Hac]). *Let $f : X \to A$ be a morphism from a smooth projective variety X to an abelian variety A. Then $R^j f_*\omega_X$ is a GV-sheaf for every $j \geq 0$.*

Remark 1.4. Theorem 1.3 implies Thm 1.1 by using Kollár's theorem on the higher direct images of ω_X.

Definition 1.5 (Torsion subvariety). Let A be an abelian variety and T a closed subvariety of A. T is called a *torsion subvariety* if T is a translate of an abelian subvariety by a torsion point.

Theorem 1.6 (Simpson [Sim]). *Let $f : X \to A$ be a morphism from a smooth projective variety X to an abelian variety A. Then $V^j_k(\omega_X, f)$ is a finite union of torsion subvarieties of* $\mathrm{Pic}^0(A)$ *for every $j \geq 0$ and $k \geq 1$.*

We investigate whether these theorems can be generalized to log pluricanonical bundles $\mathcal{O}_X(m(K_X+\Delta))$ of a log canonical pair (X, Δ).

2. Main results

We fix the following notation and convention.

- X is a smooth projective variety, A is an abelian variety, and $f : X \to A$ is a morphism.
- Δ is a boundary $\mathbb{Q}$-divisor on X with simple normal crossing support, that is, a $\mathbb{Q}$-divisor on X whose coefficients are in $[0, 1]$ and $\mathrm{Supp}\Delta$ is a simple normal crossing divisor.
- When considering a $\mathbb{Q}$-divisor $m(K_X + \Delta)$ for some positive integer m, we always assume that there exists a Cartier divisor D on X such that $D \sim_{\mathbb{Q}} m(K_X + \Delta)$. So we can consider $V^j_k(m(K_X + \Delta), f)$ and $R^jf_*\mathcal{O}_X(m(K_X + \Delta))$.
- H^j_m denotes the statement that $R^jf_*\mathcal{O}_X(m(K_X+\Delta))$ is a GV-sheaf on A, which is a generalization of Theorem 1.3.
- S^j_m denotes the statement that $V^j_k(m(K_X + \Delta), f)$ is a finite union of torsion subvarieties for every $k \geq 1$, which is a generalization of Theorem 1.6.

Theorem 2.1. *In the above notation, the following hold.*

(i) *H^j_1 holds for every $j \geq 0$.*
(ii) *H^0_m holds for every $m \geq 1$.*
(iii) *If $j \geq 1$ and $m \geq 2$, then H^j_m does not hold in general.*
(iv) *S^j_1 holds for every $j \geq 0$.*
(v) *S^0_m holds for every $m \geq 1$.*
(vi) *If $j \geq 1$ and $m \geq 2$, then S^j_m does not hold in general.*

Remark 2.2.

- (ii) is a result of Popa–Schnell [PoSc, Theorem 1.10]. (i) is immediately deduced by their argument, although it is not explicitly stated in their paper. So (i) and (ii) are not new results.
- The KLT case of (iv) was proved by Clemens–Hacon [ClHa, Theorem 8.3].
- The $\Delta = 0$ case of Theorem 2.1 (v) was proved by Chen–Hacon [ChHa, Theorem 3.2].

- (ii) and (v) holds for general projective log canonical pairs, which are easily reduced to the log smooth case by taking a log resolution.
- For (iii) and (vi), we will construct counterexamples. In fact, those counterexamples can be taken as $\Delta = 0$ (just a pluricanonical bundle, not a log pluricanonical bundle).

We can generalize the generic vanishing theorem for log canonical pairs by using Theorem 2.1 (i).

Theorem 2.3. *Let X be a smooth projective variety, Δ a boundary $\mathbb{Q}$-divisor on X with simple normal crossing support, $f : X \to A$ a morphism to an abelian variety, and D a Cartier divisor on X such that $D \sim_{\mathbb{Q}} K_X + \Delta$. Set $l = \max\{\dim V - \dim f(V) \mid V = X \text{ or } V \text{ is a log canonical center of } (X, \Delta)\}$. Then*

$$\operatorname{codim} V^i(D, f) \geq i - l$$

for any i.

Proof. Set $S = \lfloor \Delta \rfloor$ and let Δ_i be an irreducible component of S. Consider the exact sequence $\cdots \to R^j f_* \mathcal{O}_X(D - \Delta_i) \to R^j f_* \mathcal{O}_X(D) \to R^j f_* \mathcal{O}_{\Delta_i}(D|_{\Delta_i}) \to \cdots$. Then it follows that $R^j f_* \mathcal{O}_X(D) = 0$ for $j > l$ by induction of both the dimension of X and the number of irreducible components of S.

Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(A, R^q f_* \mathcal{O}_X(D) \otimes \xi) \Rightarrow H^{p+q}(X, \mathcal{O}_X(D) \otimes f^* \xi),$$

where $\xi \in \operatorname{Pic}^0(A)$. Then it follows by the spectral sequence that

$$V^i(D, f) \subset \bigcup_{q=0}^{l} V^{i-q}(R^q f_* \mathcal{O}_X(D)).$$

Furthermore, $R^q f_* \mathcal{O}_X(D)$ are GV-sheaves on A for all q by Theorem 2.1 (i), so

$$\operatorname{codim} V^{i-q}(R^q f_* \mathcal{O}_X(D)) \geq i - q$$

for $0 \leq i \leq l$. Hence $\operatorname{codim} V^i(D, f) \geq i - l$. □

Proof of Theorem 2.1 (iii). We will construct an irregular smooth projective variety of dimension ≥ 2 with big anti-canonical bundle and show that such a variety does not satisfy H^j_m for some $j \geq 1$ and $m \geq 2$.

Let A be an abelian variety. We take an ample line bundle L on A and define a vector bundle E as the direct sum of L^{-1} and $\mathcal{O}_A$. Let $\pi : X = \mathbb{P}_A(E) \to A$ be the projective bundle on A associated to E. Clearly the irregularity $q(X)$ of X is positive. The canonical bundle

ω_X is isomorphic to $\pi^*(\omega_A \otimes \det E) \otimes \mathcal{O}_X(-\operatorname{rank} E) = \pi^* L^{-1} \otimes \mathcal{O}_X(-2)$ (see [Laz, 7.3.A]).

We will see that ω_X^{-1} is big. Let ξ and l be the numerical classes of $\mathcal{O}_X(1)$ and L, respectively. Note that ξ is an effective class since $H^0(X, \mathcal{O}_X(1)) = H^0(A, E) \neq 0$. The numerical class of ω_X^{-1} is equal to

$$2\xi + \pi^* l = \frac{N-1}{N} 2\xi + \frac{1}{N} 2\xi + \pi^* l,$$

where N is a sufficiently large integer such that $(1/N)2\xi + \pi^* l$ is ample. So the numerical class of ω_X^{-1} is represented by the sum of an effective class and an ample class. Therefore ω_X^{-1} is big.

Let $f = \operatorname{alb}_X : X \to A$ be the Albanese morphism of X. Now we show that $R^j f_* \omega_X^{\otimes m}$ is not a GV-sheaf for some positive integers j and m.

Now we prove the following lemma.

Lemma 2.4. *Let X be a smooth projective variety of dimension n and D be a big Cartier divisor on X. Then*

$$V^0(mD) = \{\xi \in \operatorname{Pic}^0(X) \mid H^0(X, \mathcal{O}_X(mD + \xi)) \neq 0\} = \operatorname{Pic}^0(X)$$

for any sufficiently large and divisible m.

Proof. Since D is big, there exist a positive integer m_0, a very ample Cartier divisor H, and an effective Cartier divisor E such that $m_0 D \sim H + E$. For any positive integer m, we have

$$V^0(mm_0 D) = V^0(mH + mE) \supset V^0(mH).$$

We can take a positive integer m_1 satisfying that

$$H^i(X, \mathcal{O}_X(mH + \xi)) = 0$$

for every $\xi \in \operatorname{Pic}^0(X)$, $m \geq m_1$ and $i > 0$ (take m_1 such that $m_1 H - K_X$ is ample). According to the notion of the Castelnuovo–Mumford regularity, $mH + \xi$ is 0-regular for every $\xi \in \operatorname{Pic}^0(X)$ and $m \geq m_1 + n$, and so it is globally generated. In particular, $V^0(mH) = \operatorname{Pic}^0(X)$ for every $m \geq m_1 + n$. Therefore $V^0(mm_0 D) = \operatorname{Pic}^0(X)$ for every $m \geq m_1 + n$. $\square$

By the above lemma, we can take a positive integer m such that $V^n(\omega_X^{\otimes m}, f) = -V^0(\omega_X^{\otimes(1-m)}, f) = \operatorname{Pic}^0(A)$. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(A, R^q f_* \omega_X^{\otimes m} \otimes \alpha) \Rightarrow H^{p+q}(X, \omega_X^{\otimes m} \otimes f^* \alpha), \quad \alpha \in \operatorname{Pic}^0(A).$$

Then it follows that

$$\mathrm{Pic}^0(A) = V^n(\omega_X^{\otimes m}, f) \subset \bigcup_{i=0}^{n} V^i(R^{n-i} f_* \omega_X^{\otimes m}).$$

So $V^i(R^{n-i} f_* \omega_X^{\otimes m}) = \mathrm{Pic}^0(A)$ for some i. Note that $i > 0$ since $R^n f_* \omega_X^{\otimes m} = 0$. Hence it follows that $R^{n-i} f_* \omega_X^{\otimes m}$ is not a GV-sheaf. □

Theorem 2.1 (ii) is proved by the following vanishing theorem.

Theorem 2.5 (Popa–Schnell [PoSc, Theorem 1.7])**.** *Let (X, Δ) be a projective log canonical pair, Y a projective variety, $g : X \to Y$ a morphism, andL an ample and globally generated line bundle on Y. Take an integer $m \geq 1$. Then $H^i(Y, f_* \mathcal{O}_X(m(K_X + \Delta)) \otimes L^{\otimes l}) = 0$ for every $i > 0$ and $l \geq (m-1)(\dim Y + 1) + 1$.*

Conversely, by a similar argument, Theorem 2.1 (iii) implies that the above vanishing does not hold for higher cohomologies of pluricanonical bundles in general.

Corollary 2.6. *Let X be a smooth projective variety, Y a projective variety, $g : X \to Y$ a morphism, and L an ample and globally generated line bundle on Y. Take integers $j \geq 1$ and $m \geq 2$. Then we can not take a positive integer $N = N(j, m, \dim Y)$ depending only on j, m and $\dim Y$ such that $H^i(Y, f_* \omega_X^{\otimes m} \otimes L^{\otimes l}) = 0$ for every $i > 0$ and $l \geq N$.*

Sketch of proof of Theorem 2.1 (iv). (For a detailed proof, see [Shi, Theorem 3.5].) By assumption, there exists a Cartier divisor D such that $D \sim_{\mathbb{Q}} K_X + \Delta$. Set $C = D - (K_X + \lfloor \Delta \rfloor)$. Since $C \sim_{\mathbb{Q}} \{\Delta\}$, $NC \sim N\{\Delta\}$ for some positive integer N. Take the normalization of the cyclic cover $\mathrm{Spec} \bigoplus_{k=0}^{N-1} \mathcal{O}_X(-kC) \to X$. Then

$$\pi_* \mathcal{O}_Y = \bigoplus_{k=0}^{N-1} \mathcal{O}_X(-kC + \lfloor k\{\Delta\} \rfloor).$$

So $\pi_* \mathcal{O}_Y(-\pi^* \lfloor \Delta \rfloor)$ contains $\mathcal{O}_X(-C - \lfloor \Delta \rfloor) = \mathcal{O}_X(-(D - K_X))$ as a direct summand. Hence it follows that, if $V_k^j(-\pi^* \lfloor \Delta \rfloor, f \circ \pi)$ is a finite union of torsion subvarieties for every j and k, then $V_k^j(D, f)$ is also a finite union of torsion subvarieties for every j and k. Further, we can show that $(Y, \pi_* \lfloor \Delta \rfloor)$ is a log canonical pair.

Take a log resolution $\mu : Y' \to Y$ of $(Y, \pi^* \lfloor \Delta \rfloor)$. Set

$$\Delta_{Y'} = \mu^*(K_Y + \pi^* \lfloor \Delta \rfloor) - K_{Y'},$$

then

$$\Delta_{Y'}^{=1} = \mu_*^{-1}(\pi^* \lfloor \Delta \rfloor) + E$$

for some reduced μ-exceptional divisor E. Since $-K_{Y'/Y} = -(K_{Y'} - \mu^*K_Y)$ has no irreducible components with coefficient 1, every component of E is contained in $\mu^*\pi^*\lfloor\Delta\rfloor$. Since E is μ-exceptional, E is in fact contained in $\mu^*\pi^*\lfloor\Delta\rfloor - \mu_*^{-1}\pi^*\lfloor\Delta\rfloor$. So $F = \mu^*\pi^*\lfloor\Delta\rfloor - \mu_*^{-1}\pi^*\lfloor\Delta\rfloor - E$ is an effective and μ-exceptional divisor on Y'. By the Fujino–Kovács vanishing theorem (see [Kov] and [Fuj2]),

$$R^i\mu_*\mathcal{O}_{Y'}(-\Delta_{Y'}^{=1}) = 0$$

for $i > 0$. Therefore

$$\begin{aligned} R\mu_*\mathcal{O}_{Y'}(-\Delta_{Y'}^{=1}) &\cong \mu_*\mathcal{O}_{Y'}(-\Delta_{Y'}^{=1}) \\ &\cong \mu_*\mathcal{O}_{Y'}(-\mu_*^{-1}\pi^*\lfloor\Delta\rfloor - E) \\ &\cong \mu_*\mathcal{O}_{Y'}(-\mu^*\pi^*\lfloor\Delta\rfloor + F) \\ &\cong \mathcal{O}_Y(-\pi^*\lfloor\Delta\rfloor). \end{aligned}$$

Thus we have $V_k^j(-\Delta_{Y'}^{=1}, f\circ\pi\circ\mu) = V_k^j(-\pi^*\lfloor\Delta\rfloor, f\circ\pi)$. Moreover, $\Delta_{Y'}^{=1}$ is a simple normal crossing divisor on Y'. Then the proof is reduced to the case when Δ is a simple normal crossing divisor. This case holds due to Budur [Bud]. □

Sketch of proof of Theorem 2.1 (v). (For a detailed proof, see [Shi, Theorem 3.9].) Take any point $\xi \in V_k^0(m(K_X + \Delta), f)$. Then there exists $\xi_0 \in \mathrm{Pic}^0(A)$ such that $\xi = m\xi_0$. After replacing (X, Δ) by a suitable log resolution, we can take a Cartier divisor D_0 on X such that

- $D_0 \sim_{\mathbb{Q}} K_X + \Delta_0$, where Δ_0: a boundary $\mathbb{Q}$-divisor with SNC support,
- $\xi_0 \in V_k^0(D_0, f)$, and
- $V_k^0(D_0, f) + (m-1)\xi_0 \subset V_k^0(m(K_X + \Delta), f)$.

By Theorem 2.1 (iv), $V_k^0(D_0, f)$ is a finite union of torsion subvarieties. So there exist an abelian subvariety B of A and a torion point q of A such that $\xi_0 \in B + q \subset V_k^0(D_0, f)$. Then

$$\xi = \xi_0 + (m-1)\xi_0 \in B + q + (m-1)\xi_0 \subset V_k^0(m(K_X + \Delta), f).$$

Since $\xi_0 \in B + q$, $\xi_0 = b + q$ for some $b \in B$. So

$$B + q + (m-1)\xi_0 = B + q + (m-1)b + (m-1)q = B + mq.$$

Therefore

$$\xi \in B + mq \in V_k^0(m(K_X + \Delta), f).$$

So $V_k^0(m(K_X + \Delta), f)$ is a union of torsion subvarieties. Since the set of torsion subvarieties of A is countable, $V_k^0(m(K_X + \Delta), f)$ is in fact a finite union of torsion subvarieties. □

Corollary 2.7 (Campana–Koziarz–Păun [CKP], Kawamata [Kaw])**.** *Let (X, Δ) be a projective log canonical pair. Assume that $K_X + \Delta \equiv 0$. Then $K_X + \Delta \sim_{\mathbb{Q}} 0$.*

We give another proof of this theorem.

Proof. By taking a log resolution, we may assume that (X, Δ) is log smooth. Take $m > 0$ such that $\alpha = m(K_X + \Delta) \in \mathrm{Pic}^0(X)$. Then $h^0(X, m(K_X + \Delta) - \alpha) = h^0(X, \mathcal{O}_X) \neq 0$, so $-\alpha \in V^0(m(K_X + \Delta))$. Hence $V^0(m(K_X + \Delta))$ is non-empty. Theorem 2.1 (v) implies that there exists a torsion point $\beta \in V^0(m(K_X + \Delta))$. This means that $m(K_X + \Delta) \sim_{\mathbb{Q}} 0$. □

In addition, we give the following corollary, which is an implication of Iitaka's subadditivity conjecture.

Corollary 2.8. *Let (X, Δ) be a projective log canonical pair, A an abelian variety, $f : X \to A$ a surjective morphism with connected fibers, and F a sufficiently general fiber of f. Assume that $\kappa((K_X + \Delta)|_F) \geq 0$. Then $\kappa(K_X + \Delta) \geq 0$.*

Proof. Take $m > 0$ such that $m(K_X + \Delta)$ is Cartier and $h^0(m(K_X + \Delta)|_F) \neq 0$. Then $f_*\mathcal{O}_X(m(K_X + \Delta)) \neq 0$. Theorem 2.1 (ii) implies that $f_*\mathcal{O}_X(m(K_X + \Delta))$ is a GV-sheaf on A.

Now we need the fact that, for a GV-sheaf $\mathcal{F}$ on A, $\mathcal{F} \neq 0$ if and only if $V^0(\mathcal{F}) \neq \varnothing$. So $V^0(f_*\mathcal{O}_X(m(K_X + \Delta))) \neq \varnothing$. Then Theorem 2.1 (v) implies that there exists a torsion point $\alpha \in V^0(f_*\mathcal{O}_X(m(K_X + \Delta)))$. Take $N > 0$ such that $N\alpha = 0$. We compute

$$\begin{aligned} h^0(X, \mathcal{O}_X(Nm(K_X + \Delta))) &= h^0(X, \mathcal{O}_X(Nm(K_X + \Delta)) \otimes f^*\alpha^{\otimes N}) \\ &\geq h^0(X, \mathcal{O}_X(m(K_X + \Delta)) \otimes f^*\alpha) \\ &= h^0(A, f_*\mathcal{O}_X(m(K_X + \Delta)) \otimes \alpha) \\ &\neq 0. \end{aligned}$$

So $\kappa(K_X + \Delta) \geq 0$. □

Proof of Theorem 2.1 (vi). Let E be an elliptic curve, L a principal polarization on E (i.e. an ample line bundle on E with $h^0(L) = 1$), and A an abelian variety of dimension $g \geq 2$ including E as a proper abelian subvariety. Take a non-torsion point $a \in A$ and define a closed immersion $\iota : E \to A$ by $\iota(x) = x + a$. By definistion, $\iota(E) = E + a$. Let $\hat{A}$ be the dual abelian variety of A, and $R\Phi : D(A) \to D(\hat{A})$ and $R\Psi : D(\hat{A}) \to D(A)$ the Fourier–Mukai transforms.

Set $F = R\Phi\iota_*L \in D(\hat{A})$. Since $h^i(E, L \otimes \iota^*\alpha) = 0$ for $i > 0$ and $\alpha \in \mathrm{Pic}^0(A)$ by Kodaira vanishing, $R^i\Phi\iota_*L = 0$ for $i > 0$. So $F =$

$\Phi\iota_* L$. Furthermore $h^0(E, L\otimes\iota^*\alpha) = \chi(E, L\otimes\iota^*\alpha) = \chi(E, L) = 1$ for $\alpha\in\mathrm{Pic}^0(A)$, so F is in fact a line bundle on $\hat{A}$.

Take a vector bundle $V = F\oplus\mathcal{O}_{\hat{A}}$ on $\hat{A}$. Let $\pi : X = \mathbb{P}_{\hat{A}}(V)\to\hat{A}$ be the projective bundle over $\hat{A}$ associated to V. Then

$$\begin{aligned}\omega_X &= \pi^*(\omega_{\hat{A}}\otimes\det V)\otimes\mathcal{O}_{\hat{A}}(-\operatorname{rank} V)\\ &= \pi^* F\otimes\mathcal{O}_{\hat{A}}(-2)\end{aligned}$$

(cf. [Laz, 7.3.A]). Therefore

$$\begin{aligned}\pi_*(\omega_X^{-1}) &= F^{-1}\otimes\pi_*\mathcal{O}_{\hat{A}}(2)\\ &= F^{-1}\otimes S^2V\\ &= F^{-1}\otimes(F^2\oplus F\oplus\mathcal{O}_{\hat{A}})\\ &= F\oplus\mathcal{O}_{\hat{A}}\oplus F^{-1}.\end{aligned}$$

So

$$\begin{aligned}V^{g+1}(\omega_X^2) = V^{g+1}(\omega_X^2,\pi) &= -V^0(\omega_X^{-1},\pi) = -V^0(\pi_*\omega_X^{-1})\\ &= -V^0(F)\cup -V^0(\mathcal{O}_{\hat{A}})\cup -V^0(F^{-1})\\ &= V^g(F^{-1})\cup\{0\}\cup V^g(F)\end{aligned}$$

(note that π is the Albanese morphism of X, so we have the first equality).

First we calculate $V^g(F^{-1})$.

$$\begin{aligned}V^g(F^{-1}) &= \{a\in A | h^g(\hat{A}, F^{-1}\otimes L_a)\neq 0\}\\ &= \{a\in A | h^g(\hat{A}, (-1)^*(F^{-1}\otimes L_a))\neq 0\}\\ &= \{a\in A | h^g(\hat{A}, (-1)^*F^{-1}\otimes L_{-a})\neq 0\}\\ &= -V^g((-1)^*F^{-1})\\ &= -\mathrm{Supp} R^g\Psi(-1)^*F^{-1},\end{aligned}$$

where $(-1) : \hat{A}\to\hat{A}$ is the multiplication by -1. The last equation follows by the base change theorem. We write $R\Delta(\cdot) = R\mathscr{H}om(\cdot,\mathcal{O}_{\hat{A}})$. Then

$$\begin{aligned}R\Psi(-1)^*F^{-1} &= R\Psi(-1)^*R\Delta R\Phi\iota_* L\\ &= R\Psi(-1)^*(-1)^*R\Phi R\Delta\iota_* L[g] \quad (R\Phi R\Delta(\cdot) = (-1)^*R\Phi R\Delta(\cdot)[g])\\ &= (-1)^*R\Delta\iota_* L \quad \text{(by Mukai's theorem)}\\ &= (-1)^*R\mathscr{H}om(\iota_* L,\mathcal{O}_{\hat{A}})\\ &= (-1)^*R\iota_* R\mathscr{H}om(L,\mathcal{O}_{\hat{A}}\otimes\omega_{E/A}[1-g]) \quad \text{(Grothendieck duality)}\\ &= (-1)^*R\iota_* L^{-1}[1-g].\end{aligned}$$

So $R^g\Psi(-1)^*F^{-1} = (-1)^*R^1\iota_*L^{-1} = 0$. This implies that $V^g(F^{-1}) = -V^g((-1)^*F^{-1}) = \varnothing$ (using base change theorem).

Next, we calculate $V^g(F)$. By base change theorem, $V^g(F) = \mathrm{Supp}R^g\Psi F$. We have $R^g\Psi F = R^g\Psi R\Phi\iota_*L = (-1)^*\iota_*L$ by Mukai's theorem, so

$$V^g(F) = \mathrm{Supp}R^g\Psi F = (-1)^{-1}(\mathrm{Supp}\iota_*L) = E - a.$$

Consequently, we have

$$V^{g+1}(\omega_X^2) = \{0\} \cup E - a.$$

Therefore $V^{g+1}(\omega_X^2)$ is not a union of torsion translates. □

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Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: `tshibata@math.kyoto-u.ac.jp`