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Out-of-time-ordered correlators and purity in rational conformal field theories

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In this paper we investigate measures of chaos and entanglement in rational conformal field theories in 1+1 dimensions. First, we derive a formula for the late time value of the out-of-time-ordered correlators for this class of theories. Our universal result can be expressed as a particular combination of the modular S-matrix elements known as anyon monodromy scalar. Next, in the explicit setup of an $\mathrm{SU}(N)_k$ Wess–Zumino–Witten model, we compare the late time behavior of the out-of-time-ordered correlators and the purity. Interestingly, in the large-c limit, the purity grows logarithmically as in holographic theories; in contrast, the out-of-time-ordered correlators remain, in general, nonvanishing.

Subject Index A61, B21, B27, B30

1. Introduction

Two-dimensional conformal field theories (2D CFTs) have played an important role in understanding a number of interesting questions in theoretical physics. In this vein they've become central tools in the study of entanglement [1,2] and more recently quantum chaos. Based on earlier work on superconductors by Larkin and Ovchinnikov [3], Kitaev has proposed that chaotic behavior in quantum systems can be diagnosed by computing the expectation value of the square of commutators of local operators [4]. This essentially amounts to calculating the out-of-time-ordered thermal correlator (OTOC)

$$C_{ij}^{\beta}(t) \equiv \frac{\left\langle \mathcal{O}_i^{\dagger}(t)\mathcal{O}_j^{\dagger}\mathcal{O}_i(t)\mathcal{O}_j \right\rangle_{\beta}}{\left\langle \mathcal{O}_i^{\dagger}\mathcal{O}_i \right\rangle_{\beta}\left\langle \mathcal{O}_j^{\dagger}\mathcal{O}_j \right\rangle_{\beta}}. \tag{1}$$

If this quantity vanishes exponentially at late times for generic operators then the quantum system is chaotic. A number of universal properties of this object can be obtained for 2D CFTs. In particular, its been argued that chaotic behavior might be a telling characteristic of holographic CFTs [5–8].

On the other hand, one of the characteristic features of CFTs at large central charge is a so-called scrambling of entanglement [14]. One particular incarnation of scrambling is a logarithmic evolution

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of Rényi entanglement entropies after local operator excitation. Here we will focus on the second Rényi entropy or simply the purity. Various studies have showed that, for rational CFTs (RCFTs), the purity saturates to a constant equal to the logarithm of the quantum dimension of the local operator's conformal family [9–11]. Meanwhile, it is believed that in holographic CFTs (consistent with the Ryu–Takayanagi formula [12]) the Rényi entropies will grow logarithmically with time [13,14] (also at large c, the scrambling time can be naturally obtained, in a similar setup, from the evolution of the mutual information in CFT and holographically [15–17][1]). This means that at large c, holographic CFTs, the information about nonperturbative constants (like quantum dimensions or modular S-matrix) gets scrambled.

In this work we would like probe the similarities and differences between purity and OTOCs in the setup of RCFTs and find out which specific (nonperturbative) information about the theory is forfeit by quantum chaos. For that, we first fill the existing gap and compute the late time value of the OTOCs valid for any RCFT. Next, we consider a nontrivial integrable 2D CFT, the $\mathrm{SU}(N)_k$ Wess–Zumino–Witten (WZW) model, where a number of known results can be put in the new light of entanglement and quantum chaos measures. Moreover, we consider a large-c 't Hooft limit that shares some features with holographic CFTs and compare the evolution of purity and OTOC in this regime. We observe that, in the large-c limit, the purity grows logarithmically, while the OTOCs approach a nonvanishing constant value.

This paper is organized as follows: In Sect. 2, we compute the late time value of OTOC in RCFT and topological quantum field theory (TQFT). In Sect. 3, we revise the relationship between purity, quantum dimension, and logarithmic growth. In Sect. 4, we illustrate both OTOC and purity for an $\mathrm{SU}(N)_k$ WZW model. Finally, in Sect. 5 we study the behavior of these quantities in the 't Hooft limit. Finally, we conclude and place details in two appendices.

2. Late times of OTOC in RCFTs

In the present section we compute the late time value of the OTOC (1) with insertion points [6]

$$
\begin{aligned}
z_1 &= \exp\left[\tfrac{2\pi}{\beta}(t+i\epsilon_1)\right], & \bar{z}_1 &= \exp\left[-\tfrac{2\pi}{\beta}(t+i\epsilon_1)\right],\\
z_2 &= \exp\left[\tfrac{2\pi}{\beta}(t+i\epsilon_2)\right], & \bar{z}_2 &= \exp\left[-\tfrac{2\pi}{\beta}(t+i\epsilon_2)\right],\\
z_3 &= \exp\left[\tfrac{2\pi}{\beta}(x+i\epsilon_3)\right], & \bar{z}_3 &= \exp\left[\tfrac{2\pi}{\beta}(x-i\epsilon_3)\right],\\
z_4 &= \exp\left[\tfrac{2\pi}{\beta}(x+i\epsilon_4)\right], & \bar{z}_4 &= \exp\left[\tfrac{2\pi}{\beta}(x-i\epsilon_4)\right].
\end{aligned}
\tag{2}
$$

The main message from these points is that for the appropriate ordering of the ϵ_i (see the figures), as we increase t the cross-ratio $z=(z_{12}z_{34})/(z_{13}z_{24})$ encircles the point $z=1$ in the complex plane clockwise, and comes back to 0 (this doesn't happen with $\bar{z}$). The role of the temperature in this specific behavior of z is not crucial and it is used only to extract the universal predictions for the quantum chaos. More precisely, in chaotic CFTs, these correlators are expected to damp after the so-called scrambling time [5]. In contrast, for RCFTs, which are integrable systems, one expects $C^{\beta}_{ij}(t)$ to reach constant nonvanishing values. Indeed, as we shall see, the OTOCs are given by the

[1] The relation between scrambling and OTO correlators was also demonstrated in Ref. [29] but the connection with entanglement scrambling [14] used here is not clear to us at this moment.

succinct formula

$$C_{ij}^{\beta}(t) \rightarrow \frac{1}{d_i d_j} \frac{S_{ij}^*}{S_{00}} \tag{3}$$

at late times, where S_{ij}^* is the complex conjugate of the modular S-matrix. The argument proceeds as follows: First we write

$$\left\langle \mathcal{O}_i^\dagger(z_1,\bar{z}_1)\mathcal{O}_i(z_2,\bar{z}_2)\mathcal{O}_j^\dagger(z_3,\bar{z}_3)\mathcal{O}_j(z_4,\bar{z}_4)\right\rangle = |z_{12}|^{-4h_i}|z_{34}|^{-4h_j} f(z,\bar{z}). \tag{4}$$

Then, we express $f(z,\bar{z})$ in terms of the conformal blocks of the theory $\mathcal{F}_{jj}^{ii}(p|z)$ (and their anti-holomorphic counterparts $\bar{\mathcal{F}}_{jj}^{ii}(p|\bar{z})$)

$$f(z,\bar{z}) = \sum_p \mathcal{F}_{jj}^{ii}(p|z)\bar{\mathcal{F}}_{jj}^{ii}(p|\bar{z}). \tag{5}$$

At early times, since $z \approx 0$ and $\bar{z} \approx 0$, the contribution from the identity channel ($p = 0$) dominates; thus, $f(z,\bar{z}) \approx 1$. At late times, once again $z \approx 0$ and $\bar{z} \approx 0$. However, as time goes by, the cross-ratio z traverses a nontrivial contour around $z = 1$ in the complex plane (this is not the case for $\bar{z}$). As shown in Ref. [5], extracting this monodromy from the explicit form of the large-c conformal block [19], one can see the butterfly effect in 2D CFT. In RCFTs, the monodromy of conformal blocks is given by a finite matrix and we have

$$\mathcal{F}_{jj}^{ii}(p|z) \rightarrow \sum_q \mathcal{M}_{pq}\mathcal{F}_{jj}^{ii}(q|z). \tag{6}$$

Because the cross-ratio z goes around $z = 1$ and finally comes back to $z = 0$, the only relevant component is $\mathcal{M}_{00}$. Therefore, we obtain

$$\lim_{t\to\infty} f(z,\bar{z}) = \mathcal{M}_{00}\mathcal{F}_{jj}^{ii}(0|z)\bar{\mathcal{F}}_{jj}^{ii}(0|\bar{z}). \tag{7}$$

Moreover, for RCFTs this monodromy matrix element can be expressed in terms of the modular S-matrix as [20,21]

$$\mathcal{M}_{00} = \frac{S_{ij}^*}{S_{00}} \frac{S_{00}}{S_{0i}} \frac{S_{00}}{S_{0j}}. \tag{8}$$

We can also derive this late time value of the OTOC using 3D TQFT technology [22]. As time passes, the operators evolve as depicted in Fig. 1. Their orbits are mapped to 3D links made by the corresponding anyons as in Fig. 2. The relation between 2D CFT and 3D TQFT is given as follows. First, the initial state of 3D TQFT is determined by the sector of the conformal block we choose. In this case, we choose the identity sector in CFT and in 3D TQFT the pairs of anyons are created from the vacuum. Then, because there is a monodromy in the CFT side, there is a link in the 3D TQFT side. Finally, corresponding to taking the identity sector at late times, anyons fuse to the vacuum, which means that the final state in the 3D TQFT is given by the annihilation pair of anyons. As a result, we obtain the Hopf link of two Wilson loops. From this observation, we find that the monodromy matrix element is given by the expectation value of the Hopf link divided by the expectation value

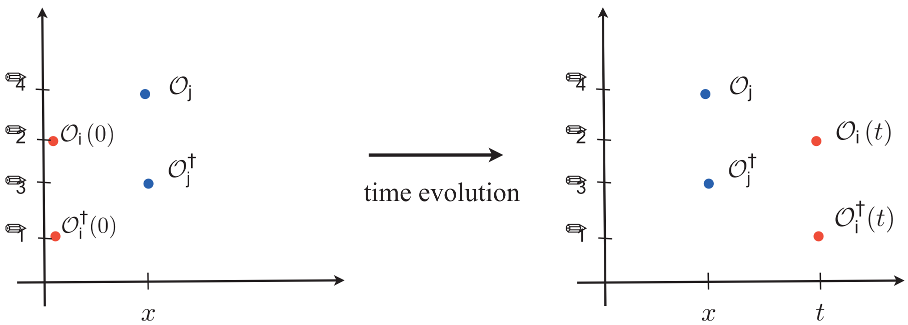


Fig. 1. The orbit of the chiral part of operators in the complex plane.

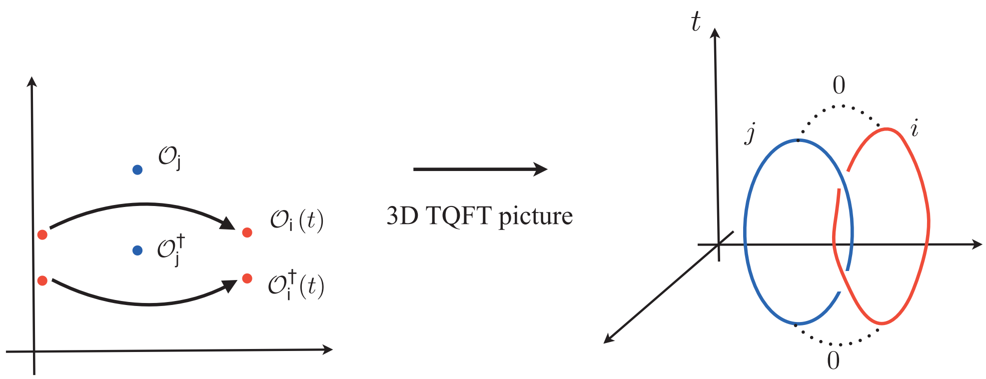
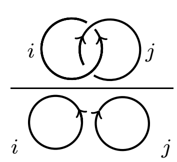


Fig. 2. 3D TQFT counterpart. Here, 0 denotes the identity channel of the conformal blocks.

of two nonlinked Wilson loops. Based on results from Ref. [22], we find

$$C^{\beta}_{ij}(t) \to$$



$$\equiv \frac{1}{d_i d_j}\frac{S^*_{ij}}{S_{00}}. \tag{9}$$

This exactly matches with the right-hand side of Eq. (8) and naturally explains why this combination appears in late time OTOC. If we apply this formula to the Ising model CFT, we reproduce exactly the results from the explicit calculation of monodromy in Ref. [5, Appendix B].

Let us finally mention that the above late time value, known as monodromy scalar, has been proposed as a measure of nonabelian anyons in interferometry experiments [23]. It would be interesting to explore this connection as a possible experimental measure of quantum chaos.

3. Purity and quantum dimension

Now, we turn our attention to entanglement. We are interested in a local quench setup where a state is excited by a local operator. More precisely, we take a pure state in a (1+1)-dimensional CFT and divide space into two halves A and $\bar{A}$. Then, we insert a local operator $\mathcal{O}$ with conformal dimension $h = \bar{h}$ into $\bar{A}$ at, say $x = -l$, and study the time evolution of entanglement in the system. In particular, we consider the evolution of the second Rényi entropy; hereafter we refer to this quantity as the purity (strictly speaking, the purity corresponds to the logarithm of the square of the reduced density matrix). Using the replica method, the purity can be extracted from the canonical 4-point function $\mathcal{G}(z,\bar{z}) \equiv \langle \mathcal{O}(0)\mathcal{O}(z,\bar{z})^{\dagger}\mathcal{O}(1)\mathcal{O}^{\dagger}(\infty)\rangle$ and it reads [18]

$$\Delta S_A^{(2)}(z,\bar{z}) = -\log\left[|z(1-z)|^{4h}\mathcal{G}(z,\bar{z})\right], \tag{10}$$

where the points entering the cross-ratios are expressed in terms of the replica points as $z_i^2 = w_i$, where

$$\begin{aligned} w_1 &= i(\epsilon - it) - l, \qquad & w_2 &= -i(\epsilon + it) - l, \\ \bar{w}_1 &= -i(\epsilon - it) - l, \qquad & \bar{w}_2 &= i(\epsilon + it) - l. \end{aligned} \tag{11}$$

As one takes $\epsilon \to 0, \bar{z} \to 0$, meanwhile, z can become either 0 or 1 for times earlier or later than l respectively.

In an RCFT, given the singularity structure of $\mathcal{G}$, this implies that $\Delta S_A^{(2)}$ vanishes at early times since only the identity channel contributes. Moreover, since early and late times are mapped to each other by the transformation $(z,\bar{z}) \to (1-z,\bar{z})$, one finds that the late time purity can be extracted from the fusion matrix element $F_{00}[\mathcal{O}]$. Furthermore, this quantity corresponds to the inverse of the quantum dimension of $\mathcal{O}$'s conformal family. Hence, at late times we have [10,18]

$$\Delta S_A^{(2)}(t) = \log d_{\mathcal{O}}. \tag{12}$$

Observe that the appearance of a constant contribution at late times for $\Delta S_A^{(2)}$ is closely related to the singular behavior

$$\mathcal{G}(z,\bar{z}) \to d_{\mathcal{O}}^{-1}((1-z)\bar{z})^{-2h} \tag{13}$$

of the 4-point function as $(z,\bar{z}) \to (1,0)$. The authors of Ref. [14] have argued that in holographic CFTs, where the Ryu–Takayanagi formula [12] is valid, such a singularity disappears due to the "scrambling of entanglement". This way, in our setup, the appearance of the (nonperturbative) quantum dimension at late times is replaced by a divergent logarithmic growth of the Rényi entropy. This fast growth of entanglement is equivalent to the breakdown of the quasi-particle picture that is characteristic of strongly coupled large-c theories. The temperature dependence can be introduced by the standard conformal map (see Ref. [16]) but the logarithmic growth with time at large c is not affected. Below, we will show how this behavior emerges in the "holographic" large-c limit of WZW models.

It is also worth mentioning that the $\log d_{\mathcal{O}}$ increase can be obtained from the topological entanglement entropy [24,25] if one of the regions contains an anyonic excitation [26]. At first sight, the late time purity and the OTOC are rather similar objects, i.e., both are captured by the vacuum conformal block. Naively, one would expect to be able to use them interchangeably as indicators of quantum chaos, or even to diagnose whether a CFT has a holographic dual. This is in fact not the case, and below we present an example where the purity displays the behavior expected from a holographic theory, while the OTOC doesn't.

4. On purity and OTOC in SU$(N)_k$ WZW

In this section we consider a WZW model with affine Lie algebra SU$(N)_k$. Just as the quantum dimension, the late time value OTOC is invariant under level-rank duality, hence, the following discussion is valid for SU$(k)_N$ as well. Knowing the modular S-matrix of the model, using Eq. (9) we could compute directly the late time OTOC. Here we follow a direct approach instead, both to illustrate the underlying mechanisms and as a consistency check. We focus on the 4-point function of operators $g^{\alpha}_{\beta}(z_i,\bar{z}_i)$ (and their conjugates) in the fundamental representation $\alpha = \{1,0,\dots,0\}$ that have conformal dimension

$$h = \bar{h} = \frac{N^2-1}{2N\kappa}, \tag{14}$$

where $\kappa = N + k$. The general correlator that we employ is

$$\left\langle g^{\alpha_1}_{\beta_1}(z_1,\bar{z}_1)(g^{-1})^{\beta_2}_{\alpha_2}(z_2,\bar{z}_2)g^{\alpha_3}_{\beta_3}(z_3,\bar{z}_3)(g^{-1})^{\beta_4}_{\alpha_4}(z_4,\bar{z}_4)\right\rangle \equiv \frac{1}{|z_{12}|^{4h}|z_{34}|^{4h}}|z|^{4h}\mathcal{G}(z,\bar{z}). \tag{15}$$

Recall that we characterized OTOCs by the function $f(z,\bar{z})$, which is related to $\mathcal{G}(z,\bar{z})$ via $f(z,\bar{z}) = |z|^{2h}\mathcal{G}(z,\bar{z})$. To apply the above correlator in our OTOC we set $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, and $\alpha_3 = \alpha_4$ with $\beta_3 = \beta_4$. On the other hand, for the purity all the α's (and β's) must be equal.

The general 4-point functions (15) are well-known solutions of the Knizhnik–Zamolodchikov equations [27]. The canonical correlator can be expanded in terms of affine conformal blocks

$$\mathcal{G}(z,\bar{z}) = \sum_{i,j} I_i \bar{I}_j \sum_n X_{nn}\mathcal{F}^{(n)}_i(z)\mathcal{F}^{(n)}_j(\bar{z}), \tag{16}$$

with $i,j,n \in \{1,2\}$ and SU(N) factors $I_1 = \delta^{\alpha_2}_{\alpha_1}\delta^{\alpha_4}_{\alpha_3}$, $I_2 = \delta^{\alpha_4}_{\alpha_1}\delta^{\alpha_2}_{\alpha_3}$. In our arguments we will only use $X_{11} = 1$; more details can be found in Ref. [28].

Let us compute the purity first. In order to extract the late time value, we apply the fusion transformation that mixes conformal blocks

$$\mathcal{G}(1-z,\bar{z}) = \sum_{i,j} I_i \bar{I}_j \sum_{n,m} X_{nn}c_{nm}\mathcal{F}^{(m)}_{3-i}(z)\mathcal{F}^{(n)}_j(\bar{z}), \tag{17}$$

where the relevant coefficient is

$$c_{11} = N\frac{\Gamma(N/\kappa)\Gamma(-N/\kappa)}{\Gamma(1/\kappa)\Gamma(-1/\kappa)} = [N]^{-1} = d_g^{-1}, \tag{18}$$

with d_g being the quantum dimension for the fundamental representation, where the quantum numbers are defined as

$$[x] = \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}}, \qquad q = \exp\left[-\frac{2\pi i}{N+k}\right]. \tag{19}$$

Taking the limit of the conformal blocks for $(z,\bar{z}) \to (1,0)$ (see Appendix A) leaves us with the log of the quantum dimension multiplied by the appropriate singularity such that we get $\log[N]$ at late times. It is also interesting to see that even though the 4-point correlator is expanded in terms of the affine conformal blocks, that are sums of the Virasoro blocks, the relevant constant is still hidden in the vacuum block. Moreover, from the definition, we have $[N] = [k]$, which is in fact a consequence of the level-rank duality for quantum dimensions inherited by the purity.

Now, let us study the OTOC. Extracting the monodromy around $z = 1$ brings us to

$$f(z,\bar{z}) = \exp[-2\pi i(h_\theta - 2h)]\sum_{i,j} I_i\bar{I}_j \sum_{n,m} X_{nn}B_{nm}f^{(m)}_i(z)f^{(n)}_j(\bar{z}), \tag{20}$$

where B_{nm} are the monodromy matrix elements of the solutions of the hypergeometric equation (see, e.g., Ref. [31]). Taking the limit of $(z, \bar{z}) \to (0, 0)$ leaves only the terms from $f_1^{(1)}$ and we are left with the overall exponent prefactor and the coefficient B_{11} given by

$$B_{11} = 1 - 2i \exp[-i\pi(1 - \tfrac{N}{\kappa})] \frac{\sin^2(\frac{\pi}{\kappa})}{\sin(\pi(1 - \frac{N}{\kappa}))}. \tag{21}$$

After some algebra, and expressing the answer in terms of quantum numbers we find that at late times

$$C_{ij}^{\beta}(t) \to \exp[-2\pi i(h_\theta - 2h)]B_{11} = q^{(1/N)+(1/2)} \frac{\left(q^{-(N+2)/2} + [N-1]\right)}{[N]}. \tag{22}$$

We can compare this answer with our RCFT result Eq. (3). Indeed, the S-matrix element for the present example has been computed in Ref. [26] and it reads

$$\frac{S^*_{\alpha\alpha}}{S_{00}} = q^{(1/N)+(1/2)} \left(q^{-(N+2)/2} + [N-1]\right)[N], \tag{23}$$

and inserting $d_i = d_j = [N]$ beautifully matches Eq. (3). For example, for the $\mathrm{SU}(2)_k$ model, the late time OTO Eq. (22) reduces to

$$C_{ij}^{\beta}(t) \to \cos\left(\frac{2\pi}{k+2}\right)\cos^{-1}\left(\frac{\pi}{k+2}\right), \tag{24}$$

which can be extracted from the explicit form of the $\mathrm{SU}(2)_k$ modular S-matrix

$$S_{ij} = \left(\frac{2}{2+k}\right)^{1/2} \sin\left(\frac{(i+1)(j+1)\pi}{k+2}\right), \tag{25}$$

by setting $i = j = 1$. Note that, in general, the elements of the modular S-matrix can be complex (except the first row that are related to quantum dimensions, which are real).

Summarizing, we have shown that late time values of the purity and OTOC are given in terms of the quantum dimensions as well as the modular S-matrix. It is interesting that, in RCFTs, OTOCs give us the access to the entire modular S-matrix, whereas Rényi entropies only give us access to the first row S_{0i}. It is also interesting to consider the classical limit ($k \to \infty$) of WZW models where the purity becomes the log of the dimension of the fundamental, and the OTOC equals 1.

5. OTOC and purity in the large-c limit

Finally, it is interesting to compare the behavior of the purity and the OTOC in the large-c limit. In the $\mathrm{SU}(N)_k$ WZW model, the central charge is given by

$$c = \frac{k(N^2-1)}{k+N}. \tag{26}$$

By introducing the ’t Hooft coupling constant

$$\lambda = \frac{N}{k}, \tag{27}$$

we can define a ’t Hooft limit of large central charge with the coupling fixed (weak or strong). The 4-point correlator has been analyzed in detail in this limit by Ref. [30] and we apply their analysis in our context. For $c \to \infty$, the 4-point correlator becomes Eq. A1 (see Appendix A). (Note that here, unlike in Ref. [5], all our operators are light: $h/c \to 0$ as $c \to \infty$.) Using this correlator, one can see that for a large central charge, the singularities leading to the quantum dimension are absent, which leads to logarithmic growth of the purity,

$$\Delta S_A^{(2)}(t) \simeq 2h \log\left(\frac{2t}{\epsilon}\right) - \log(2). \tag{28}$$

This behavior comes from discarding terms proportional to $1/\sqrt{c}$. However, if we include such corrections, then the late time answer becomes the logarithm of the quantum dimension in the large-c limit. It is illustrative to verify this in the strong coupling regime, where $h = 1/2$ and the correlator (see Appendix A) can be computed by approximating the operators $g^{\alpha}_{\beta}(z_i, \bar{z}_i) \simeq \frac{1}{k}\sum_{i=1}^{k} \psi^{\alpha}(z_i)\bar{\psi}^{\beta}(\bar{z}_i)$ with complex fermions. In this limit we have

$$\mathcal{G}(z,\bar{z}) \simeq \frac{I_1\bar{I}_1}{|z|^2} + \frac{I_2\bar{I}_2}{|1-z|^2} + \sqrt{\frac{\lambda}{c}}\left(\frac{I_1\bar{I}_2}{z(1-\bar{z})} + \frac{I_2\bar{I}_1}{(1-z)\bar{z}}\right). \tag{29}$$

Using Eq. (10), it is clear that neglecting the last two terms in the above expression leads to logarithmic growth of the purity in the large-c limit, which is sometimes known as *scrambling of entanglement*. Another way to look at this order of limits issue is that, at strong coupling, the timescale at which the purity reaches $\log d_O$ can be estimated as $t - l \simeq (c^{1/4}/2\lambda^{1/4})\epsilon$. If we then take the large-c limit first (like in holography), we will not reach the finite constant and we are left with logarithmic growth with time (see also the discussion in Ref. [13]).

On the other hand, the late time value of the OTOC comes from the first term in Eq. (29) (irrespectively of the weak or strong coupling) and in $f(z,\bar{z})$ it is simply 1. For different operators $\alpha_1 = \alpha_2 \neq \alpha_3 = \alpha_4$ only I_2 vanishes so the result remains the same. Thus, OTOC is a good indicator of integrability even when entanglement scrambles.

6. Conclusion

We have shown that OTOC in RCFTs approach to a universal constant at late times, which is completely determined in terms of the modular S-matrix of the theory. Moreover, we have pointed out that this quantity is potentially observable in experimental setups. We provided a nontrivial example in the integrable $\mathrm{SU}(N)_k$ WZW model. We also argued in this setup, that in the large-c limit the purity displays logarithmic growth that is characteristic of holographic models, but the OTOCs remain constant, as a good measure of quantum chaos should since the theory is far from chaotic. It would be interesting to understand how chaos and scrambling are related and emerge in large-c CFT with holographic duals. Extending our work to operators in higher representations, nonrational CFTs, and (non)integrable theories in higher dimensions might shed more light on these issues.

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Appendix A. 4-point function in SU$(N)_k$ WZW

From the Knizhnik–Zamolodchikov equations, one can derive the canonical 4-point function that is written in terms of the cross-ratios and the dimensions $h = (N^2 - 1)/(2N(k+N))$ and $h_\theta = N/(N+k)$. More precisely, the affine conformal blocks are expressed as

$$\begin{aligned}
\mathcal{F}_1^{(1)}(z) &= z^{-2h}(1-z)^{h_\theta-2h}u_1(z),\\
\mathcal{F}_1^{(2)}(z) &= z^{-2h}(1-z)^{h_\theta-2h}u_2(z),\\
\mathcal{F}_2^{(1)}(z) &= \frac{1}{k}z^{1-2h}(1-z)^{h_\theta-2h}\tilde{u}_1(z),\\
\mathcal{F}_2^{(2)}(z) &= -Nz^{1-2h}(1-z)^{h_\theta-2h}\tilde{u}_2(z),
\end{aligned}$$

with $u(z)_i$ and $\tilde{u}_i(z)$ being the standard solutions of the hypergeometric equations $u_1(z) = {}_2F_1(a,b,c;z)$ and $u_2(z) = z^{1-c}\,{}_2F_1(a-c+1, b-c+1, 2-c; z)$, where u_1 and u_2 are parametrized by $a = 1/\kappa$, $b = -1/\kappa$, and $c = 1-(N/\kappa)$ such that $1-c = N/\kappa = h_\theta$, and $\tilde{u}_1$ and $\tilde{u}_2$ are parametrized by $a = 1+1/\kappa$, $b = 1-1/\kappa$ and $c = 2-N/\kappa$ such that $1-c = (N/\kappa)-1 = h_\theta - 1$. The monodromy of the conformal blocks under the loop that encircles $z=1$ is a combination of the contribution from the pre-factors as well as the monodromy of the hypergeometric functions (see Ref. [31]).

The function $\mathcal{G}(z,\bar{z})$ in Eq. (16) admits the large-c expansion given by

$$\mathcal{G}(z,\bar{z}) \simeq \frac{I_1\bar{I}_1}{|z|^{4h}} + \frac{I_2\bar{I}_2}{|1-z|^{4h}} + \frac{\lambda}{\sqrt{c(1+\lambda)}}\left[\left(\gamma(z,\bar{z})\bar{I}_1 I_2 + \text{c.c.}\right)\right], \tag{A1}$$

where

$$\gamma(z,\bar{z}) = \frac{{}_2F_1\left(1,1,\frac{2+\lambda}{1+\lambda};z\right)}{\bar{z}^{2h}z^{2h-1}} - \frac{{}_2F_1\left(\frac{\lambda}{1+\lambda},\frac{\lambda}{1+\lambda},\frac{1+2\lambda}{1+\lambda};\bar{z}\right)}{\lambda(1-z)^{2h}},$$

with $2h = \lambda/(1+\lambda)$. Notice that around $(z,\bar{z}) \approx (1,0)$,

$$\gamma(z,\bar{z}) \approx \left(\frac{\pi}{1+\lambda}\right)\frac{\csc\left(\frac{\lambda}{1+\lambda}\right)}{\bar{z}^{2h}(1-z)^{2h}}.$$

Plugging the above expression into $\mathcal{G}$ and afterwards in Eq. (10), one finds a constant contribution to the late time purity; this constant duly corresponds to the logarithm of the first term in the large-c expansion of the quantum dimension.

Appendix B. OTOC in Liouville theory

It is also interesting to "naively" apply our formula for late time value of OTO in a (nonrational) Liouville 2D CFT with central charge $c = 1 + 6Q^2$. From the explicit form of the analog of the S-matrix [32] (see also Refs. [34,35]), the quantum dimension of a nondegenerate operator with weight $\Delta_p = p^2 + \frac{1}{4}Q^2$ reads $d_p = \sinh(\pi p b)\sinh\left(\pi p b^{-1}\right)$. Moreover, the S-matrix element between two such nondegenerate operators is given by $\mathcal{S}_p^q = \sqrt{2}\cos(\pi pq)$. Plugging these into Eq. (9) yields at large c,

$$C_{pq}^{\beta}(t) \sim \Lambda_{p,q}\, c\, \exp\left[-\pi\,(p+q)\left(\frac{c}{6}\right)^{1/2}\right],$$

where $\Lambda_{p,q}$ is a constant that depends on p and q. Observe that the above expression is damped exponentially as we increase the central charge, in contrast with the RCFT case.

References

[1] P. Calabrese and J. L. Cardy, J. Stat. Mech. **0406**, P06002 (2004).
[2] C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B **424**, 443 (1994).
[3] A. Larkin and Y. Ovchinnikov, J. Exp. Theor. Phys. **28**, 1200 (1969).
[4] A. Kitaev, Hidden correlations in the Hawking radiation and thermal noise, talk given at the Fundamental Physics Prize Symposium, Nov. 10, 2014. Available at: https://www.youtube.com/watch?v=OQ9qN8j7EZI.
[5] D. A. Roberts and D. Stanford, Phys. Rev. Lett. **115**, 131603 (2015).
[6] D. A. Roberts, D. Stanford, and L. Susskind, J. High Energy Phys. **1503**, 051 (2015).
[7] J. Maldacena, S. H. Shenker, and D. Stanford, arXiv:1503.01409 [hep-th] [Search INSPIRE].
[8] A. L. Fitzpatrick and J. Kaplan, arXiv:1601.06164 [hep-th] [Search INSPIRE].
[9] M. Nozaki, T. Numasawa, and T. Takayanagi, Phys. Rev. Lett. **112**, 111602 (2014).
[10] P. Caputa and A. Veliz-Osorio, Phys. Rev. D **92**, 065010 (2015).
[11] P. Caputa and M. M. Rams, arXiv:1609.02428 [cond-mat.str-el] [Search INSPIRE].
[12] S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96** 181602 (2006).
[13] P. Caputa, M. Nozaki, and T. Takayanagi, Prog. Theor. Exp. Phys. **2014**, 093B06 (2014).
[14] C. T. Asplund, A. Bernamonti, F. Galli, and T. Hartman, J. High Energy Phys. **1509**, 110 (2015) [arXiv:1506.03772 [hep-th]] [Search INSPIRE].
[15] S. H. Shenker and D. Stanford, J. High Energy Phys. **1403**, 067 (2014).
[16] P. Caputa, J. Simon, A. Stikonas, and T. Takayanagi, J. High Energy Phys. **1501**, 102 (2015).
[17] P. Caputa, J. Simon, A. Stikonas, T. Takayanagi, and K. Watanabe, J. High Energy Phys. **1508**, 011 (2015).
[18] S. He, T. Numasawa, T. Takayanagi, and K. Watanabe, Phys. Rev. D **90**, 041701 (2014).
[19] A. L. Fitzpatrick, J. Kaplan, and M. T. Walters, J. High Energy Phys. **1408**, 145 (2014).
[20] G. W. Moore and N. Seiberg, Nucl. Phys. B **313**, 16 (1989).
[21] G. W. Moore and N. Seiberg, Phys. Lett. B **212**, 451 (1988).
[22] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[23] P. Bonderson, K. Shtengel, and J. K. Slingerland, Phys. Rev. Lett. **97**, 016401 (2006).
[24] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[25] M. Levin and X. G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[26] S. Dong, E. Fradkin, R. G. Leigh, and S. Nowling, J. High Energy Phys. **0805**, 016 (2008).
[27] V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B **247**, 83 (1984).
[28] P. Di Francesco, P. Mathieu, and D. Senechal (1997) *Conformal Field Theory* (Springer, New York, 1997).
[29] P. Hosur, X. L. Qi, D. A. Roberts, and B. Yoshida, arXiv:1511.04021 [hep-th] [Search INSPIRE].
[30] E. Kiritsis and V. Niarchos, J. High Energy Phys. **1104**, 113 (2011).
[31] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions, Vol. 1* (McGraw-Hill, New York, 1953), p. 93.
[32] A. B. Zamolodchikov and A. B. Zamolodchikov, arXiv:hep-th/0101152 [Search INSPIRE].
[33] Y. Gu and X. L. Qi, arXiv:1602.06543 [hep-th] [Search INSPIRE].
[34] S. Jackson, L. McGough, and H. Verlinde, Nucl. Phys. B **901**, 382 (2015).
[35] L. McGough and H. Verlinde, J. High Energy Phys. **1311**, 208 (2013).