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Violation of cosmic censorship in the gravitational collapse of a dust cloud in five dimensions

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We analyze the null geodesic equations in five-dimensional spherically symmetric spacetime with collapsing inhomogeneous dust cloud. By using a new method, we prove the existence and non-existence of solutions to null geodesic equations emanating from the central singularity for smooth initial distribution of dust. Moreover, we also show that the null geodesics can extend to null infinity in a certain case, which implies violation of the cosmic censorship conjecture.

Subject Index A13, E00, E01, E04, E31

1. Introduction

Black hole spacetime is one of the most fascinating objects in gravitational theory. In particular, it is quite interesting that black holes can contain singularities inside their event horizons. At singularities the spacetime curvature often diverges, and the physics breaks down because gravitational theory is described in terms of curvature.

Observers outside black holes cannot see such breakdown because no information can come out of black holes, at least at the classical level. However, black hole singularities may cause serious effects on an observer inside a black hole and/or the final fate of black hole evapolation due to Hawking radiation. Here we define a singularity that is visible for an observer inside a black hole as a "locally naked singularity." If a singularity is not wrapped by the horizon, it may cause serious effects on the physics. We define this as a "globally naked singularity."

In order to certify the predictability of physics, it is very important to ask whether they can be naked or not. It is usually supposed that no naked singularity will appear in a physical situation. In this context, Penrose proposed the so-called cosmic censorship conjecture (CCC) [1]. More precisely, there are two types of CCC, strong CCC and weak CCC. Strong CCC states that no locally naked singularities form during gravitational collapse; weak CCC states that no globally naked singularities form during collapse.

The cosmic censorship conjecture is assumed in proving several key theorems on black hole spacetime, such as the event horizon topology theorem, uniqueness theorem, and so on (see [2] for the details). The conjecture has been investigated by many authors in a variety of setups, but it is still controversial. In [3], Oppenheimer and Snyder considered a spherical collapse of homogeneous pressureless fluid, which is called dust; they found that no naked singularities form. But many subsequent works, for example [4–11], reported the violation of strong CCC in several situations. In [12], Christodoulou examined the global nakedness of singularities in four-dimensional spacetime. He considered the spherical collapse of inhomogeneous dust and proved that a singularity can be globally naked in some situations, i.e. violation of weak CCC occurs in general.

Motivated by fundamental theories such as string theory, those works have been extended to higher-dimensional spacetime [13–16]. It was shown that strong CCC always holds for the spherical collapse of a dust cloud with smooth initial data in spacetimes with dimensions higher than five, i.e. any observer cannot see singularities formed. One simple explanation for this fact is that, in higher dimensions, gravity near the singularity is stronger than the lower-dimensional case and the event horizon appears earlier. On the other hand, in general, it was also shown that the strong CCC does not hold in five-dimensional spacetimes.

As far as we know, there has been no work on a global analysis of collapsing spacetime in *five* dimensions. It is natural to ask whether the weak CCC is actually violated in five dimensions. And, if violated, it is important to clarify in what conditions naked singularities form. In this paper we focus on analysis of five-dimensional inhomogeneous spherically symmetric dust collapse and give a new method to examine the nakedness of a singularity. To do so we have to know whether a causal geodesic emanating from the singularity exists or not. So, we give a method to investigate the existence of a solution of the null geodesic equation. Furthermore, we examine the spacetime structure in detail, and give a necessary and sufficient condition for naked singularity formation. We also examine the condition that a singularity is not only locally naked but also globally naked. Finally, we will see the dependence of the global nakedness of a singularity on the initial density distribution.

The organization of this paper is as follows. In Sect. 2, we present the setting and the fundamental nature of inhomogeneous spherically symmetric dust collapse in five-dimensional spacetime. Then, we derive the differential equation for the null geodesic in terms of dimensionless quantities. In Sect. 3, we examine the existence condition for a solution to the differential equation for the null geodesic. By virtue of the Schauder fixed-point theorem [17], we show that a solution of the differential equation for the null geodesic exists near the singularity. This means that the singularity can be at least locally naked. In Sect. 4, we analyze the spacetime structure around the singularity. We identify the earliest null geodesic emanating from the central singularity, and give a necessary and sufficient condition for the singularity to be naked. In Sect. 5, we consider the global nakedness of the singularity. We will show that a class of initial density distributions leads to a globally naked singularity.

2. Five-dimensional Lemaître–Tolman–Bondi spacetime and the equation of the null line

We consider spherically symmetric dust collapse in five dimensions. This is known as the Lemaître–Tolman–Bondi (LTB) solution in higher dimensions. In the comoving coordinate of the dust, the metric of this spacetime is written as

$$ds^2 = -dt^2 + e^{2\omega(t,r)}dr^2 + R(t,r)^2 d\Omega^2, \tag{1}$$

where R is the area radius of the $r = const.$ three-sphere and we set r by $R(0,r) = r$. Then, in this coordinate, the Einstein equations become

$$\rho(t,r) = \frac{3}{2}\frac{M'(r)}{R^3 R'}, \tag{2}$$

$$\dot{R}^2 = \frac{M(r)}{R^2} + E(r), \tag{3}$$

$$e^{2\omega} = \frac{R'^2}{1+E(r)}, \tag{4}$$

where $\rho(t,r)$ is the energy density of dust, which is proportional to the Ricci scalar, $M(r)$ is an arbitrary C^1 function of r, and $E(r)$ is an arbitrary function of r. Dot "˙" and prime "′" mean partial derivatives with respect to t and r, respectively. $M(r)$ corresponds to the total mass in the region surrounded by the $r = const.$ surface. Actually, $M(r)$ is proportional to the Misner–Sharp quasi-local mass [18], and $E(r)$ is the initial energy of the dust shell.

As mentioned in the introduction, the point where spacetime curvature diverges is a singularity. From Eq. (2), we can find the two types of singularities, i.e. $R = 0$ and $R' = 0$. A singularity at $R = 0$ ($R' = 0$) is called a shell focusing singularity (a shell crossing singularity). If we introduce pressure to the fluid, the shell crossing singularity may disappear, so they are regarded as unphysical singularities. Throughout this paper, we only consider the shell focusing singularity, that is, we assume $R' > 0$. In addition, we also assume that the initial velocity of the shells at $t = 0$ is zero, that is,

$$\dot{R}^2(0,r) = \frac{M(r)}{r^2} + E(r) = 0. \tag{5}$$

Now we can solve Eq. (3) as

$$t(R,r) = -\int_r^R dR \frac{1}{\sqrt{M(r)\left(\frac{1}{R^2} - \frac{1}{r^2}\right)}} = \frac{r^2}{\sqrt{M(r)}}\sqrt{1 - \frac{R^2}{r^2}}. \tag{6}$$

From this equation and (2), we see that the singularity appears at

$$t_S(r) = \frac{r^2}{\sqrt{M(r)}}. \tag{7}$$

Using Eqs. (1), (3), and (4), we can compute the expansion of outgoing null geodesics on the $r = const.$ surface $\Theta(r)$ as

$$\begin{aligned}\Theta(r) \propto \frac{dR(t, r_{\rm null}(t))}{dt} &= \dot{R} + R'\frac{dr_{\rm null}}{dt} \\ &= \dot{R} + R' e^{-\omega} \\ &= \sqrt{1 - \frac{M(r)}{r^2}} - \sqrt{\frac{M(r)}{R^2} - \frac{M(r)}{r^2}}.\end{aligned} \tag{8}$$

In the second equality we used the equation $-dt^2 + e^{2\omega}dr^2 = 0$, which holds for null geodesics along the outer radial direction. This equation implies that the apparent horizon ($\Theta = 0$) is located at $R = \sqrt{M(r)}$. Since we are interested in naked singularity formation during the dust collapse, we assume that there is no apparent horizon initially. Then, Eq. (8) with the setting of $R(0, r) = r$ tells us that

$$1 - \frac{M(r)}{r^2} > 0 \tag{9}$$

is required for all r in this coordinate patch. From Eqs. (1), (4), and (5), this is equivalent to the condition that r is a spacelike coordinate. In addition, from Eq. (6), we also see that the apparent horizon appears at

$$t_{\rm AH}(r) = \frac{r^2}{\sqrt{M(r)}}\sqrt{1 - \frac{M(r)}{r^2}}. \tag{10}$$

From Eqs. (7) and (10), if $r \neq 0$, it is easy to see that

$$t_{\rm AH}(r) < t_{\rm S}(r) \tag{11}$$

holds. This means that the apparent horizon appears before the singularity does, and the singularity is surrounded by the apparent horizon. So only the singularity at $r = 0$ could be naked, and another singularities are covered by the event horizon [2]. In order to examine the possibility of the occurrence of a naked singularity, we assume that the singularity at $r = 0$ appears within non-zero finite time $t_{\rm S}(0)$, that is,

$$t_{\rm S}(0) = \lim_{r\to 0} t_{\rm S}(r) = \lim_{r\to 0} \frac{r^2}{\sqrt{M(r)}}. \tag{12}$$

Therefore, using a C^0 function $A(r)$ with $A(0) \neq 0$, we can write $M(r)$ as

$$M(r) = A(r) r^4. \tag{13}$$

In the above, $A(r)$ corresponds to the mean density for the region surrounded by the $r = const.$ shell. From Eqs. (2) and (13), we have

$$A(r) = \frac{2}{3}\int_0^1 dv v^3 \rho(0, vr). \tag{14}$$

Moreover, we assume that the initial density $\rho(0, r)$ is a C^∞ function of compact support which monotonically decreases with respect to r, and $\rho'(0, r)$ is continuous at $r = 0$ on the initial time slice $[\rho'(0, r) \leq 0,\ \rho'(0, 0) = 0]$. Then, we see that $A(r)$ satisfies[1]

$$A(r) = \alpha - \frac{\beta}{2} r^2 + O(r^3) \tag{15}$$

near $r = 0$, and

$$A'(r) \leq 0, \tag{16}$$

[1] In this paper, $f(x) = O(x^a)$ means that the absolute value of the function $f(x)$ is bounded by a constant times x^a as $x \to 0$, i.e. that there exists a positive x_0 and c in $\mathbb{R}^+$ such that $|f(x)| \leq c x^a$ holds for all $x \in [0, x_0]$.

where $\alpha, \beta \in \mathbb{R}$ are parameters satisfying $\alpha > 0, \beta \geq 0$. Using $A(r)$, $t_{\mathrm{S}}(r)$ and $t_{\mathrm{AH}}(r)$ are rewritten as

$$t_{\mathrm{S}}(r) = \sqrt{\frac{1}{A(r)}}, \tag{17}$$

$$t_{\mathrm{AH}}(r) = \sqrt{\frac{1}{A(r)}(1 - A(r)r^2)}. \tag{18}$$

If $\beta = 0$ in Eq. (15), we can immediately show that the central singularity must be covered by the apparent horizon.

Theorem 1 If $\beta = 0$, the strong cosmic censorship holds.

Proof. $\beta = 0$ implies

$$A(r) - \alpha = O(r^3). \tag{19}$$

Then,

$$\begin{aligned} t_{\mathrm{AH}}(r) - t_{\mathrm{S}}(0) &= \sqrt{\frac{1}{A(r)}(1 - A(r)r^2)} - \sqrt{\frac{1}{\alpha}} \\ &= \frac{1}{\sqrt{\alpha A(r)}}\left(\sqrt{\alpha} - \frac{\sqrt{\alpha^3}}{2}r^2 - \sqrt{A(r)} + O(r^3)\right) \\ &= -\frac{\sqrt{\alpha}}{2}r^2 + O(r^3). \end{aligned} \tag{20}$$

Thus, there exists $r_0 \in \mathbb{R}$ such that

$$t_{\mathrm{AH}}(r) - t_{\mathrm{S}}(0) < 0 \tag{21}$$

holds for arbitrary $r \in [0, r_0]$. Therefore, there exists an apparent horizon around the past of the central singularity, so null geodesics cannot emanate from the singularity and the strong cosmic censorship holds. □

This fact is already known in Refs. [13–15]. Accordingly, in order to figure out the condition for naked singularity formation, we suppose $\beta > 0$ in the following discussion. Let us consider the future-directed null geodesics along the outer radial direction. Because of the spherical symmetry, the differential equation for future-directed null geodesics along the outer radial direction is given by

$$\begin{aligned} \frac{dt}{dr} &= e^{\omega} \\ &= \frac{R'}{\sqrt{1 - A(r)r^2}} \\ &= \frac{1}{\sqrt{(1 - A(r)r^2)(1 - A(r)t^2)}}\left(1 - A(r)t^2 - \frac{A'(r)}{2}rt^2\right). \end{aligned} \tag{22}$$

We used Eq. (1) in the first line, Eqs. (4), (5), and (13) in the second line, and the explicit expression of R' derived from Eq. (6) in the third.

Now, we introduce the dimensionless functions and parameters to write Eq. (22) in dimensionless form. Let $a(x)$ be a function in $C^{\infty}[0, \infty)$ such that

$$a \begin{cases} (0) = 1 & (23) \\ \dfrac{da(0)}{dx} = 0 & (24) \\ \dfrac{d^2a(0)}{dx^2} = -1 & (25) \\ \dfrac{da(x)}{dx} \leq 0 & (26) \\ a(x) = \dfrac{m}{x^4} \quad (x \geq l). & (27) \end{cases}$$

In the above, m and l are dimensionless parameters that are related to the total mass of the dust and the dust cloud radius, respectively. These conditions imply that $a(x)$ is written as

$$a(x) = 1 - \frac{x^2}{2} + O(x^3). \tag{28}$$

Using $a(x)$, we write $A(r)$ as

$$A(r) = \alpha a\left(\sqrt{\frac{\beta}{\alpha}}r\right). \tag{29}$$

We can easily show that this $A(r)$ satisfies the conditions (15) and (16). In the above, α and β are non-zero positive parameters satisfying

$$\max_{0 \leq x \leq l} a(x)x^2 \equiv \eta < \frac{\beta}{\alpha^2}. \tag{30}$$

This condition comes from Eq. (9). In this way, $A(r)$ is parameterized by the two parameters α, β. From Eq. (14), the initial density $\rho(0, r)$ is also parameterized as

$$\rho(0, r) = \frac{3\alpha}{2r^3}\frac{d}{dr}\left(r^4 a\left(\sqrt{\frac{\beta}{\alpha}}r\right)\right). \tag{31}$$

In the following, the function $a(x)$ is given so that it satisfies Eqs. (23)–(27), and the initial density distribution is parameterized by α and β. Using α, β, and $a(x)$, Eq. (22) is rewritten as

$$\frac{dt}{dr} = \frac{1}{\sqrt{\left(1 - \alpha a(\sqrt{\frac{\beta}{\alpha}}r)r^2\right)\left(1 - \alpha a(\sqrt{\frac{\beta}{\alpha}}r)t^2\right)}}\left(1 - \alpha a\Big(\sqrt{\frac{\beta}{\alpha}}r\Big)t^2 - \frac{\alpha r t^2}{2}\frac{d}{dr}a\Big(\sqrt{\frac{\beta}{\alpha}}r\Big)\right). \tag{32}$$

Moreover, using dimensionless coordinates

$$x \equiv \sqrt{\frac{\beta}{\alpha}}r \tag{33}$$

and

$$\zeta \equiv \sqrt{\alpha}(t - t_{\rm S}(0)) = \sqrt{\alpha}\left(t - \frac{1}{\sqrt{\alpha}}\right), \tag{34}$$

we obtain the dimensionless equation for the null line

$$\frac{d\zeta}{dx} = \frac{1}{\sqrt{\gamma\left(1 - \frac{a(x)x^2}{\gamma}\right)\left(1 - a(x)(\zeta+1)^2\right)}}\left(1 - a(x)(\zeta+1)^2 - \frac{x}{2}(\zeta+1)^2\frac{d}{dx}a(x)\right), \tag{35}$$

where

$$\gamma \equiv \frac{\beta}{\alpha^2}. \tag{36}$$

Equation (30) is also rewritten as

$$\gamma > \eta. \tag{37}$$

Note here that the right-hand side of Eq. (35) is not a Lipschitz continuous function in the region that contains $x = \zeta = 0$. If this equation has a solution that starts from $x = \zeta = 0$, then, at least, the singularity is locally naked. Moreover, if the solution could extend to $x \to \infty$, the singularity would be visible at null infinity, that is, it would be globally naked. From now on we will ask if this differential equation has a solution. For convenience, we define the dimensionless coordinate θ as

$$\theta x^2 \equiv \zeta. \tag{38}$$

Then Eq. (35) becomes

$$\frac{d\theta}{dx} + \frac{2\theta}{x} = \frac{1}{x^2\sqrt{\gamma\left(1 - \frac{a(x)x^2}{\gamma}\right)\left(1 - a(x)(\theta x^2+1)^2\right)}}\left(1 - a(x)(\theta x^2+1)^2 - \frac{x}{2}(\theta x^2+1)^2\frac{d}{dx}a(x)\right). \tag{39}$$

In the current expression, from Eqs. (17), (18), and the relation $\theta = \frac{\sqrt{\alpha}t - 1}{x^2}$, we see that the singularity and the apparent horizon are located at

$$\theta_{\rm S}(x) = \frac{1}{x^2}\left(\frac{1}{\sqrt{a(x)}} - 1\right) \tag{40}$$

and

$$\theta_{\rm AH}(x;\gamma) = \frac{1}{x^2}\left(\frac{1}{\sqrt{a(x)}}\sqrt{1 - \frac{a(x)x^2}{\gamma}} - 1\right), \tag{41}$$

respectively, and

$$\theta_{\rm S}(0) = \frac{1}{4}, \tag{42}$$

$$\theta_{\rm AH}(0;\gamma) = \frac{1}{4} - \frac{1}{2\gamma}. \tag{43}$$

Note that, from the coordinate transformations (34) and (38), we see that the region $x = 0$ is singular for arbitrary θ. But any future-directed curve does not emanate from the central singularity located in $\theta < 0$ because any point in the region satisfying $x \neq 0$ and $\theta < 0$ is not in the future time slice of the central singularity. Then, for our purposes, we focus on the central singularity located in $\theta \geq 0$.

If the differential equation (39) has a C^1 solution $\theta(x)$ for $x \in [0, l]$, the multiplication of x with the right-hand side of Eq. (35) for its solution behaves around $x = 0$ as

$$\lim_{x\to 0} \frac{1}{x\sqrt{\gamma\left(1 - \frac{a(x)x^2}{\gamma}\right)\left(1 - a(x)(\theta x^2+1)^2\right)}} \left(1 - a(x)(\theta x^2+1)^2 - \frac{x}{2}(\theta x^2+1)^2 \frac{d}{dx}a(x)\right)$$
$$= \frac{1 - 2\theta(0)}{\sqrt{\gamma\left(\frac{1}{2} - 2\theta(0)\right)}}. \tag{44}$$

Thus, if

$$2\theta(0) = \frac{1 - 2\theta(0)}{\sqrt{\gamma\left(\frac{1}{2} - 2\theta(0)\right)}} \tag{45}$$

holds, then the first-order pole of Eq. (39) is cancelled out and does not appear. As with Christodoulou [12], let us introduce a real number λ satisfying

$$2\lambda = \frac{1 - 2\lambda}{\sqrt{\gamma\left(\frac{1}{2} - 2\lambda\right)}}. \tag{46}$$

Since γ is a positive real number [see below Eqs. (29) and (36)], we have

$$0 < \lambda < \frac{1}{4}. \tag{47}$$

In addition, since

$$\frac{d\sqrt{\gamma(\lambda)}}{d\lambda} = \frac{d}{d\lambda}\left(\frac{1 - 2\lambda}{2\lambda\sqrt{\left(\frac{1}{2} - 2\lambda\right)}}\right) = \frac{-4\lambda^2 + 6\lambda - 1}{4\lambda^2\left(\frac{1}{2} - 2\lambda\right)^{\frac{3}{2}}}, \tag{48}$$

we see

$$\min_{0<\lambda<\frac{1}{4}} \sqrt{\gamma(\lambda)} = \sqrt{\gamma\left(\lambda_M\right)} = \sqrt{11 + 5\sqrt{5}} \equiv \sqrt{\gamma_{\rm min}}, \tag{49}$$

where

$$\lambda_{\rm M} \equiv \frac{3 - \sqrt{5}}{4}, \tag{50}$$

that is, γ has the minimum at $\lambda = \lambda_M$. If $\gamma > \gamma_{\min}$ holds, we have the two solution to Eq. (46) for given γ, $\lambda_-(\gamma)$, and $\lambda_+(\gamma)$, satisfying $0 < \lambda_-(\gamma) < \lambda_M < \lambda_+(\gamma) < \frac{1}{4}$. If $\gamma = \gamma_{\min}$, Eq. (46) has the single solution $\lambda_\pm(\gamma) = \lambda_M$. If $\gamma < \gamma_{\min}$, Eq. (46) has no solution. From Eq. (46), it is easy to see that $\lambda_-(\gamma)$ and $\lambda_+(\gamma)$ satisfy $\lim_{\gamma\to\infty} \lambda_-(\gamma) = 0$ and $\lim_{\gamma\to\infty} \lambda_+(\gamma) = \frac{1}{4}$.

If λ satisfying Eq. (46) exists, we rewrite Eq. (39) as

$$\frac{d\theta}{dx} - 2\frac{(\lambda-\theta)}{x} = \frac{1 - a(x)(\theta x^2+1)^2 - \frac{x}{2}(\theta x^2+1)^2 \frac{d}{dx}a(x)}{x^2\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)\left(1-a(x)(\theta x^2+1)^2\right)}} - \frac{2\lambda}{x}$$
$$\equiv \lambda f(x, \theta; \lambda). \tag{51}$$

The formal solution to this equation is given by

$$\theta(x) = \lambda\left(1 + x\int_0^1 dv v^2 f(vx, \theta(vx); \lambda)\right)$$
$$\equiv T_\lambda(\theta)(x), \tag{52}$$

where T_λ is a nonlinear map on a functional space. Equations (51) and (52) imply that the fixed point of T_λ can be a solution to Eq. (51). Thus, if T_λ has a fixed point on a proper subset of C^0 on a proper domain, we can prove the existence of the solution satisfying (51), which emanates from the central singularity, and this means that the singularity is naked. In the next section, using the fixed-point theorem for a compact operator introduced soon, we examine the condition that the central singularity is naked.

3. Existence of null geodesics emanating from the central singularity

3.1. Preparation

In the four-dimensional case [12], the existence of null geodesics is shown by using the fixed-point theorem for contraction mapping [19]. In the five-dimensional case, however, we cannot use the same method (see Appendix B). Thus, we have to innovate.

First, we introduce a fixed-point theorem that is suitable for the current issue [20].

Theorem (Schauder fixed-point theorem [17]) Let D be a nonempty, closed, bounded, convex subset of a Banach space X, and suppose $T : D \to D$ is a compact operator. Then T has a fixed point.

In the above, a compact operator is defined as follows.

Definition (Compact operator) An operator T is compact if and only if:

(1) T is continuous.
(2) T maps a bounded set into a relatively compact set.

Here, “relatively compact” means that the closure is compact. Note that T does not have to be a linear operator.

We also use the Arzelà–Ascoli theorem to show the relative compactness of the image of T.

Theorem (Arzelà–Ascoli theorem) Let G be a nonempty open set in $\mathbb{R}^n$. A subset M in $C^0(\bar{G})$ is relatively compact if and only if the following two conditions hold:

(i) Uniform boundedness:

$$\sup_{f\in M} \left(\sup_{x\in\bar{G}} |f(x)| \right) < \infty. \tag{53}$$

(ii) Equicontinuity: For arbitrary $\epsilon > 0$, there exists $\delta > 0$, which depends only on ϵ, such that

$$\sup_{f\in M} |f(x) - f(y)| < \epsilon \tag{54}$$

for each $x, y \in \bar{G}$ satisfying $|x - y| < \delta$.

We apply the Schauder fixed-point theorem to the nonlinear operator T_λ defined in Eq. (52) and ask if the equation for the null line has a solution. First of all, we introduce a domain such that T_λ maps its domain into itself. So let us define

$$D_{\lambda,b,c,d} \equiv \{\theta \mid \theta \in C^0[0,d],\ |\theta - \lambda| \le bx^c\}, \tag{55}$$

where $b, c, d \in \mathbb{R}$ satisfy $b > 0$, $c > 0$, $l \ge d > 0$, and $\lambda + bx^c < \theta_S(x)$ for all x in $[0, l]$. The last inequality can always be satisfied for sufficiently small d because θ_S is a continuous function and $\lambda < \frac{1}{4} = \theta_S(0)$ always holds. We can control the maximum norm of the elements of $D_{\lambda,b,c,d}$ by parameter b and the speed of their convergence as $x \to 0$ by parameter c, respectively. Here we introduce the uniform norm $\|\theta\| \equiv \sup_{x\in[0,d]} \theta(x)$ such that $D_{\lambda,b,c,d}$ becomes a subset of a Banach space $C^0[0, d]$. Then we can show the following lemma.

Lemma 1 For all $\lambda < \frac{9-\sqrt{33}}{16}$, there exist $c(\lambda) \in (0, 1)$, which depend only on λ, and $\bar{d} \in (0, l]$, such that $T_\lambda : D_{\lambda,b,c,d} \to D_{\lambda,b,c,d}$ for all $c \in [c(\lambda), 1)$ and $d \in (0, \bar{d}]$.

Proof. For $\theta \in D_{\lambda,b,c,d}$, we estimate the right-hand side of Eq. (51) as

$$\begin{aligned} |\lambda f(x,\theta(x);\lambda)| &= \left| \frac{1 - a(x)(\theta(x)x^2+1)^2 - \frac{x}{2}(\theta(x)x^2+1)^2 \frac{d}{dx}a(x)}{x^2\sqrt{\gamma\left(1 - \frac{a(x)x^2}{\gamma}\right)\left(1 - a(x)(\theta(x)x^2+1)^2\right)}} - \frac{2\lambda}{x} \right| \\ &\le \frac{2\lambda}{\sqrt{\gamma}\left(\frac{1}{2} - 2\lambda\right)^{\frac{3}{2}}} bx^{c-1} + O(1) + O(x^{2c-1}), \end{aligned} \tag{56}$$

where we used (46) and (55) (the details are shown in Appendix A) and the terms $O(1)$ and $O(x^{2c-1})$ do not depend on θ. Then we see that

$$|T_\lambda(\theta) - \lambda| \le x \int_0^1 dvv^2 |\lambda f(vx, \theta(vx);\lambda)| = \frac{2\lambda}{\sqrt{\gamma}(c+2)\left(\frac{1}{2} - 2\lambda\right)^{\frac{3}{2}}} bx^c + O(x^{2c}) + O(x). \tag{57}$$

Hence, if

$$0 < c < 1 \tag{58}$$

and

$$\frac{2\lambda}{\sqrt{\gamma}(c+2)\left(\frac{1}{2}-2\lambda\right)^{\frac{3}{2}}} < 1 \tag{59}$$

hold and d is sufficiently small, the second and third terms in the right-hand side of Eq. (57) can be much smaller than the first term and ignorable, and $|T_\lambda(\theta)-\lambda| \le bx^c$ holds for all x in $[0,d]$. This means that there exists a positive number $\bar{d}$ such that $T_\lambda(\theta)$ is in $D_{\lambda,b,c,d}$ for arbitrary d in $(0,\bar{d}]$, that is, T_λ maps $D_{\lambda,b,c,d}$ into itself for such d.

There exists c such that Eqs. (58) and (59) hold if and only if

$$\lambda < \frac{9-\sqrt{33}}{16} \tag{60}$$

holds. In this case, Eqs. (58) and (59) imply

$$\frac{8\lambda^2}{(1-2\lambda)(1-4\lambda)} - 2 < c < 1, \tag{61}$$

and then we can take the parameter c in this range so as to satisfy (58) and (59). For example, $c(\lambda)$ in Lemma 1 is a number slightly larger than $\frac{8\lambda^2}{(1-2\lambda)(1-4\lambda)} - 2$. □

Remark: the restriction for c, $c < 1$, in Lemma 1 comes from the circumstance that one wants to control the matter initial distribution by Eq. (15) or (28).

From (50) and (60), for all $\lambda_-(\gamma)$, we can take some c satisfying Eqs. (58) and (59) because of $\lambda_-(\gamma) \le \lambda_M < \frac{9-\sqrt{33}}{16}$. Thus, we can choose c and d such that $T_\lambda : D_{\lambda,b,c,d} \to D_{\lambda,b,c,d}$ for all $\lambda_-(\gamma)$.

Next, we evaluate $T_\lambda(\theta)$ and show that it is uniformly continuous on $D_{\lambda,b,c,d}$.

Lemma 2 $T_\lambda : D_{\lambda,b,c,d} \to D_{\lambda,b,c,d}$ is uniformly continuous.

Proof. First, we evaluate the absolute value of the difference of the integrand in $T_\lambda(\theta)$ for different θ_1 and θ_2 in $D_{\lambda,b,c,d}$. For convenience, let us define

$$g_i \equiv 1 - a(x)\Big(\theta_i(x)x^2+1\Big)^2, \tag{62}$$

where the index i takes 1 or 2. Since we have $\theta_i(x) < \theta_S(x)$ from the definition (55), and Eq. (40) shows us $1 - a(x)(\theta_S(x)x^2+1)^2 = 0$, g_i is always positive. Then, we obtain

$$\begin{aligned}
&|\lambda f(x,\theta_1(x);\lambda) - \lambda f(x,\theta_2(x);\lambda)| \\
&= \left|1 + \frac{x\frac{d}{dx}a(x)}{2a(x)}\left(1+\frac{1}{\sqrt{g_1g_2}}\right)\right| \frac{\left|\sqrt{g_1}-\sqrt{g_2}\right|}{x^2\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}} \\
&\le \left\{1 + \left|\frac{x\frac{d}{dx}a(x)}{2a(x)}\right|\left(1+\frac{1}{\sqrt{g_1g_2}}\right)\right\}\left(\frac{1}{\sqrt{g_1}+\sqrt{g_2}}\right)\frac{|g_1-g_2|}{x^2\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}} \\
&\le \left\{1 - \frac{x\frac{d}{dx}a(x)}{2a(x)}\left(1+\frac{1}{1-a(x)\{(\lambda+bx^c)x^2+1\}^2}\right)\right\}
\end{aligned}$$

$$\times \frac{a(x)\{x^4(\lambda + bx^c) + x^2\}}{\sqrt{1 - a(x)\{(\lambda + bx^c)x^2 + 1\}^2}} \frac{|\theta_1(x) - \theta_2(x)|}{x^2\sqrt{\gamma\left(1 - \frac{a(x)x^2}{\gamma}\right)}}$$
$$= \left(\frac{4\lambda}{1-4\lambda}x^{-1} + B_1(x)x^{\delta-1}\right)|\theta_1(x) - \theta_2(x)|, \tag{63}$$

where $\delta = \min(1, c)$ and $B_1(x) \in C^0[0, d]$ is a positive function that does not depend on d, θ_1, or θ_2, and we used Eqs. (26) and (46) and the fact that

$$|g_1 - g_2| = |a(x)||(\theta_1(x)x^2 + 1)^2 - (\theta_2(x)x^2 + 1)^2|$$
$$\leq 2|a(x)||x^4(\lambda + bx^c) + x^2||\theta_1(x) - \theta_2(x)| \tag{64}$$

(see Appendix A for the details). Using (63), we obtain

$$\|T_\lambda(\theta_1) - T_\lambda(\theta_2)\| = \sup_{0\leq x\leq d}\left|x\int_0^1 v^2\Big\{\lambda f(vx, \theta_1(vx); \lambda) - \lambda f(vx, \theta_2(vx); \lambda)\Big\}dv\right|$$
$$\leq \sup_{0\leq x\leq d}\frac{1}{x^2}\int_0^x y^2\left(\frac{4\lambda}{1-4\lambda}y^{-1} + B_1(y)y^{\delta-1}\right)|\theta_1(y) - \theta_2(y)|dy$$
$$\leq \|\theta_1 - \theta_2\|\sup_{0\leq x\leq d}\frac{1}{x^2}\int_0^x\left(\frac{4\lambda}{1-4\lambda}y + B_1(y)y^{\delta+1}\right)dy$$
$$= \|\theta_1 - \theta_2\|\left(\frac{2\lambda}{1-4\lambda} + \sup_{0\leq x\leq d}\frac{1}{x^2}\int_0^x B_1(y)y^{\delta+1}dy\right). \tag{65}$$

The integral of the right-hand side of the inequality is finite because $B_1 \in C^0[0, d]$ and $\delta > 0$. Therefore, $T_\lambda : D_{\lambda,b,c,d} \to D_{\lambda,b,c,d}$ is uniformly continuous. □

Moreover, for specific initial conditions, T_λ becomes a contraction mapping. In this case, as below, we can immediately show the existence of a solution to Eq. (52).

THEOREM 2 For all $\lambda < \frac{1}{6}$, there exist $d \in (0, l]$ (l corresponds to the surface of the dust cloud) and a unique solution $\theta \in C^\infty(0, d]$ to the integral equation (52), which is continuous at $x = 0$ and satisfies $\theta(0) = \lambda$.

Proof. The last term of (65) is estimated as

$$\frac{1}{x^2}\int_0^x B_1(y)y^{\delta+1}dy = O(x^\delta). \tag{66}$$

Here, if

$$\frac{2\lambda}{1-4\lambda} < 1, \tag{67}$$

we can choose sufficiently small d such that

$$\frac{2\lambda}{1-4\lambda} + \sup_{0\leq x\leq d}\frac{1}{x^2}\int_0^x B_1(y)y^{\delta+1}dy < 1, \tag{68}$$

that is, $T_\lambda : D_{\lambda,b,c,d} \to D_{\lambda,b,c,d}$ is contractive. Thus, by the fixed-point theorem for contraction mapping [19], T_λ has a unique fixed point $\theta \in D_{\lambda,b,c,d}$. Note that the condition (67) is equivalent to $\lambda < \frac{1}{6}$. □

Since the integrand of the right-hand side of Eq. (52) is a C^∞ function in the region except for the singularity, the solution θ must be a C^∞ function in $(0, d]$. On the other hand, the solution $\theta(x) \in C^0[0, d]$ is not differentiable at $x = 0$ in general. However, $\zeta(x) = x^2\theta(x)$ is in $C^1[0, d]$ because $\zeta(x) \in C^\infty(0, d]$, and $\frac{d}{dx}\zeta(x)$ is $O(x)$ from (35). This means that the differential equation of the null line (35) has a solution $\zeta(x) \in C^1[0, d]$ which is a future-directed outgoing null geodesic emanating from the central singularity for all $\lambda < \frac{1}{6}$.

For $\gamma > 24$, Eq. (46) tells us $\lambda_-(\gamma) < \frac{1}{6}$. Then, in this case, there exists a null line emanating from the central singularity, that is, it is a locally naked singularity at least.

3.2. A proof of the existence of the null geodesics

In the case of $\gamma_{\min} \le \gamma \le 24$, that is, $\frac{1}{6} \le \lambda_-(\gamma)$ holds, we cannot use Theorem 2 to show the existence of a solution to Eq. (52). Then we have to develop another method. As we mentioned already, we can show the existence of a solution to Eq. (52) by using the Schauder fixed-point theorem.

Theorem 3 For all $\lambda < \frac{9-\sqrt{33}}{16}$, there exist $d \in (0, l]$ and a solution $\theta \in C^\infty(0, d]$ to the integral equation (52), which is continuous at $x = 0$ and satisfies $\theta(0) = \lambda$.

Proof. From Schauder fixed-point theorem, if $D_{\lambda,b,c,d}$ is a nonempty, closed, bounded, convex subset of a Banach space, and T_λ maps $D_{\lambda,b,c,d}$ into itself and is a compact operator, then T_λ has a fixed point. We already showed in Lemma 1 that we can take certain numbers $d \in (0, l]$ and $c \in (0, 1)$ such that T_λ maps $D_{\lambda,b,c,d}$ into itself for all $\lambda < \frac{9-\sqrt{33}}{16}$. Moreover, Lemma 2 tells us that T_λ is continuous. By the definition (55), $D_{\lambda,b,c,d}$ is obviously a nonempty, closed, bounded subset of a Banach space $C^0[0, d]$. Furthermore, for all $\theta_1, \theta_2 \in D_{\lambda,b,c,d}$ and $0 < \kappa < 1$,

$$\lambda - bx^c \le \kappa\theta_1(x) + (1-\kappa)\theta_2(x) \le \lambda + bx^c. \tag{69}$$

This means that $D_{\lambda,b,c,d}$ is convex. Then all we have to show is that $T_\lambda(D_{\lambda,b,c,d})$ is a relatively compact set. By virtue of the Arzelà–Ascoli theorem, the remaining task is to show uniform boundedness and equicontinuity of $T_\lambda(D_{\lambda,b,c,d})$. Uniform boundedness results from the boundedness of $D_{\lambda,b,c,d}$ as follows:

$$\begin{aligned}\sup_{T_\lambda(\theta)\in T_\lambda(D_{\lambda,b,c,d})} \left(\sup_{x\in[0,d]} |T_\lambda(\theta)(x)| \right) &\le \sup_{T_\lambda(\theta)\in T_\lambda(D_{\lambda,b,c,d})} \left(\sup_{x\in[0,d]} \lambda + bx^c \right) \\ &= \sup_{T_\lambda(\theta)\in T_\lambda(D_{\lambda,b,c,d})} (\lambda + bd^c) \\ &= \lambda + bd^c. \end{aligned} \tag{70}$$

Next, to show equicontinuity, we evaluate

$$\begin{aligned}|T_\lambda(\theta)(x) - T_\lambda(\theta)(y)| &= \left| \frac{1}{x^2}\int_0^x s^2\lambda f(s,\theta(s);\lambda)ds - \frac{1}{y^2}\int_0^y s^2\lambda f(s,\theta(s);\lambda)ds \right| \\ &= \left| \left(\frac{1}{x^2} - \frac{1}{y^2}\right)\int_0^x s^2\lambda f(s,\theta(s);\lambda)ds - \frac{1}{y^2}\int_x^y s^2\lambda f(s,\theta(s);\lambda)ds \right|. \end{aligned} \tag{71}$$

Here, note that we can assume $x < y$ without loss of generality. Then, using Eq. (56),

$$|T_\lambda(\theta)(x) - T_\lambda(\theta)(y)| \leq \left(\frac{1}{x^2} - \frac{1}{y^2}\right)\int_0^x s^2|\lambda f(s,\theta(s);\lambda)|ds + \frac{1}{y^2}\int_x^y s^2|\lambda f(s,\theta(s);\lambda)|ds$$
$$\leq h(x)|x^c - y^c| + y^c|h(x) - h(y)| + 2^{n+1}h(x)y^{c-\frac{1}{2^{n-1}}}\left|x^{\frac{1}{2^{n-1}}} - y^{\frac{1}{2^{n-1}}}\right|, \tag{72}$$

where $h(x)$ is a C^0 positive function in the range $[0, d]$ that does not depend on θ, and n is an arbitrary natural number (the details of the calculation are in Appendix A). So we take n such that

$$c - \frac{1}{2^{n-1}} > 0. \tag{73}$$

Then, since $h(x)$, y^c, and $2^{n+1}h(x)y^{c-\frac{1}{2^{n-1}}}$ are continuous functions for (x, y) in $[0, d] \times [0, d]$, there exist real numbers Δ_1, Δ_2, and Δ_3 such that

$$|T_\lambda(\theta)(x) - T_\lambda(\theta)(y)| < \Delta_1|x^c - y^c| + \Delta_2|h(x) - h(y)| + \Delta_3\left|x^{\frac{1}{2^{n-1}}} - y^{\frac{1}{2^{n-1}}}\right|. \tag{74}$$

Since all continuous functions on a compact set are uniformly continuous, for all $\epsilon > 0$ there exist $\delta_1 > 0$, $\delta_2 > 0$, and $\delta_3 > 0$, independent of x and y, such that $\Delta_1|x^c - y^c| < \frac{\epsilon}{3}$, $\Delta_2|h(x) - h(y)| < \frac{\epsilon}{3}$, $\Delta_3\left|x^{\frac{1}{2^{n-1}}} - y^{\frac{1}{2^{n-1}}}\right| < \frac{\epsilon}{3}$ hold whenever $|x - y| < \delta_1$, $|x - y| < \delta_2$, $|x - y| < \delta_3$, respectively. Thus, $|T_\lambda(\theta)(x) - T_\lambda(\theta)(y)| < \epsilon$ holds whenever $|x - y| < \delta \equiv \min\{\delta_1, \delta_2, \delta_3\}$ holds. This means that $T_\lambda(D_{\lambda,b,c,d})$ is equicontinuous. Therefore, $T_\lambda(D_{\lambda,b,c,d})$ is a relatively compact set. Since any closed set included in a compact set is also compact, T maps any bounded set into a relatively compact set. Thus T is a compact operator. □

In the same way as the discussion below Theorem 2, Theorem 3 means that the differential equation for null line (35) has a solution $\zeta \in C^1[0, d]$ which is a future-directed outgoing null geodesic emanating from the central singularity for all $\lambda < \frac{9-\sqrt{33}}{16}$. Thus, in the following, if the function that is a solution to Eq. (51) converges to a finite value as $x \to 0$, we consider the function as a solution to Eq. (51) that is also defined at $x = 0$. Note that the solution found in Theorem 2 is unique, but it is not necessary that the solution found in Theorem 3 is unique.

If λ satisfying Eq. (46) exists, $\lambda_-(\gamma)$ always satisfies $\lambda_-(\gamma) \leq \lambda_M < \frac{9-\sqrt{33}}{16}$. From Theorem 3, this fact means that $T_{\lambda_-(\gamma)}$ has a fixed point which converges to $\lambda_-(\gamma)$ as $x \to 0$, that is, there exists a null geodesic emanating from the central singularity. Thus, the central singularity must be locally naked at least if λ satisfying Eq. (46) exists.

4. Spacetime structure around the singularity

In this section, we show the existence of the earliest null geodesic θ_{n_0} emanating from the central singularity for all $\gamma \geq \gamma_{\min} = 11 + 5\sqrt{5}$. Since such a null geodesic determines the causal structure around the naked singularity and the global nakedness of the singularity, θ_{n_0} plays an important role in our analysis. On the other hand, for $\gamma < \gamma_{\min}$, we also show that there is no causal geodesic emanating from the central singularity.

In the case of four dimensions, we can show that g_{rr} of the LTB spacetime is a strictly monotonically decreasing function with respect to t near the singularity. Using this nature, we can immediately specify θ_{n_0} (see Ref. [12]). By contrast, in the case of five dimensions, g_{rr} becomes a monotonically increasing function with respect to t near the singularity. Therefore, we need to develop another method to specify θ_{n_0} that is general to some extent.

First, we show that any future-directed null geodesic along the outer radial direction cannot emanate from the central singularity located at $\theta < \lambda_-(\gamma)$.

LEMMA 3 If λ satisfying Eq. (46) exists and a future-directed null geodesic along the outer radial direction, $\theta(x)$, converges as $x \to 0$, then $\theta(0) \geq \lambda_-(\gamma)$ holds. On the other hand, if λ satisfying Eq. (46) does not exist, any future-directed outgoing causal line, $\theta_c(x)$, satisfies $\theta_c(x) \to -\infty$ as $x \to 0$.

Proof. Let us assume that a solution to Eq. (51), $\theta(x)$, is a C^0 function in the range $(0, d]$ and is a bounded function. Bearing Eq. (44) in mind, Eq. (51) for $\theta(x)$ can be written as

$$\begin{aligned}\frac{d\theta(x)}{dx} &= \frac{1 - a(x)(\theta(x)x^2+1)^2 - \frac{x}{2}(\theta(x)x^2+1)^2\frac{d}{dx}a(x)}{\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)\left(1-a(x)(\theta(x)x^2+1)^2\right)}} - \frac{2\theta(x)}{x}\\ &= \frac{1-2\theta(x)+O(x)}{x\sqrt{\gamma\left(\frac{1}{2}-2\theta(x)+O(x)\right)}} - \frac{2\theta(x)}{x}.\end{aligned} \tag{75}$$

Moreover, if the solution $\theta(x)$ converges as $x \to 0$, then, for all $\epsilon > 0$, there exists $d_0 > 0$ such that

$$\begin{aligned}\frac{1-2\theta(x)+O(x)}{x\sqrt{\gamma\left(\frac{1}{2}-2\theta(x)+O(x)\right)}} - \frac{2\theta(x)}{x} &> \frac{1-2\theta(0)}{x\sqrt{\gamma\left(\frac{1}{2}-2\theta(0)\right)}} - \frac{2\theta(0)}{x} - \frac{\epsilon}{x}\\ &\equiv \frac{g(\theta(0);\gamma)}{x} - \frac{\epsilon}{x}\end{aligned} \tag{76}$$

for arbitrary $x \in (0, d_0]$, where

$$g(\theta;\gamma) \equiv \frac{1-2\theta}{\sqrt{\gamma(\frac{1}{2}-2\theta)}} - 2\theta. \tag{77}$$

In the case that λ satisfying Eq. (46) exists, $g(\theta;\gamma)$ satisfies

$$\begin{cases} g(\lambda_\pm(\gamma);\gamma) = 0 & \\ g(\theta;\gamma) < 0 & \lambda_-(\gamma) < \theta < \lambda_+(\gamma) \\ g(\theta;\gamma) > 0 & \theta < \lambda_-(\gamma), \lambda_+(\gamma) < \theta < \frac{1}{4}. \end{cases} \tag{78}$$

In the case that $\lambda_\pm(\gamma)$ does not exist, for all $\theta < \frac{1}{4}$,

$$g(\theta;\gamma) > 0 \tag{79}$$

and $\inf_{\theta<\frac{1}{4}} g(\theta;\gamma) > 0$ hold.

Now we assume that λ satisfying Eq. (46) exists, and a solution to Eq. (51), $\theta(x)$, satisfies $\theta(0) < \lambda_-(\gamma)$. In this case, we can choose ϵ such that

$$g(\theta(0);\gamma) - \epsilon = \kappa_0^2, \tag{80}$$

where κ_0 is a nonzero real number. Then, if the solution $\theta(x)$ converges as $x \to 0$, from Eqs. (75), (76), and (80), there exists $d_0 > 0$ such that

$$\frac{d\theta(x)}{dx} > \frac{\kappa_0^2}{x} \tag{81}$$

for arbitrary $x \in (0, d_0]$. Integrating this equation, we obtain

$$\theta(x) - \theta(0) > \lim_{x_0 \to 0} \kappa_0^2 \Big[\ln x \Big]_{x_0}^{x}. \tag{82}$$

In the above, the right-hand side diverges. This contradicts the assumption that $\theta(x)$ is bounded. Thus, $\theta(0) \geq \lambda_-(\gamma)$, that is, any future-directed null geodesic along the outer radial direction does not converge to the central singularity that satisfies $\theta < \lambda_-(\gamma)$.

On the other hand, in the case that λ satisfying Eq. (46) does not exist, Eqs. (75) and (79) hold for any bounded solution $\theta(x)$ satisfying $\theta(x) < \frac{1}{4}$. In addition, since any future-directed null geodesics cannot emerge from the apparent horizon determined by Eq. (41), any solution that can extend $x \to 0$ must not enter the region $\theta > \theta_{\mathrm{AH}}(x;\gamma)$ as $x \to 0$. Then, there exist $d_1 > 0$ and $\mu > 0$ such that the solution satisfies $\theta(x) < \frac{1}{4} - \mu$ for all $x \in [0, d_1]$, because $\theta_{\mathrm{AH}}(x;\gamma)$ is continuous and $\theta_{\mathrm{AH}}(0;\gamma) = \frac{1}{4} - \frac{1}{2\gamma}$ holds. In this region, $g(\theta;\lambda)$ is defined. Then, for all $\epsilon > 0$, there exists d_2 satisfying $d_1 \geq d_2 > 0$ such that

$$\begin{aligned} \frac{d\theta(x)}{dx} &= \frac{1 - 2\theta(x) + O(x)}{x\sqrt{\gamma\left(\frac{1}{2} - 2\theta(x) + O(x)\right)}} - \frac{2\theta(x)}{x} > \frac{g(\theta(x);\gamma)}{x} - \frac{\epsilon}{x} \\ &\geq \frac{\inf_{\theta<\frac{1}{4}} g(\theta;\gamma)}{x} - \frac{\epsilon}{x} \end{aligned} \tag{83}$$

for arbitrary $x \in (0, d_2]$. Note that we do not assume the solution $\theta(x)$ converges as $x \to 0$ here. Since λ satisfying Eq. (46) does not exist, $\inf_{\theta<\frac{1}{4}} g(\theta;\gamma) > 0$. Then we can choose ϵ such as

$$\inf_{\theta<\frac{1}{4}} g(\theta;\gamma) - \epsilon = \kappa_1^2, \tag{84}$$

where κ_1 is a nonzero real number. Thus, in the same way as the case that λ satisfying Eq. (46) exists, we can show that any solution $\theta(x)$ cannot be bounded. Therefore, any future-directed null geodesic along the outer radial direction must diverge to $-\infty$ as $x \to 0$. Since the future-directed null geodesic along the outer radial direction is obviously the earliest line that emerged from $x = 0$ at arbitrary time, any future-directed outgoing causal line must diverge to $-\infty$ at $x = 0$. □

Thus, for the case that λ satisfying Eq. (46) *does not exist*, there is no causal line which emanates from the central singularity, that is, strong cosmic censorship holds. In contrast, for the case that λ satisfying Eq. (46) *exists*, we have just discussed converged null geodesics and have not yet shown anything about other causal lines that emanate from the central singularity. To address this point, we will first present Lemmas 4 and 5.

LEMMA 4 Let $\theta(x)$ be a future-directed outgoing null geodesic along the radial direction that oscillates as $x \to 0$. If λ satisfying Eq. (46) exists, then $\underline{\lim}_{x\to 0}\, \theta(x) \geq \lambda_-(\gamma)$ holds.

Proof. We assume that λ satisfying Eq. (46) exists and a future-directed outgoing null geodesic along the radial direction, $\theta(x)$, is a C^0 function in the range $(0, d]$ and oscillates as $x \to 0$. If $\underline{\lim}_{x\to 0}\,\theta(x) < \lambda_-(\gamma)$ holds, we can choose $\mu_1, \mu_2 \in \mathbb{R}$ such that

$$\underline{\lim}_{x\to 0}\,\theta(x) < \mu_1 < \mu_2 < \min\{\overline{\lim}_{x\to 0}\,\theta(x), \lambda_-(\gamma)\}. \tag{85}$$

Here let us define

$$\tilde{\theta}(x) = \begin{cases} \mu_1 & \theta(x) \le \mu_1 \\ \theta(x) & \mu_1 < \theta(x) < \mu_2 \\ \mu_2 & \mu_2 \le \theta(x). \end{cases} \tag{86}$$

It is easy to show that $\tilde{\theta}$ is a C^0 bounded function in the range $(0, d]$ and satisfies $\underline{\lim}_{x\to 0}\,\tilde{\theta}(x) = \mu_1$, $\overline{\lim}_{x\to 0}\,\tilde{\theta}(x) = \mu_2$, and

$$\frac{d\tilde{\theta}}{dx}(x) = \begin{cases} 0 & \theta(x) < \mu_1 \\ \dfrac{1-a(x)(\theta(x)x^2+1)^2 - \frac{x}{2}(\theta(x)x^2+1)^2\frac{d}{dx}a(x)}{\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)\left(1-a(x)(\theta(x)x^2+1)^2\right)}} - \dfrac{2\theta(x)}{x} & \mu_1 < \theta(x) < \mu_2 \\ 0 & \mu_2 < \theta(x). \end{cases} \tag{87}$$

For all x which satisfy $\mu_1 < \theta(x) < \mu_2 < \lambda_-(\gamma) < \frac{1}{4}$, the function $g(\theta(x);\gamma)$ exists as a real function. Then, as with the proof of Lemma 3, for all $\epsilon > 0$, there exists a $d_0 > 0$ such that

$$\begin{aligned} \frac{1-a(x)(\theta(x)x^2+1)^2 - \frac{x}{2}(\theta(x)x^2+1)^2\frac{d}{dx}a(x)}{\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)\left(1-a(x)(\theta(x)x^2+1)^2\right)}} - \frac{2\theta(x)}{x} &> \frac{g(\theta(x);\gamma)}{x} - \frac{\epsilon}{x} \\ &\ge \frac{\inf_{\mu_1<\theta<\mu_2} g(\theta;\gamma)}{x} - \frac{\epsilon}{x} \end{aligned} \tag{88}$$

for arbitrary $x \in (0, d_0]$, which satisfy $\mu_1 < \theta(x) < \mu_2$. Here, $\inf_{\mu_1<\theta<\mu_2} g(\theta;\gamma) > 0$ because of $\mu_2 < \lambda_-(\gamma)$. Then we can choose ϵ such that

$$\inf_{\mu_1<\theta<\mu_2} g(\theta;\gamma) - \epsilon = \kappa^2, \tag{89}$$

where κ is a real number. Equations (87), (88), and (89) imply

$$\frac{d\tilde{\theta}}{dx}(x) \ge \begin{cases} 0 & \theta(x) < \mu_1 \\ \frac{\kappa^2}{x} & \mu_1 < \theta(x) < \mu_2 \\ 0 & \mu_2 < \theta(x) \end{cases} \tag{90}$$

for arbitrary $x \in (0, d_0]$. Thus, from Eqs. (86) and (90), $\tilde{\theta}(x)$ is a monotonically increasing function. This contradicts the assumption that $\theta(x)$ oscillates as $x \to 0$. Therefore, the solution $\theta(x)$ must satisfy $\underline{\lim}_{x\to 0}\,\theta(x) \ge \lambda_-(\gamma)$. □

Thus, from Lemmas 3 and 4, we conclude that any future-directed outgoing null geodesic along the radial direction, $\theta(x)$, satisfies (i) $\underline{\lim}_{x\to 0}\,\theta(x) \ge \lambda_-(\gamma)$, or (ii) $\theta(x)$ diverges to $-\infty$ as $x \to 0$ if λ satisfying Eq. (46) exists. However, unlike the case when λ satisfying Eq. (46) does not exist, it does not immediately mean that any future-directed outgoing causal line comes to satisfy $\theta \geq \lambda_-(\gamma)$ or $\theta \to -\infty$ as $x \to 0$, because a null line that converges to the central singularity exists in this case. Hence, we have to carefully examine the geodesics in this case.

For $d > 0$, let us define $G_{\lambda_-(\gamma),d} \subset C^0[0,d]$ as the set of the solutions to Eq. (51) for γ that converge to $\lambda_-(\gamma)$ as $x \to 0$ and do not enter the singularity at a point in $(0,d]$. Then, we can show the following lemma for $G_{\lambda_-(\gamma),d}$.

LEMMA 5 If λ satisfying Eq. (46) exists, then there exists a solution to Eq. (51), $\theta_{n_0}(x;\gamma)$, such that $\theta_{n_0}(x;\gamma) \leq \theta(x)$ for all $\theta \in G_{\lambda_-(\gamma),d}$ and $x \in [0,d]$, where d is an arbitrary positive number. Moreover, if the curve $\theta = \theta_{n_0}(x;\gamma)$ enters the singularity at $x = d_1 > 0$, $G_{\lambda_-(\gamma),d}$ is empty for all d satisfying $d \geq d_1$.

Proof. We suppose that λ satisfying Eq. (46) exists. Let us define

$$G_{\lambda_-(\gamma),d}(x) \equiv \{\theta(x) | \theta \in G_{\lambda_-(\gamma),d}\}, \tag{91}$$

where $x \leq d$. By Theorem 3, there exists $d_0 > 0$ such that $G_{\lambda_-(\gamma),d_0}(d_0)$ is not empty. If a solution to Eq. (51) exists in $[0,d_0]$, then, for arbitrary $d \in (0,d_0]$, this solution also exists in $[0,d]$. Then, $G_{\lambda_-(\gamma),d}(x)$ is not empty for all $d \in (0,d_0]$ and $x \in (0,d]$.

Now we suppose that $G_{\lambda_-(\gamma),d_0}(x)$ is not empty for all $x \in (0,d_0]$. Since the right-hand side of Eq. (51) satisfies the Lifshitz condition on an arbitrary closed set that does not contain the singularity, all solutions to Eq. (51) do not intersect each other and can extend arbitrarily in any open set that does not contain the singularity. This fact means that the ordering of the solution orbits with respect to the coordinate θ is conserved.

Let us define $\theta(x;x_0,\theta_0)$ as the solution to Eq. (51) that passes through (x_0,θ_0). Then, for $d_0 \geq x_0 > 0$, Lemmas 3 and 4 and the above discussion tell us that $\theta(x;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0))$ must converge to $\lambda_-(\gamma)$ or diverge to $-\infty$ as $x \to 0$, because $\theta(x;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0))$ cannot intersect any element of $G_{\lambda_-(\gamma),d}$. Let us assume that $\theta(x;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0))$ diverges to $-\infty$ as $x \to 0$. Then, there exists $0 < x_1 < x_0$ such that $\theta(x_1;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0)) < 0$ holds. On such x_1, there exists a future-directed solution to Eq. (51), $\theta_1(x)$, such that $\theta(x_1;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0)) < \theta_1(x_1) < 0$. Since the region of $\theta < 0$ is not a future of the slice $\theta = 0$, Lemmas 3 and 4 imply that $\theta_1(x)$ must diverge to $-\infty$ as $x \to 0$, that is, $\theta_1(x) \notin G_{\lambda_-(\gamma),d_0}$. Since the ordering of the solution orbits with respect to the coordinate θ is conserved, $\theta(x;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0)) < \theta_1(x)$ holds for an arbitrary point in the domain of θ_1. On the other hand, from the definition of $\theta(x;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0))$, for arbitrary $\epsilon > 0$, there exits $\theta_\epsilon(x) \in G_{\lambda_-(\gamma),d_0}$ such that $0 \leq \theta_\epsilon(x_0) - \theta(x_0;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0)) < \epsilon$ holds. $\theta_\epsilon(x)$ also satisfies $\theta_\epsilon(x) \geq \theta(x;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0))$ for arbitrary $x \in (0,x_0]$.

Let us assume that $\theta_1(x)$ enters the singularity at $x = x_S(\theta_1)$ in $(x_1,x_0]$. Since $\theta_\epsilon(x) \in G_{\lambda_-(\gamma),d_0}$, $\theta_\epsilon(x)$ satisfies $0 < \theta_\epsilon(x) < \theta_S(x)$ for all $x \in (0,x_0]$. Then $\lim_{x \to x_S(\theta_1)} \theta_1(x) = \theta_S(x_S(\theta_1)) > \theta_\epsilon(x_S(\theta_1))$ and $\theta_\epsilon(x_1) > 0 > \theta_1(x_1)$ hold, that is, $\theta_1(x)$ and $\theta_\epsilon(x)$ intersect at a point in $(x_1,x_S(\theta_1))$. This contradicts the fact that the right-hand side of Eq. (51) satisfies the Lifshitz condition. Then $\theta_1(x)$ does not enter the singularity in the range $(x_1,x_0]$. In this case, there exists $\epsilon' > 0$ such that $\theta(x_0;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0)) \leq \theta_{\epsilon'}(x_0) < \theta_1(x_0)$ holds. Since $\theta_{\epsilon'}(x_1) > 0 > \theta_1(x_1)$ holds, this means that $\theta_1(x)$ and $\theta_{\epsilon'}(x)$ intersect at a point in (x_1,x_0), and this leads to a contradiction. Thus, $\theta(x;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0))$ must converge to $\lambda_-(\gamma)$ as $x \to 0$, that is, $\theta(x;x_0,\inf G_{\lambda_-(\gamma),d_0}(x_0)) \in G_{\lambda_-(\gamma),d_0}$.

Since the ordering of the solution orbits with respect to the coordinate θ is conserved and the solutions can extend arbitrarily in any open set that does not contain the singular points, for arbitrary d, $\theta(x; x_0, \inf G_{\lambda_-(\gamma),d_0}(x_0))$ must satisfy $\theta_G(x) \geq \theta(x; x_0, \inf G_{\lambda_-(\gamma),d_0}(x_0))$ for arbitrary $\theta_G \in G_{\lambda_-(\gamma),d}$ and $x \in [0, d]$. This means that $\theta(x; x_0, \inf G_{\lambda_-(\gamma),d_0}(x_0))$ must be $\theta_{n_0}(x; \gamma)$. Moreover, if the curve $\theta = \theta_{n_0}(x; \gamma)$ enters the singularity line $\theta = \theta_S(x)$ at some $x = d_1 > 0$, then all other solution lines that emanate from the singularity are surrounded by the curve $\theta = \theta_{n_0}(x; \gamma)$ and $\theta = \theta_S(x)$. Since $\theta_{n_0}(x; \gamma)$ and any other solution line do not intersect each other except at the singularity, all solutions must intersect with the singularity at a point in $(0, d_1]$. Then $G_{\lambda_-(\gamma),d}$ is empty for $d \geq d_1$. □

Lemmas 3, 4, and 5 imply the following theorem.

Theorem 4 (i) If λ satisfying Eq. (46) exists, then $\theta_{n_0}(x; \gamma)$ defined in Lemma‘5 exists and is the earliest of all future-directed causal lines emanating from the central singularity. $\theta_{n_0}(x; \gamma)$ converges to $\lambda_-(\gamma)$ as $x \to 0$.

(ii) If λ satisfying Eq. (46) does not exist, then strong cosmic censorship holds.

Furthermore, (i) and (ii) mean that λ satisfying Eq. (46) exists if and only if the central singularity is naked.

Proof. We have already shown (ii) below Lemma 3. Then we will focus on (i). We suppose that a future-directed causal line $\theta_c(x)$ satisfies $\theta_c(x_0) < \theta_{n_0}(x_0, \gamma)$ at a point $x = x_0 > 0$. From Lemmas 3, 4, and 5, $\theta(x; x_0, \theta_c(x_0))$ must diverge to $-\infty$ as $x \to 0$. Since $\theta(x; x_0, \theta_c(x_0))$ corresponds to a future-directed outgoing null geodesic along the radial direction, $\theta_c(x)$ also diverges to $-\infty$ as $x \to 0$, that is, the line with $\theta_c(x)$ must emanate from the regular center. Then there is no future-directed causal line that emanates from the central singularity before $\theta_{n_0}(x; \gamma)$. □

From Theorem 4, if $\theta_{n_0}(x; \gamma)$ can extend to $x = l$ and $\theta_{n_0}(l; \gamma) < \theta_{AH}(l; \gamma)$ holds, the central singularity must be globally naked. Using this fact, in the next section we consider the global structure of this spacetime.

5. Global spacetime structure and the globally naked singularity

In this section, we consider global properties of a singularity. We will see the dependence of the nakedness of the central singularity on the initial density distribution characterized by γ and $a(x)$ (see Eq. (31) for the definitions). The discussion in this section is similar to the four-dimensional case [12].

Lemma 6 For any initial density distribution parameterized as (31), there exists γ_0 such that the solution $\theta_{n_0}(x; \gamma)$ can extend to $x = l$ (corresponding to the surface of the dust cloud) and $\theta_{n_0}(l; \gamma) < \theta_{AH}(l; \gamma)$ holds for all $\gamma \in [\gamma_0, \infty]$. θ_{n_0} is defined in Lemma 5.

Proof. Since the outer region of the $x = l$ surface is the Schwarzschild spacetime and the event horizon is identical to the apparent horizon in the Schwarzschild spacetime, $\theta_{n_0}(l; \gamma) < \theta_{AH}(l; \gamma)$ means that the null line corresponding to $\theta_{n_0}(x; \gamma)$ arrives at the outer region of the event horizon of the Schwarzschild spacetime, that is, the null line $\theta_{n_0}(x; \gamma)$ will attain the future null infinity and then the central singularity is globally naked.

To prove this lemma, by virtue of Theorem 3, it is enough to show that, for sufficiently large γ, there exist b and c such that (i) $\lambda_-(\gamma)+bx^c < \theta_S(x)$ holds for all x in $[0,l]$; (ii) $|T_{\lambda_-(\gamma)}(\theta)(x)-\lambda_-(\gamma)| \le bx^c$ holds for all x in $[0,l]$ and all θ in $C^0[0,l]$ that satisfy $|\theta(x)-\lambda_-(\gamma)| \le bx^c$; and (iii) $\theta_{n_0}(l;\gamma) < \theta_{AH}(l;\gamma)$ holds (see the proof of Theorem 3 for the details).

The conditions (23), (25), and (26) imply $a(x) < 1$ for all x in $(0,l]$. Since θ_S is continuous in the range $[0,l]$, we see

$$\min_{0\le x\le l} \theta_S(x) = \min_{0\le x\le l} \frac{1}{x^2}\left(\frac{1}{\sqrt{a(x)}}-1\right) \equiv \theta_{S,\min} > 0. \tag{92}$$

Since $\lambda_-(\gamma)$ is a monotonically decreasing function such that it satisfies $\lim_{\gamma\to\infty}\lambda_-(\gamma)=0$, there exists γ_1 such that $\lambda_-(\gamma) < \theta_{S,\min}$ holds for all $\gamma \in [\gamma_1,\infty)$. Then we can take a positive real number $b(\gamma_1)$ such that $\lambda_-(\gamma_1)+b(\gamma_1)l^c < \theta_{S,\min}$ holds. For such $b(\gamma_1)$,

$$\lambda_-(\gamma)+b(\gamma_1)x^c \le \lambda_-(\gamma_1)+b(\gamma_1)l^c < \theta_{S,\min} \le \theta_S(x) \tag{93}$$

holds for all γ in $[\gamma_1,\infty)$. This means that the condition (i) for all γ in $[\gamma_1,\infty)$ holds for this $b(\gamma_1)$.

Next, we confirm that the condition (ii) holds for sufficiently large γ. For $\theta \in D_{\lambda_-(\gamma),b(\gamma_1),c,l}$, we evaluate

$$\begin{aligned}
|\lambda_-(\gamma)f(x,\theta(x);\lambda_-(\gamma))| &\le \left|\frac{1-a(x)(\theta(x)x^2+1)^2-\frac{x}{2}(\theta(x)x^2+1)^2\frac{d}{dx}a(x)}{x^2\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)\left(1-a(x)(\theta(x)x^2+1)^2\right)}}-\frac{2\lambda_-(\gamma)}{x}\right| \\
&= \left|\frac{C_3(x,\theta(x))-2\lambda_-(\gamma)\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}\sqrt{C_4(x,\theta(x))}}{x\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)C_4(x,\theta(x))}}\right| \\
&\le \frac{F_1(x,\theta(x),\lambda_-(\gamma))|\theta(x)-\lambda_-(\gamma)|+xF_2(x,\theta(x),\lambda_-(\gamma))}{\sqrt{\gamma}xF_3(x,\theta,\lambda_-(\gamma))},
\end{aligned} \tag{94}$$

where

$$\begin{aligned}
x^2C_3(x,\theta) &\equiv x^2-2\theta x^2+x^3C_1(x,\theta) \\
&\equiv 1-a(x)(\theta x^2+1)^2-\frac{x}{2}(\theta x^2+1)^2\frac{d}{dx}a(x),
\end{aligned} \tag{95}$$

$$x^2C_4(x,\theta) \equiv \frac{x^2}{2}-2\theta x^2+x^3C_2(x,\theta) \equiv 1-a(x)(\theta x^2+1)^2, \tag{96}$$

$$F_1(x,\theta,\lambda_-(\gamma)) \equiv 4\left|(\theta-\lambda_-(\gamma))-1+2\lambda_-(\gamma)-xC_1(x,\theta)+2\lambda_-^2(\gamma)\gamma\right|, \tag{97}$$

$$\begin{aligned}
F_2(x,\theta,\lambda_-(\gamma)) \equiv \Big|&2(1-2\lambda_-(\gamma))C_1(x,\theta)+C_1^2(x,\theta)-4\lambda_-^2(\gamma)\gamma C_2(x,\theta) \\
&+4\lambda_-^2(\gamma)a(x)x^2\left(\frac{1}{2}-2\theta+xC_2(x,\theta)\right)\Big|,
\end{aligned} \tag{98}$$

and

$$F_3(x,\theta,\lambda_-(\gamma)) \equiv \sqrt{\left(1-\frac{a(x)x^2}{\gamma}\right)C_4(x,\theta(x))} \times \left(C_3(x,\theta(x)) + 2\lambda_-(\gamma)\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}\sqrt{C_4(x,\theta(x))}\right), \tag{99}$$

respectively (see Appendix A for the details of the above evaluations). From Eq. (28), $C_1(x,\theta)$, $C_2(x,\theta)$, $C_3(x,\theta)$, and $C_4(x,\theta)$ defined in the above equations are C^∞ functions in any region. Here let us define

$$\tilde{D}_{\gamma_1,b(\gamma_1),c,l} \equiv \bigcup_{\lambda\in[0,\lambda_-(\gamma_1)]} \{(x,\theta(x)) \mid x\in[0,l],\ \theta\in D_{\lambda,b(\gamma_1),c,l}\}, \tag{100}$$

where $b(\gamma_1)$ is introduced just before Eq. (93). Note that $\tilde{D}_{\gamma_1,b(\gamma_1),c,l}$ is a closed compact subset and does not contain the singular line $\theta = \theta_S(x)$. Since (26) and $1 - a(x)(\theta x^2+1)^2 > 0$ hold in $\tilde{D}_{\gamma_1,b(\gamma_1),c,l}$ except for $x = 0$, $x^2C_3(x,\theta)$ and $x^2C_4(x,\theta)$ can be zero only at $x = 0$ in $\tilde{D}_{\gamma_1,b(\gamma_1),c,l}$. On the other hand, $C_3(0,\theta) = \frac{1}{2} - 2\theta$ and $C_4(0,\theta) = 1 - 2\theta$ do not vanish in $\tilde{D}_{\gamma_1,b(\gamma_1),c,l}$. Then, $C_3(x,\theta) > 0$ and $C_4(x,\theta) > 0$ hold in $\tilde{D}_{\gamma_1,b(\gamma_1),c,l}$. In addition, from definition (46), the $2\lambda_-(\gamma)\sqrt{\gamma}$ appearing in $F_1(x,\theta,\lambda_-(\gamma))$, $F_2(x,\theta,\lambda_-(\gamma))$, and $F_3(x,\theta,\lambda_-(\gamma))$ above are strictly positive and bounded for γ such that $\lambda_-(\gamma)$ is in $[0,\lambda_-(\gamma_1)]$, and, using condition (30), we can see easily that $1-\frac{a(x)x^2}{\gamma}$ is also strictly positive and bounded for γ such that $\lambda_-(\gamma)$ is in $[0,\lambda_-(\gamma_1)]$. Thus, we conclude that there exist the strictly positive values ν_1, ν_2, and ν_3 defined by

$$\nu_1 \equiv \max_{\lambda_-(\gamma)\in[0,\lambda_-(\gamma_1)]}\left\{\max_{(x,\theta)\in\tilde{D}_{\gamma_1,b(\gamma_1),c,l}} F_1(x,\theta,\lambda_-(\gamma))\right\}, \tag{101}$$

$$\nu_2 \equiv \max_{\lambda_-(\gamma)\in[0,\lambda_-(\gamma_1)]}\left\{\max_{(x,\theta)\in\tilde{D}_{\gamma_1,b(\gamma_1),c,l}} F_2(x,\theta,\lambda_-(\gamma))\right\} \tag{102}$$

and

$$\nu_3 \equiv \min_{\lambda_-(\gamma)\in[0,\lambda_-(\gamma_1)]}\left\{\min_{(x,\theta)\in\tilde{D}_{\gamma_1,b(\gamma_1),c,l}} F_3(x,\theta,\lambda_-(\gamma))\right\}, \tag{103}$$

respectively. Using these values, for arbitrary $x\in[0,l]$, $\theta\in D_{\lambda_-(\gamma),b(\gamma_1),c,l}$, and $\lambda_-(\gamma)\in[0,\lambda_-(\gamma_1)]$, we see

$$|\lambda_-(\gamma)f(x,\theta(x);\lambda_-(\gamma))| \le \frac{\nu_1|\theta(x)-\lambda_-(\gamma)| + \nu_2 x}{\sqrt{\gamma}\nu_3 x} \le \frac{\nu_1 b(\gamma_1)x^c + \nu_2 x}{\sqrt{\gamma}\nu_3 x}. \tag{104}$$

Then, we have

$$|T_{\lambda_-(\gamma)}(\theta) - \lambda_-(\gamma)| \le \frac{\nu_1 b(\gamma_1)}{(c+2)\sqrt{\gamma}\nu_3}x^c + \frac{\nu_2}{3\sqrt{\gamma}\nu_3}x. \tag{105}$$

Since the right-hand side of this equation converges to 0 as $\gamma \to 0$, there exists γ_2 in $[\gamma_1, \infty)$ such that, for arbitrary $\gamma \in [\gamma_2, \infty)$,

$$\frac{\nu_1 b(\gamma_1)}{(c+2)\sqrt{\gamma}\nu_3} + \frac{\nu_2}{3\sqrt{\gamma}\nu_3} l^{1-c} \leq b(\gamma_1) \tag{106}$$

holds. Thus, for a $c \in (0,1)$ and arbitrary $\gamma \in [\gamma_2, \infty)$, we obtain

$$\begin{aligned} |T_{\lambda_-(\gamma)}(\theta) - \lambda_-(\gamma)| &\leq \frac{\nu_1 b(\gamma_1)}{(c+2)\sqrt{\gamma}\nu_3} x^c + \frac{\nu_2}{3\sqrt{\gamma}\nu_3} x \\ &\leq \frac{\nu_1 b(\gamma_1)}{(c+2)\sqrt{\gamma}\nu_3} x^c + \frac{\nu_2}{3\sqrt{\gamma}\nu_3} l^{1-c} x^c \\ &\leq b(\gamma_1) x^c. \end{aligned} \tag{107}$$

This means that $T_{\lambda_-(\gamma)}$ maps $D_{\lambda_-(\gamma), b(\gamma_1), c, l}$ into itself, that is, the condition (ii) holds for a $c \in (0,1)$ and arbitrary $\gamma \in [\gamma_2, \infty)$. Therefore, from Theorems 3 and 4, $\theta_{n_0}(x;\gamma)$, which is the earliest of all future-directed causal lines emanating from the central singularity, exists in the range $[0, l]$.

Finally, we will examine the condition (iii) for sufficiently large γ. For arbitrary $\epsilon > 0$, there exists γ_3 such that

$$\begin{aligned} \theta_{\mathrm{AH}}(l;\gamma) &= \frac{1}{l^2}\left(\frac{1}{\sqrt{a(l)}}\sqrt{1 - \frac{a(l)l^2}{\gamma}} - 1\right) \\ &> \frac{1}{l^2}\left(\frac{1}{\sqrt{a(l)}} - 1\right) - \epsilon \\ &\geq \theta_{\mathrm{S,min}} - \epsilon \end{aligned} \tag{108}$$

holds for arbitrary γ in $[\gamma_3, \infty)$. In the above, we used Eqs. (40) and (41) and definition (92) for $\theta_{\mathrm{S,min}}$. Now we choose $\epsilon > 0$ such that $\theta_{\mathrm{S,min}} - \epsilon > \lambda_-(\gamma_1) + b(\gamma_1) l^c$ holds. Then, we have

$$\theta_{\mathrm{AH}}(l;\gamma) > \theta_{\mathrm{S,min}} - \epsilon > \lambda_-(\gamma_1) + b(\gamma_1) l^c \geq \lambda_-(\gamma) + b(\gamma_1) l^c \geq \theta_{n_0}(l;\gamma) \tag{109}$$

for any γ in $[\gamma_0, \infty)$, where $\gamma_0 \equiv \max\{\gamma_1, \gamma_2, \gamma_3\}$. For the last inequality, we used the fact that there is a solution in $D_{\lambda_-(\gamma), b(\gamma_1), c, l}$ and $\theta_{n_0}(x;\gamma)$ is the earliest of all future-directed causal lines emanating from the central singularity. □

Therefore, for all $\gamma \in [\gamma_0, \infty)$, $\theta_{n_0}(x;\gamma)$ arrives at the surface of the dust cloud before the event horizon appears there; that is, the central singularity is globally naked in this case.

On the other hand, for γ sufficiently close to η defined by (30), we show that the central singularity is surrounded by the event horizon, that is, the central singularity is only locally naked.

LEMMA 7 (i) For any initial density distribution which is parameterized by Eq. (31) and satisfies $\eta \geq \gamma_{\min} = \sqrt{11 + 5\sqrt{5}}$ [$\gamma_{\min}$ is defined by Eq. (49)], there exists γ_1 such that $\gamma_1 \to \infty$ for $a(l) \to 1$ and the central singularity is only locally naked for arbitrary $\gamma \in (\eta, \gamma_1]$.

(ii) For any initial density distribution which is parameterized by eq. (31) and satisfies $\eta < \gamma_{\min}$, if there exists x_0 in $[0, l]$ that satisfies $\frac{\gamma_{\min}}{\gamma_{\min} + x_0^2} < a(x_0)$, then there exists γ_2 such that $\gamma_2 \to \infty$ for $a(l) \to 1$ and the central singularity is only locally naked for arbitrary $\gamma \in [\gamma_{\min}, \gamma_2]$.

Proof. We suppose that λ satisfying Eq. (46) exists. In this case, from Theorem 3, the central singularity is locally naked at least. Let us define x_η as

$$a(x_\eta)x_\eta^2 \equiv \max_{x\in[0,l]} a(x)x^2 = \eta. \tag{110}$$

Note that $x_\eta \neq 0$ because $a(0)$ is finite and $a(x)x^2 > 0$ except for $x = 0$. At $x = x_\eta$, the apparent horizon appears at

$$\theta_{\mathrm{AH}}(x_\eta;\gamma) = \frac{1}{x_\eta^2}\left(\frac{1}{\sqrt{a(x_\eta)}}\sqrt{1-\frac{\eta}{\gamma}}-1\right). \tag{111}$$

If $\eta \geq \gamma_{\mathrm{min}}$ holds, there exists γ_1 such that the right-hand side of this equation becomes negative for arbitrary γ in $(\eta, \gamma_1]$. Additionally, since $\sqrt{1-\frac{\eta}{\gamma}} < 1$ always holds, $\theta_{\mathrm{AH}}(x_\eta;\gamma)$ would be negative if $a(x_\eta)$ were equal to 1. This fact and $a(l) \leq a(x_\eta) \leq 1$ tell us that $\gamma_1 \to \infty$ for $a(l) \to 1$. Since $\theta_{\mathrm{AH}}(x_\eta;\gamma) < 0$ means that the apparent horizon can exist at an earlier timeslice than the central singularity appears, null geodesics emanating from the central singularity cannot arrive at future null infinity for arbitrary γ in $(\eta, \gamma_1]$; that is, the central singularity is only locally naked.

On the other hand, for $\eta < \gamma_{\mathrm{min}}$, γ cannot approach η because of the condition (37). But if there exists x_0 in $[0, l]$ that satisfies $\frac{\gamma_{\mathrm{min}}}{\gamma_{\mathrm{min}}+x_0^2} < a(x_0)$, then $\theta_{\mathrm{AH}}(x_0;\gamma_{\mathrm{min}}) < 0$ holds from Eq. (111). Since $\theta_{\mathrm{AH}}(x;\gamma)$ is continuous with respect to γ, there exists γ_2 such that $\theta_{\mathrm{AH}}(x_0;\gamma) < 0$ holds for arbitrary $\gamma \in [\gamma_{\mathrm{min}}, \gamma_2]$, that is, the central singularity is only locally naked in these cases. In addition, since $\frac{\gamma_{\mathrm{min}}}{\gamma_{\mathrm{min}}+x_0^2} < 1$ and $a(l) \leq a(x_0) \leq 1$ always hold, we have $\gamma_2 \to \infty$ for $a(l) \to 1$. □

Furthermore, we can show the monotonicity of $\theta_{n_0}(x;\gamma)$ with respect to γ at each x. Let us define $\theta(x;\gamma)$ as a solution to Eq. (51) for γ, which converges to $\lambda_-(\gamma)$ as $x \to 0$.

Lemma 8 For any initial density distribution parameterized as (31), $\theta(x;\gamma_s) > \theta(x;\gamma_l)$ holds for γ_s and γ_l such that $\gamma_s < \gamma_l$, and all x such that $\theta(x;\gamma_s)$ exists. In particular, $\theta_{n_0}(x;\gamma)$ defined in Lemma 5 is a monotonically decreasing function of γ at each x.

Proof. We suppose that $\gamma_s < \gamma_l$ and $\theta(x;\gamma_s)$ exists in the range $[0, d_s)$. Now let us define

$$I \equiv \{x \mid \theta(x;\gamma_s) > \theta(x;\gamma_l)\}. \tag{112}$$

I is the union of intervals and not empty because $\theta(x;\gamma_s)$ and $\theta(x;\gamma_l)$ are continuous and $\theta(0;\gamma_s) = \lambda_-(\gamma_s) > \lambda_-(\gamma_l) = \theta(0;\gamma_l)$. We shall show that $[0, d_0) \subset I$ implies $d_0 \in I$ for arbitrary $d_0 < d_s$ in the following. This implies $I = [0, d_s)$ because any interval contained in I must not be a closed proper subset in $[0, d_s)$ by definition.

Now we suppose that $[0, d_0) \subset I$ and $0 < x_1 < x_2 < d_0 < d_s$. Let us define

$$f_0(x,\theta) \equiv \frac{1-a(x)(\theta x^2+1)^2-\frac{x}{2}(\theta x^2+1)^2\frac{d}{dx}a(x)}{x\sqrt{1-a(x)(\theta x^2+1)^2}}. \tag{113}$$

From (26) and the fact that $f_0(x,\theta)$ is differentiable except for the singularity, it is positive and $\frac{\partial f_0(x,\theta)}{\partial\theta}$ is finite for arbitrary $x \in [x_1, d_0]$ and $\theta \in [\theta(x;\gamma_l), \theta(x;\gamma_s)]$. Then, from Eq. (51), we have

$$\begin{aligned}
&\theta(x_1;\gamma_s) - \theta(x_1;\gamma_l)\\
&= \theta(x_2;\gamma_s) - \theta(x_2;\gamma_l) + \int_{x_1}^{x_2} \frac{2}{x}(\theta(x;\gamma_s) - \theta(x;\gamma_l))dx\\
&\quad - \int_{x_1}^{x_2} \left(\frac{1}{\sqrt{\gamma_s - a(x)x^2}} - \frac{1}{\sqrt{\gamma_l - a(x)x^2}}\right)\frac{f_0(x,\theta(x;\gamma_s))}{x}dx\\
&\quad - \int_{x_1}^{x_2} \Big(f_0(x,\theta(x;\gamma_s)) - f_0(x,\theta(x;\gamma_l))\Big)\frac{1}{x\sqrt{\gamma_l - a(x)x^2}}dx.\\
&< \theta(x_2;\gamma_s) - \theta(x_2;\gamma_l) + \int_{x_1}^{x_2} \frac{2}{x}(\theta(x;\gamma_s) - \theta(x;\gamma_l))dx\\
&\quad + \int_{x_1}^{x_2} |f_0(x,\theta(x;\gamma_s)) - f_0(x,\theta(x;\gamma_l))|\frac{1}{x\sqrt{\gamma_l - a(x)x^2}}dx.\\
&\le \theta(x_2;\gamma_s) - \theta(x_2;\gamma_l) + \int_{x_1}^{x_2} \frac{2}{x}(\theta(x;\gamma_s) - \theta(x;\gamma_l))dx\\
&\quad + \int_{x_1}^{x_2} \sup_{\theta(x;\gamma_l)\le\theta\le\theta(x;\gamma_s)} \left|\frac{\partial f_0(x,\theta)}{\partial\theta}\right| (\theta(x;\gamma_s) - \theta(x;\gamma_l))\frac{1}{x\sqrt{\gamma_l - a(x)x^2}}dx.\\
&= \theta(x_2;\gamma_s) - \theta(x_2;\gamma_l) + \int_{x_1}^{x_2} F(x)(\theta(x;\gamma_s) - \theta(x;\gamma_l))dx,
\end{aligned} \tag{114}$$

where $F(x)$ is the positive function defined as

$$F(x) \equiv \frac{2}{x} + \sup_{\theta(x;\gamma_l)\le\theta\le\theta(x;\gamma_s)} \left|\frac{\partial f_0(x,\theta)}{\partial\theta}\right| \frac{1}{x\sqrt{\gamma_l - a(x)x^2}}. \tag{115}$$

For the first inequality in the above, we used the fact that $\frac{1}{\sqrt{\gamma_s - a(x)x^2}} - \frac{1}{\sqrt{\gamma_l - a(x)x^2}}$ is positive because of $\gamma_s < \gamma_l$. Thus, we obtain

$$\theta(x_2;\gamma_s) - \theta(x_2;\gamma_l) > (\theta(x_1;\gamma_s) - \theta(x_1;\gamma_l)) \exp\Big(-\int_{x_1}^{x_2} F(x)dx\Big). \tag{116}$$

As $x_2 \to d_0$, this inequality becomes

$$\theta(d_0;\gamma_s) - \theta(d_0;\gamma_l) > (\theta(x_1;\gamma_s) - \theta(x_1;\gamma_l)) \exp\Big(-\int_{x_1}^{d_0} F(x)dx\Big) > 0 \tag{117}$$

because $F(x)$ is bounded in the range $[x_1, d_0]$ and we supposed $\theta(x_1;\gamma_s) > \theta(x_1;\gamma_l)$. This means $d_0 \in I$. □

Here let us define N as the set of real numbers $\lambda_-(\gamma)$ such that $G_{\lambda_-(\gamma),d}$ contains more than one element for some d. Then, from Lemma 8, we have the following corollary:

COROLLARY 1 N is countable.

Proof. We suppose $\gamma_s < \gamma_l$ again. From Lemma 8, $\theta(x;\gamma_s) > \theta(x;\gamma_l)$ holds for arbitrary $\theta(x;\gamma_s)$ and $\theta(x;\gamma_l)$ at arbitrary x such that $\theta(x;\gamma_s)$ exists. So the geodesics $\theta(x;\gamma_s)$ and $\theta(x;\gamma_l)$ do not intersect in the domain of $\theta(x;\gamma_s)$. This means that $G_{\lambda_-(\gamma_s),d}(x) \bigcap G_{\lambda_-(\gamma_l),d}(x)$ is an empty set for arbitrary d and x in the domain of $\theta(x;\gamma_s)$. In addition, since Eq. (51) satisfies the Lifshitz condition on an arbitrary compact set which does not contain the singularity, the elements in $G_{\lambda_-(\gamma),d}$ do not intersect each other in the region which does not contain the singularity. So if $G_{\lambda_-(\gamma),d}$ contains two different functions $\theta_1(x;\gamma)$ and $\theta_2(x;\gamma)$ which satisfy $\theta_1(x_0;\gamma) < \theta_2(x_0;\gamma)$ for an x_0, an arbitrary solution to Eq. (51), $\theta(x)$, which satisfies $\theta_1(x_0;\gamma) < \theta(x_0) < \theta_2(x_0;\gamma)$ must be contained in $G_{\lambda_-(\gamma),d}$. Thus, for nonzero x, $G_{\lambda_-(\gamma),d}(x)$ is always an interval in $\mathbb{R}$ if $G_{\lambda_-(\gamma),d}$ contains more than one element.

Now we assume that N is uncountable. From Theorem 3, a solution $\theta(x;\gamma(\lambda_{\mathrm{M}}))$ exists in the range $[0, d_{\mathrm{M}}]$, where d_{M} is a positive number. From Lemma 8, for arbitrary λ satisfying $\lambda < \lambda_{\mathrm{M}}$, all solutions $\theta(x;\gamma(\lambda))$ also exist in $[0, d_{\mathrm{M}}]$ because the region $\theta < \theta(x;\gamma(\lambda_{\mathrm{M}}))$ does not contain the singularity at $\theta = \theta_{\mathrm{S}}(x)$. Thus, $G_{\lambda,d_{\mathrm{M}}}(d_{\mathrm{M}})$ is an interval for all λ in N. Here we define $|G_{\lambda,d}(x)|$ as the Lebesgue measure of $G_{\lambda,d}(x)$. $|G_{\lambda,d_{\mathrm{M}}}(d_{\mathrm{M}})|$ is nonzero for arbitrary λ in N. We can evaluate the sum of $|G_{\lambda,d_{\mathrm{M}}}(d_{\mathrm{M}})|$ for λ in N as

$$\sum_{\lambda\in N} \left|G_{\lambda,d_{\mathrm{M}}}(d_{\mathrm{M}})\right| \leq \left|\bigcup_{\lambda\in N} G_{\lambda,d_{\mathrm{M}}}(d_{\mathrm{M}})\right| \\ \leq |[0,\theta_{\mathrm{S}}(d_{\mathrm{M}})]| = \theta_{\mathrm{S}}(d_{\mathrm{M}}). \tag{118}$$

For the first inequality, we used the fact that $G_{\lambda,d_{\mathrm{M}}}(d_{\mathrm{M}})$ is an interval for all λ in N and does not have a common part with each other for different λ. For the second one, we used the fact that the line $\theta = \theta(x;\gamma(\lambda))$ does not enter the noncentral singularity in the range $[0, d_{\mathrm{M}}]$ for all λ in N and the region $\theta < 0$ at $x = d_{\mathrm{M}}$, which is not in the future of the central singularity. However, since the sum of uncountable infinite numbers of strictly positive real numbers must diverge, $\sum_{\lambda\in N} |G_{\lambda,d_{\mathrm{M}}}(d_{\mathrm{M}})|$ must diverge. This contradicts the inequality (118). Thus, N is countable. □

Since the existence theorem is based on the fixed-point theorem for contraction mapping in the four-dimensional case [12], one could immediately see that the solution to the differential equation for the null geodesic that has a certain initial value at the central singularity is unique. By contrast, in the five-dimensional case, it is not necessary that the solution found in Theorem 3 is unique because we use the Schauder fixed-point theorem for the proof of the existence of the solution. However, this corollary guarantees that the solution which converges $\lambda_-(\gamma)$ as $x \to 0$ is unique for almost every γ at least.

In Lemma 8, we proved the monotonicity of the solutions to Eq. (51) with respect to γ. In addition, we can easily show the monotonicity of $\theta_{\mathrm{AH}}(x;\gamma)$ with respect to γ at each x.

LEMMA 9 For any initial density distribution parameterized as (31), $\theta_{\mathrm{AH}}(x;\gamma)$ is a monotonically increasing function of γ at each x.

Proof. It is obvious from Eq. (41). □

From Lemmas 6, 7, 8, and 9, we obtain the following theorem.

THEOREM 5 (i) For any initial density distribution which is parameterized as (31) and satisfies $\eta \geq \gamma_{\min}$, there exists γ_C which satisfies $\eta < \gamma_C$ and $\gamma_C \to \infty$ for $a(l) \to 1$ such that (a) for arbitrary $\gamma \in (\gamma_C, \infty)$, $\theta_{n_0}(x;\gamma)$ defined in Lemma 5 goes to future null infinity, that is, the central singularity is globally naked and weak CCC does not hold, and (b) for all $\gamma \in (\eta, \gamma_C)$, the central singularity is only locally naked, that is, weak CCC holds and the outer region of the event horizon is regular.

(ii) For any initial density distribution which is parameterized as (31) and satisfies $\eta < \gamma_{\min} = 11 + 5\sqrt{5}$, if there exists x_0 in $[0, l]$ such that $\frac{\gamma_{\min}}{\gamma_{\min}+x_0^2} < a(x_0)$ holds, then there exists γ_C which satisfies $\eta < \gamma_C$ and $\gamma_C \to \infty$ for $a(l) \to 1$ such that the above (a) and (b) hold. Otherwise, there exists γ_0 satisfying $\eta < \gamma_0$, such that, for all $\gamma \in [\gamma_0, \infty)$, $\theta_{n_0}(x;\gamma)$ goes to future null infinity, that is, the central singularity is globally naked.

Proof. (i) Let us assume that the initial density distribution is parameterized as (31) and satisfies $\eta \geq \gamma_{\min}$. Then, from Lemmas 8 and 9, $\theta_{n_0}(x;\gamma)$ is a decreasing function of γ and $\theta_{\rm AH}(x;\gamma)$ is a continuous increasing function of γ for each x. In addition, from Lemma 6, there exists γ_0 such that the solution $\theta_{n_0}(x;\gamma)$ can extend to $x = l$ and $\theta = \theta_{n_0}(x;\gamma)$ does not intersect with $\theta = \theta_{\rm AH}(x;\gamma)$ for all $\gamma \in [\gamma_0, \infty]$, while from Lemma 7, if $\eta \geq \gamma_{\min}$ holds, there exists γ_1 such that $\gamma_1 \to \infty$ for $a(l) \to 1$ and $\theta = \theta_{n_0}(x;\gamma)$ intersects with $\theta = \theta_{\rm AH}(x;\gamma)$ somewhere in the dust cloud for arbitrary $\gamma \in (\eta, \gamma_1]$. Thus there exists γ_C such that $\gamma_0 \geq \gamma_C \geq \gamma_1$ and $\theta = \theta_{n_0}(x;\gamma)$ does not intersect with $\theta = \theta_{\rm AH}(x;\gamma)$ for all $\gamma \in (\gamma_C, \infty)$ and $\theta = \theta_{n_0}(x;\gamma)$ intersects with $\theta = \theta_{\rm AH}(x;\gamma)$ somewhere in the dust cloud for arbitrary $\gamma \in (\eta, \gamma_C)$. If $\theta = \theta_{n_0}(x;\gamma)$ does not intersect with $\theta = \theta_{\rm AH}(x;\gamma)$, then $\theta = \theta_{n_0}(x;\gamma)$ can extend to future null infinity because the outer region of the $x = l$ surface is the Schwarzschild spacetime. Thus, in this case, the central singularity is globally naked. If $\theta = \theta_{n_0}(x;\gamma)$ intersects with $\theta = \theta_{\rm AH}(x;\gamma)$ somewhere in the dust cloud, then $\theta = \theta_{n_0}(x;\gamma)$ cannot extend to future null infinity and will enter the singularity. This means that the central singularity is only locally naked and the outer region of the event horizon is regular because $\theta = \theta_{n_0}(x;\gamma)$ is the earliest line in all future-directed causal lines emanating from the central singularity.

(ii) Let us assume that the initial density distribution is parameterized as (31) and satisfies $\eta < \gamma_{\min} = 11 + 5\sqrt{5}$. If there exists x_0 in $[0, l]$ such that $\frac{\gamma_{\min}}{\gamma_{\min}+x_0^2} < a(x_0)$ holds, in the same way as in the proof of (i), we can show that (ii) holds. On the other hand, if such x_0 does not exist in $[0, l]$, we cannot use Lemma 7. Then all we could show in this regard is Lemma 6 only. □

6. Conclusion and discussion

In this paper, we analyzed five-dimensional inhomogeneous spherically symmetric dust collapse. By virtue of the Schauder fixed-point theorem, we proved an existence theorem for null geodesics in singular spacetime. Moreover, by using it, we showed a necessary and sufficient condition for the singularity to be naked and saw the dependence of the global nakedness of the central singularity on the initial density distribution.

In Sect. 2, we fixed the initial energy distribution of the dust so that the initial velocity of the shells is zero. This assumption is not critical for our method. Therefore, we can also discuss the nakedness of the singularity without this assumption. To prove the existence of a null geodesic emanating from the central singularity in this general case, we have to find an appropriate domain such that the operator T_λ maps its domain into itself. We expect that, for some class of energy distribution, $D_{\lambda,b,c,d}$ defined by (55) can be such domain for certain b, c, and d, and the argument will follow in a similar manner to this paper.

In specific dimensional spherically symmetric dust collapse in Lovelock gravity, or particularly in nine-dimensional spherically symmetric dust collapse in Einstein–Gauss–Bonnet gravity [21],[2] we cannot use Christodoulou's method and discussion to show the existence of null geodesics emanating from the central singularity because the singular term in the differential equation for the null geodesic does not take the form of a simple function. In contrast, our method may be used to examine the nakedness of a singularity for the above cases because our existence theorem improved Christodoulou's method [12].

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Appendix A. The evaluation details for equations (56), (63), (72), and (94)

Appendix A.1. Equation (56)

$$
\begin{aligned}
|\lambda f(x,\theta(x);\lambda)| &= \left| \frac{1-a(x)(\theta(x)x^2+1)^2 - \frac{x}{2}(\theta(x)x^2+1)^2\frac{d}{dx}a(x)}{x^2\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)\left(1-a(x)(\theta(x)x^2+1)^2\right)}} - \frac{2\lambda}{x}\right| \\
&\le \left| \frac{1-2\theta(x)+O(x)-2\lambda\sqrt{\gamma}\sqrt{\frac{1}{2}-2\theta(x)+O(x)}}{\sqrt{\gamma}x\sqrt{\frac{1}{2}-2\theta(x)+O(x)}} \right| \\
&= \left| \frac{(1-2\theta(x)+O(x))^2-4\lambda^2\gamma\left(\frac{1}{2}-2\theta(x)+O(x)\right)}{\sqrt{\gamma}x\sqrt{\frac{1}{2}-2\theta(x)+O(x)}\left(1-2\theta(x)+O(x)+2\lambda\sqrt{\gamma}\sqrt{\frac{1}{2}-2\theta(x)+O(x)}\right)} \right| \\
&= \left| \frac{\begin{array}{c}2\{-2(\theta(x)-\lambda)+O(x)\}(1-2\lambda)+\{-2(\theta(x)-\lambda)+O(x)\}^2\\ -4\lambda^2\gamma\{-2(\theta(x)-\lambda)+O(x)\}\end{array}}{\sqrt{\gamma}x\sqrt{\frac{1}{2}-2\theta(x)+O(x)}\left(1-2\theta(x)+O(x)+2\lambda\sqrt{\gamma}\sqrt{\frac{1}{2}-2\theta(x)+O(x)}\right)} \right| \\
&\le \frac{|-4(1-2\lambda)+8\lambda^2\gamma||\theta(x)-\lambda|+O(|\theta(x)-\lambda|^2)+O(x)}{\sqrt{\gamma}x\sqrt{\frac{1}{2}-2\lambda+O(x^c)+O(x)}\left(1-2\lambda+2\lambda\sqrt{\gamma}\sqrt{\frac{1}{2}-2\lambda}+O(x^c)+O(x)\right)}
\end{aligned}
$$

[2] In the Lovelock gravity case, by employing the analysis in [21], it is easy to find that we cannot apply the Christodoulou theorem in $D = 4k + 1$-dimensional spacetime, where k is the highest order of nonvanishing Lovelock coefficients [22].

$$\leq \frac{|-4(1-2\lambda)+8\lambda^2\gamma|bx^{c-1}}{\sqrt{\gamma}\sqrt{\frac{1}{2}-2\lambda}\left(1-2\lambda+2\lambda\sqrt{\gamma}\sqrt{\frac{1}{2}-2\lambda}\right)}+\frac{O(x^{2c})+O(x)}{x}$$

$$\leq \frac{2\lambda}{\sqrt{\gamma}\left(\frac{1}{2}-2\lambda\right)^{\frac{3}{2}}}bx^{c-1}+O(1)+O(x^{2c-1}). \tag{A.1}$$

In the right-hand side of theh first inequality, we can choose functions O that are independent of θ because $\theta(x)$ in $D_{\lambda,b,c,d}$ is uniformly bounded by the constants $\lambda + bl^c$ and $\lambda - bl^c$. In the same way, we can choose functions O that are independent of θ.

Appendix A.2. Equation (63)

$$\begin{aligned}
&|\lambda f(x,\theta_1(x);\lambda)-\lambda f(x,\theta_2(x);\lambda)| \\
&= \frac{1}{x^2\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}}\left|\sqrt{g_1}-\sqrt{g_2}-\frac{x\frac{d}{dx}a(x)}{2a(x)}\left(\frac{1-g_1}{\sqrt{g_1}}-\frac{1-g_2}{\sqrt{g_2}}\right)\right| \\
&= \frac{1}{x^2\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}}\left|1+\frac{x\frac{d}{dx}a(x)}{2a(x)}\left(1+\frac{1}{\sqrt{g_1g_2}}\right)\right|\left|\sqrt{g_1}-\sqrt{g_2}\right| \\
&\leq \frac{1}{x^2\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}}\left\{1+\left|\frac{x\frac{d}{dx}a(x)}{2a(x)}\right|\left(1+\frac{1}{\sqrt{g_1g_2}}\right)\right\}\left(\frac{1}{\sqrt{g_1}+\sqrt{g_2}}\right)|g_1-g_2| \\
&\leq \frac{|\theta_1(x)-\theta_2(x)|}{x^2\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}}\left\{1-\frac{x\frac{d}{dx}a(x)}{2a(x)}\left(1+\frac{1}{1-a(x)((\lambda+bx^c)x^2+1)}\right)\right\} \\
&\quad\times\frac{a(x)\{x^4(\lambda+bx^c)+x^2\}}{\sqrt{(1-a(x)((\lambda+bx^c)x^2+1)^2}} \\
&= \frac{|\theta_1(x)-\theta_2(x)|}{x^2\sqrt{\gamma}}\left\{1+\frac{x^2}{x^2-4\lambda x^2}\right\}\frac{x^2}{\sqrt{\frac{x^2}{2}-2\lambda x^2}}+B_1(x)x^{\delta-1}|\theta_1(x)-\theta_2(x)| \\
&= \frac{1-2\lambda}{\sqrt{\gamma(\frac{1}{2}-2\lambda)^3}}x^{-1}|\theta_1(x)-\theta_2(x)|+B_1(x)x^{\delta-1}|\theta_1(x)-\theta_2(x)| \\
&= \left(\frac{4\lambda}{1-4\lambda}x^{-1}+B_1(x)x^{\delta-1}\right)|\theta_1(x)-\theta_2(x)|,
\end{aligned} \tag{A.2}$$

where $B_1(x)$ is introduced as in the text.

Appendix A.3. Equation (72)

$$|T_\lambda(\theta)(x)-T_\lambda(\theta)(y)| \leq \left(\frac{1}{x^2}-\frac{1}{y^2}\right)\int_0^x s^2|\lambda f(s,\theta(s);\lambda)|ds+\frac{1}{y^2}\int_x^y s^2|\lambda f(s,\theta(s);\lambda)|ds$$

$$\leq \left(\frac{1}{x^2}-\frac{1}{y^2}\right)\int_0^x s^2\left\{\frac{2\lambda}{\sqrt{\gamma}\left(\frac{1}{2}-2\lambda\right)^{\frac{3}{2}}}bs^{c-1}+O(1)+O(s^{2c-1})\right\}ds$$
$$+\frac{1}{y^2}\int_x^y s^2\left\{\frac{2\lambda}{\sqrt{\gamma}\left(\frac{1}{2}-2\lambda\right)^{\frac{3}{2}}}bs^{c-1}+O(1)+O(s^{2c-1})\right\}ds$$
$$=\left(\frac{1}{x^2}-\frac{1}{y^2}\right)h(x)x^{2+c}+\frac{1}{y^2}(h(y)y^{2+c}-h(x)x^{2+c})$$
$$\leq h(x)|x^c-y^c|+y^c|h(y)-h(x)|+\frac{2h(x)}{y^2}|y^{2+c}-x^{2+c}|$$
$$< h(x)|x^c-y^c|+y^c|h(y)-h(x)|+2h(x)y^c\left|\frac{y^2-x^2}{y^2}\right|$$
$$\leq h(x)|x^c-y^c|+y^c|h(y)-h(x)|+2^{n+1}h(x)y^{c-\frac{1}{2^{n-1}}}\left|y^{\frac{1}{2^{n-1}}}-x^{\frac{1}{2^{n-1}}}\right|. \tag{A.3}$$

Appendix A.4. Equation (94)

$$|\lambda_-(\gamma)f(x,\theta(x);\lambda_-(\gamma))|$$
$$\leq\left|\frac{1-a(x)(\theta(x)x^2+1)^2-\frac{x}{2}(\theta(x)x^2+1)^2\frac{d}{dx}a(x)}{x^2\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)\left(1-a(x)(\theta(x)x^2+1)^2\right)}}-\frac{2\lambda_-(\gamma)}{x}\right|$$
$$=\left|\frac{1-2\theta(x)+xC_1(x,\theta(x))-2\lambda_-(\gamma)\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}\sqrt{\frac{1}{2}-2\theta(x)+xC_2(x,\theta(x))}}{x\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)\left(\frac{1}{2}-2\theta(x)+xC_2(x,\theta(x))\right)}}\right|$$
$$\equiv\left|\frac{C_3(x,\theta(x))-2\lambda_-(\gamma)\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}\sqrt{C_4(x,\theta(x))}}{x\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)C_4(x,\theta(x))}}\right|$$
$$=\left|\frac{(C_3(x,\theta(x)))^2-4\lambda_-^2(\gamma)\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)C_4(x,\theta(x))}{x\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)C_4(x,\theta(x))}\left(C_3(x,\theta(x))+2\lambda_-(\gamma)\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}\sqrt{C_4(x,\theta(x))}\right)}\right|$$
$$\leq\left|\frac{\left\{(\theta(x)-\lambda_-(\gamma))-1+2\lambda_-(\gamma)-xC_1(x,\theta(x))+2\lambda_-^2(\gamma)\gamma\right\}4(\theta(x)-\lambda_-(\gamma))}{x\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)C_4(x,\theta(x))}\left(C_3(x,\theta(x))+2\lambda_-(\gamma)\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}\sqrt{C_4(x,\theta(x))}\right)}\right|$$

$$+\left|\frac{H(x,\theta(x))}{\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)C_4(x,\theta(x))}\left(C_3(x,\theta(x))+2\lambda_-(\gamma)\sqrt{\gamma\left(1-\frac{a(x)x^2}{\gamma}\right)}\sqrt{C_4(x,\theta(x))}\right)}\right|$$

$$=\frac{F_1(x,\theta(x),\lambda_-(\gamma))|\theta(x)-\lambda_-(\gamma)|+xF_2(x,\theta(x),\lambda_-(\gamma))}{\sqrt{\gamma}xF_3(x,\theta(x),\lambda_-(\gamma))}, \tag{A.4}$$

where $C_1,\ldots,C_4$, F_1, F_2, and F_3 are defined as in the text.

Appendix B. Four-dimensional case

In this appendix, we give an overview of Christodolou's paper [12] which examined the global nakedness of a singularity in four-dimensional LTB spacetime, and see the difference between Christodoulou's and our discussions on the existence of null geodesics near the singuality. In the four-dimensional case, after change of variables, the dimensionless differential equation for future-directed null geodesics along the outer radial direction is given as

$$\frac{d\hat{\theta}}{d\hat{x}}+\frac{7\hat{\theta}}{\hat{x}}=\frac{7\hat{\lambda}}{\hat{x}}+\hat{\lambda}f_4(\hat{x},\hat{\theta};\hat{\lambda}), \tag{B.1}$$

where $\hat{\theta}$ and $\hat{x}$ are dimensionless coordinates, which correspond to θ and x defined by (38) and (33) respectively, $\hat{\lambda}$ is a certain constant, and f_4 is a C^∞ function. $\hat{\lambda}$ and f_4 are also the variables that correspond to λ and f defined in (46) and (51) in the five-dimensional case, respectively. In order not to contain a noncentral singularity, $\hat{\theta}$ is restricted in the range $0\le\hat{\theta}<\sigma(\hat{x})$, where σ is a certain function which satisfies $\sigma(\hat{x})\ge\frac{\epsilon_4}{\hat{x}}$ for a positive constant ϵ_4.

The formal solution to this differential equation is given by

$$\begin{aligned}\hat{\theta}(\hat{x})&=\lambda\left(1+\hat{x}\int_0^1 dvv^7f_4(v\hat{x},\hat{\theta}(v\hat{x});\hat{\lambda})\right)\\&\equiv T_{4,\lambda}(\hat{\theta})(\hat{x}).\end{aligned} \tag{B.2}$$

Let us define

$$D_{\hat{d},\mu}\equiv\{\hat{\theta}\mid\hat{\theta}\in C^0[0,\hat{d}],\ 0\le\hat{\theta}\le\mu\}, \tag{B.3}$$

where μ is a positive real number satisfying $\mu<\sigma(\hat{x})$ for all $\hat{x}\in[0,\hat{d}]$. $D_{\hat{d},\mu}$ becomes a subset of a Banach space by the uniform norm.

After some discussion on the nature of $T_{4,\lambda}$, as with the five-dimensional case, we can conclude that $T_{4,\lambda}$ maps $D_{\hat{d},\mu}$ into itself for sufficiently small $\hat{d}$. Furthermore, we obtain

$$\begin{aligned}\|T_{4,\hat{\lambda}}(\hat{\theta}_1)-T_{4,\hat{\lambda}}(\hat{\theta}_2)\|&=\sup_{0\le\hat{x}\le\hat{d}}\left|\hat{x}\int_0^1 v^7\hat{\lambda}\left\{f_4(v\hat{x},\hat{\theta}_1(v\hat{x});\hat{\lambda})-f_4(v\hat{x},\hat{\theta}_2(v\hat{x});\hat{\lambda})\right\}dv\right|\\&\le\frac{\hat{d}\hat{\lambda}\Delta}{8}\|\hat{\theta}_1-\hat{\theta}_2\|,\end{aligned} \tag{B.4}$$

where Δ is defined as

$$\Delta \equiv \sup_{0\le\hat{x}\le\hat{d}} \left\{ \sup_{0\le\hat{\theta}\le\mu} \left| \frac{\partial f_4}{\partial\hat{\theta}}(\hat{x},\hat{\theta};\hat{\lambda}) \right| \right\}. \tag{B.5}$$

Δ is finite because f_4 is a C^∞ function in $[0,\hat{d}] \times [0,\mu]$. Here we choose $\hat{d}_0$ so that it satisfies $\hat{d}_0 \le \hat{d}$ and

$$\hat{d}_0 < \frac{8}{\hat{\lambda}\Delta}; \tag{B.6}$$

then $T_{4,\hat{\lambda}}$ becomes a contraction mapping from $D_{\hat{d}_0,\mu}$ into itself. Therefore, by the fixed-point theorem for contraction mappings [19], we can conclude that $T_{4,\hat{\lambda}}$ has a unique fixed point, that is, a null geodesic emanating from the central singularity exists and the singularity is naked.

By contrast, in the five-dimensional case, what we can do is only to deform the differential equation for the null geodesic near the central singularity like

$$\begin{aligned} \frac{d\theta}{dx} + \frac{2\theta}{x} &= \frac{2\lambda}{x} + \frac{2\left(g(\theta;\gamma)+\theta-\lambda\right)}{x} + \lambda f_5(x,\theta;\gamma) \\ &\equiv \frac{2\lambda}{x} + \frac{2\lambda g_5(\theta;\gamma)}{x} + \lambda f_5(x,\theta;\gamma), \end{aligned} \tag{B.7}$$

where $g(\theta;\gamma)$ is defined by (77) and f_5 is a function such that $xf_5(x,\theta;\gamma)$ converges to 0 as $x \to 0$ in the region $\theta < \theta_S(x)$. In the four-dimensional case, the right-hand side of the differential equation for the null geodesic has a constant coefficient pole at first order only. However, in the five-dimensional case, the coefficient of the pole of the right-hand side of (B.7) is a function with respect to θ. Thus, the variable that corresponds to Δ in (B.4) is not finite in five dimensions and we cannot directly use the method employed for the four-dimensional case [12].

As above, we can apply the method in [12] to the case that the geodesic equation has a constant coefficient pole at first order only. On the other hand, our method can be applied to the more general case that the geodesic equation can be deformed to an expression having a general pole at first order.

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