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Bifurcation diagram for interior single-peak solutions in a Neumann problem for $u''+\lambda(-u+u^{p})=0$ with $p\in \mathbb{R}$ and $p>1$*

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1 Introduction

Let $p\in \mathbb{R}$ and $p>1$. We study a Neumann problem for a second-order differential equation,

$$u''+\lambda(-u+u^{p})=0 \quad \text{in } (-1,1), \qquad u'(\pm 1)=0, \tag{1}$$

where $\lambda>0$ is a constant and represents a control parameter. Eq. (1) has a trivial solution $u=1$.

We often encounter (1) in several situations. As an example, we consider the Keller-Segel model for chemotaxis aggregation,

$$\begin{aligned} &u_{t}=D_{1}u_{xx}-c(u(\log v)_{x})_{x}, \quad v_{t}=D_{2}v_{xx}-av-bu \quad \text{in } (-1,1),\\ &u_{x}, v_{x}=0 \text{ at } x=\pm 1, \end{aligned} \tag{2}$$

where D_{1}, D_{2}, a, b, c are constants. The stationary problem for (2) becomes

$$D_{2}v_{xx}-av-b\mu v^{c/D_{1}}=0, \quad v_{x}=0 \text{ at } x=\pm 1 \tag{3}$$

since $D_{1}u_{x}-cu(\log v)_{x}=0$ by the first equation, so that $u=\mu v^{c/D_{1}}$ for some constant μ. Eq. (3) is transformed to (1). Another example is related to the Gierer-Meinhardt model for biological pattern formations,

$$\begin{aligned} &u_{t}=D_{1}u_{xx}-\mu_{1}u+\rho_{1}\left(c_{1}\frac{u^{p_{1}}}{v^{q_{1}}}+\rho_{0}\right), \quad v_{t}=D_{2}v_{xx}-\mu_{2}v+\rho_{2}c_{2}\frac{u^{p_{2}}}{v^{q_{2}}} \quad \text{in } (-1,1),\\ &u_{x}, v_{x}=0 \text{ at } x=\pm 1, \end{aligned} \tag{4}$$

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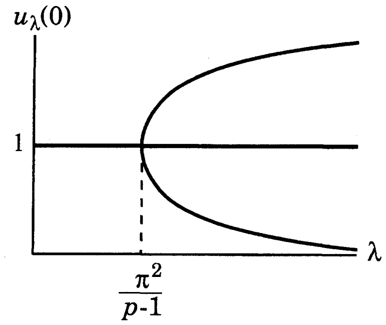

Figure 1: Bifurcation diagram for the Neumann problem (1)

where $D_i, p_i, q_i, \mu_i, \rho_i$, $i = 1, 2$, are constants. As $D_2 \to \infty$, $v_{xx} \to 0$ so that $v_x \to 0$ by the boundary conditions. Hence, in this limit, we have

$$\int_0^1 \left(\mu_2 v - \rho_2 c_2 \frac{u^{p_2}}{v^{q_2}}\right) \mathrm{d}x = 0,$$

so that

$$v^{q_2+1} = \frac{\rho_2 c_2}{\mu_2} \int_0^1 u^{p_2} \mathrm{d}x$$

by regarding v as a constant. Thus, for the stationary problem for (4), we obtain the shadow system,

$$D_1 u_{xx} - \mu_1 u + \rho_1 \left(c_1 \frac{u^{p_1}}{\xi^{q_1}} + \rho_0\right) = 0, \quad u_x = 0 \text{ at } x = \pm 1,$$

which is transformed to (1) like (3).

The following theorem for (1) was proved for $p \in \mathbb{Z}$ in [1] and for $p \in \mathbb{R} \setminus \mathbb{Z}$ in [2].

Theorem 1. *The branch of interior single-peak solutions emanates from* $(\lambda, u) = (\pi^2/(p-1), 1)$ *and the bifurcation is a supercritical pitchfork one. The branch is a graph of* λ *and unbounded in* λ. *Moreover, each solution of the branch is non-degenerate and the Morse index is two.*

Here the Morse index is the number of strictly positive eigenvalues for the associated linear problem

$$\phi'' + \lambda(-1 + p u_\lambda(y)^{p-1})\phi = \mu\phi \quad \text{in } (-1, 1), \quad \phi'(\pm 1) = 0,$$

where $u_\lambda(y)$ represents a solution of the Neumann problem. The last part of Theorem 1 is obvious from the other parts since $\mu = \lambda(p-1), \lambda(p-1) - \frac{1}{4}\pi^2$ are positive eigenvalues of the linear problem for the trivial solution $u = 1$ when $\lambda < \pi^2/(p-1)$, and so is $\mu = \lambda(p-1) - \pi^2$ when $\lambda > \pi^2/(p-1)$. The bifurcation diagram stated in Theorem 1 is sketched in Fig. 1. The upper branch represents interior single-peak solutions.

In the rest of this article we outline the proof of Theorem 1.

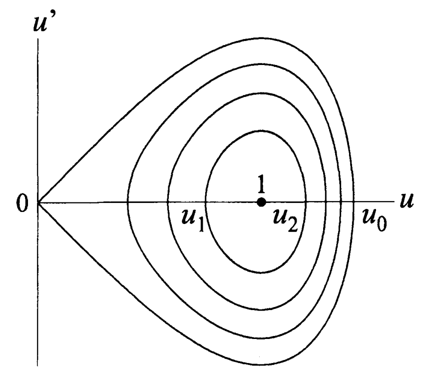

Figure 2: One-parameter family of periodic orbits in (5)

2 Monotonicity of the period functions

Using a transformation $x \mapsto x/\sqrt{\lambda}$, we rewrite (1) as

$$u'' - u + u^p = 0. \tag{5}$$

For any $u_1 \in (0,1)$ Eq. (5) has a periodic solution satisfying $(u(0), u'(0)) = (u_1, 0)$. Let $T(u_1)$ denote its period and let $u_2 \in (1, u_0)$ satisfy $F(u_2) = F(u_1)$, where

$$F(u) = -\tfrac{1}{2}u^2 + \frac{1}{p+1}u^{p+1} + \frac{p-1}{2(p+1)}, \quad u_0 = \sqrt[p-1]{\tfrac{1}{2}(p+1)}.$$

Then we have $(u(\frac{1}{2}T(u_1)), u'(\frac{1}{2}T(u_1))) = (u_2, 0)$. Note that

$$F(u_0) = F(0) = \frac{p-1}{2(p+1)}.$$

As shown in Fig. 2, there exists a one-parameter family of periodic orbits in (5).

The periodic solution $u(x)$ in (5) gives an interior single-peak solution in the Neumann problem (1) when $\frac{1}{2}T(u_1) = 2\sqrt{\lambda}$. Hence, except the last part, Theorem 1 immediately follows from the following theorem.

Theorem 2. *The period function $T(u_1)$ in (5) is strictly decreasing on $(0,1)$.*

To prove this theorem, we use a result of Chicone [3]. We first recall his result. Let $\xi_1 < 0 < \xi_2$ and let $I = (\xi_1, \xi_2) \subset \mathbb{R}$. Suppose that $V : I \to \mathbb{R}$ is a C^3 function satisfying $V(\xi_1) = V(\xi_2)$ and having a minimum $V(0) = 0$ as its only extremum. Consider second-order differential equations of the form

$$\xi'' + \frac{\mathrm{d}V}{\mathrm{d}\xi}(\xi) = 0. \tag{6}$$

Eq. (6) has the trivial solution $\xi = 0$, and any solution $\xi = \xi(t)$ of (6) with $\xi(0) \in I\backslash\{0\}$ and $\xi'(0) = 0$ is periodic. Let $\bar{T}(h)$ be its period with $h = V(\xi(0))$, and define a function $\varphi(\xi)$ as

$$\varphi(\xi) = \frac{V(\xi)}{V'(\xi)^2}. \tag{7}$$

Chicone [3] essentially proved the following result.

Proposition 3 (Chicone [3]). *Suppose that*

$$\varphi''(\xi) \geq 0 \quad \text{for } \xi \in I \setminus \{0\} \tag{8}$$

and the inequality holds in a punctured neighborhood of $\xi = 0$. *Then* $\bar{T}(h)$ *is strictly increasing on* $(0, h_0)$, *where* $h_0 = V(\xi_1)\,(= V(\xi_2))$.

3 Proof of Theorem 2

Using a transformation $u = \xi + 1$ we rewrite (5) as the form of (6) with $\xi_1 = -1$, $\xi_2 = u_0 - 1 > 0$ and $V(\xi) = F(\xi + 1)$. We compute (7) as

$$\varphi''(\xi) = \frac{(p-1)g(\xi+1)}{(p+1)(\xi+1)^4((\xi+1)^{p-1}-1)^4}, \tag{9}$$

where

$$\begin{aligned} g(u) =& pu^{3p-1} - (2p^2 - 3p + 3)u^{2p} + p(2p+1)u^{2p-2} \\ & - p(p-2)u^{p+1} + p(p-7)u^{p-1} + 3. \end{aligned}$$

We begin with the case of $p \in \mathbb{Z}$ with $p > 1$. We easily see that the function $g(u)$ is divisible by $(u-1)^4$ and define a $(3p-5)$-th order polynomial $\bar{g}(u) = g(u)/(u-1)^4$. After some highly nontrivial computations, we prove the following (see [1] for the proof).

Lemma 4. *All coefficients of* $\bar{g}(u)$ *are positive.*

From Lemma 4 and (9) we see that

$$\varphi''(u-1) = \frac{(p-1)\bar{g}(u)}{(p+1)u^4\left(\sum_{j=0}^{p-2} u^j\right)^4} > 0 \quad \text{for } u \in (0, u_0),$$

i.e., condition (8) holds.

We next assume that $p \in \mathbb{Q} \setminus \mathbb{Z}$ with $p > 1$. Let $p = m/n > 1$, where m, n are relatively prime integers and $n \geq 2$. We set $v = u^{1/n}$, $k = m - n > 0$ and $\psi(v) = n^2 g(v^n)$ to have

$$\begin{aligned} \psi(v) =& n(n+k)v^{2n+3k} - (2k^2 + kn + 2n^2)v^{2n+2k} + (n-k)(n+k)v^{2n+k} \\ & + (n+k)(3n+2k)v^{2k} - (n+k)(6n-k)v^k + 3n^2. \end{aligned}$$

We easily see that the polynomial $\psi(v)$ is factorized as $\psi(v) = (v-1)^4\bar{\psi}(v)$, where $\bar{\psi}(v)$ is a $(2n+3k-4)$-th order polynomial. We also prove the following (see [2] for the proof).

Lemma 5. *All coefficients of* $\bar{\psi}(v)$ *are positive.*

From Lemma 5 and (9) we see that

$$\varphi''(v^n - 1) = \frac{(p-1)\bar{\psi}(v)}{(p+1)n^2 v^{4n}\left(\sum_{j=0}^{k-1} v^j\right)^4} > 0 \quad \text{for } v \in \left(0, \sqrt[n]{u_0}\right).$$

i.e., condition (8) holds again.

We turn to the case of $p \in \mathbb{R} \setminus \mathbb{Q}$ with $p > 1$. Take a sequence $\{p_j\}_{j=0}^{\infty}$ such that $p_j \in \mathbb{Q}$ and $\lim_{j \to \infty} p_j = p$. We easily see that condition (8) holds for $p \in \mathbb{R} \setminus \mathbb{Q}$ since it does for $p = p_j$. This completes the proof of Theorem 2 by Proposition 3.

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