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Uniqueness and non-degeneracy of positive radial solutions of quasilinear Schrödinger equations

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1. Introduction and Main results

We consider the following quasilinear elliptic problem:

$$-\Delta u+\lambda u-\kappa\Delta(u^{2})u=g(u) \text{ in } \mathbb{R}^{N}, \tag{1.1}$$

where $\lambda>0$, $\kappa>0$ and $N\geq 2$. Typical examples of the nonlinearity $g(s)$ are given by $g(s)=s^{p}$ for $N\geq 3$ and $g(s)=e^{s}-1$ for $N=2$. In this note, we review recent results on the uniqueness and the non-degeneracy of positive radial solutions of (1.1).

Equation (1.1) can be obtained as a stationary problem of the following modified Schrödinger equation:

$$i\frac{\partial z}{\partial t}=-\Delta z-\kappa\Delta(|z|^{2})z-h(z),\ (t,x)\in(0,\infty)\times \mathbb{R}^{N}, \tag{1.2}$$

where z is a complex-valued function and h has the Gauge invariance, that is, $h(e^{i\theta}z)=h(z)$ for all $\theta\in \mathbb{R}^{N}$. Equation (1.2) appears in the study of plasma physics. (See [**6**, **10**] for the derivations.) Especially if we consider the standing wave of (1.2) of the form $z(t,x)=u(x)e^{i\lambda t}$, then $u(x)$ satisfies (1.1) provided $g(s)=h(s)-\lambda s$.

Equation (1.1) has a variational structure, that is, one can obtain solutions of (1.1) as critical points of the associated functional I defined by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^{N}}(1+2\kappa u^{2})|\nabla u|^{2}+\lambda u^{2}\,dx-\int_{\mathbb{R}^{N}}G(u)\,dx,$$

where $G(s)=\int_{0}^{s}g(t)\,dt$. In applications, the most important solution is the so-called *ground state*, which is a solution of (1.1) having the least energy among all non-trivial solutions. When we study the stability of the standing wave, the uniqueness and the non-degeneracy of the ground state play an important role.

As for the existence of ground states, we have the following result.

Theorem 1.1 [1, 8]. *Let $\lambda > 0$, $\kappa > 0$ and suppose $g(s) = s^p$, $1 < p < \frac{3N+2}{N-2}$ for $N \geq 3$ and $g(s) = e^s - 1$ for $N = 2$. Then there exists a ground state of* (1.1). *Moreover any ground state is of the class $C^2(\mathbb{R}^N)$, positive, radially symmetric and decreasing with respect to $r = |x|$ (up to translation).*

Remark 1.2. *We can obtain the existence of a ground state for more general nonlinearities. (See [**4**, **5**] for details.)*

By Theorem 1.1, we can see that if we could show the uniqueness and the non-degeneracy of positive radial solutions of (1.1), then the ground state of (1.1) is also unique and non-degenerate. However, the uniqueness and the non-degeneracy of positive solutions of (1.1) seem to be difficult and are less studied. In [**2**, **4**, **5**], they proved the uniqueness and non-degeneracy if κ is sufficiently small by applying the perturbation method. In this note, we show the uniqueness and the non-degeneracy of the positive radial solution for another range of parameters λ and κ. Indeed, we have the following result.

Theorem 1.3.

(i) *Suppose $N \geq 3$, $g(s) = s^p$ and $1 < p < \frac{3N+2}{N-2}$. There exists $c_0 = c_0(p) > 0$ such that if $\kappa\lambda^{\frac{2}{p-1}} \geq c_0$, then* (1.1) *has a unique positive radial solution.*

(ii) *Suppose $N = 2$, $\kappa > 0$ and $g(s) = e^s - 1 - s$. There exists $\lambda^* > 0$ independent of κ such that if $\lambda \geq \lambda^*$, then* (1.1) *has a unique positive radial solution.*

Theorem 1.4. *Under the assumptions of Theorem 1.3, the kernel of the linearized operator around the unique positive radial solution w is given by*

$$\mathrm{Ker}(L) = \mathrm{span}\left\{\frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_N}\right\}.$$

Especially w is non-degenerate in $H^1_{rad}(\mathbb{R}^N)$, that is, if $L(\phi) = 0$ and $\phi \in H^1_{rad}(\mathbb{R}^N)$, then $\phi \equiv 0$. Here the linearized operator L of (1.1) *defined by*

$$L(\phi) = -\Delta\phi + \lambda\phi - g'(w)\phi - 2\kappa\ \mathrm{div}(w^2\nabla\phi) - \kappa(4w\Delta w + 2w|\nabla w|^2)\phi.$$

Remark 1.5. *Theorem 1.3 (i) means that if either κ or λ is sufficiently large, then the uniqueness holds. On the other hand in Theorem 1.3 (ii), the uniqueness holds only when λ is sufficiently large. In the case $g(s) = s^p$, we have a nice scaling. Namely for a solution u of* (1.1), *we rescale $\tilde{u}(x)$ as $u(x) = \lambda^{\frac{1}{p-1}}\tilde{u}(\lambda^{\frac{1}{2}}x)$. Then we can see that* (1.1) *is reduced to*

$$-\Delta\tilde{u} + \tilde{u} - \kappa\lambda^{\frac{2}{p-1}}\Delta(\tilde{u}^2)\tilde{u} = \tilde{u}^p \quad \text{in } \mathbb{R}^N.$$

Thus in the case $g(s) = s^p$, we can describe the condition for the uniqueness in terms of $\kappa\lambda^{\frac{2}{p-1}}$. In the case $g(s) = e^s - 1$, such a scaling seems not to work well.

We prove Theorems 1.3-1.4 by applying the shooting method. However since equation (1.1) is quasilinear, it seems to be difficult to consider (1.1) directly. To avoid this difficulty, we adapt *dual approach* as in [**1**, **7**, **12**]. More precisely, we convert our quasilinear equation into a semilinear equation by using a suitable translation f. We will see that the set of positive radial solutions has one-to-one correspondence to that of the semilinear problem. This enables us to apply uniqueness results [**15**, **16**, **17**] for semilinear elliptic equations. We will also see in Lemma 2.3 and Proposition 2.4 below, there is a strong relation between the linearized operator of the original quasilinear equation and that of the converted semilinear equation. By this relation, we have only to study the non-degeneracy for the semilinear problem.

2. Dual approach

In this section, we introduce a dual variational formulation of (1.1). Firstly we study some properties of the unique solution of the ODE related to (1.1). As we will see later, this unique solution gives one-to-one correspondence between (1.1) and a semilinear elliptic problem (2.2) below.

Let $f(t)$ be a solution of the following ODE:

$$f'(t) = \frac{1}{\sqrt{1+2\kappa f(t)^2}} \text{ on } [0,\infty), \ f(0) = 0. \tag{2.1}$$

For $t < 0$, we put $f(t) = -f(-t)$. By the standard theory of ODE, we can see that f is uniquely determined, of class C^2 and invertible on $\mathbb{R}$.

From (2.1), we can show the following.

Lemma 2.1 [**1**]. *$f(t)$ satisfies the following properties:*

(i) $0 \le f(t) \le t$, $0 < f'(t) \le 1$ *for all* $t \ge 0$. $t \le f(t) \le 0$, $0 < f'(t) \le 1$ *for all* $t \le 0$.

(ii) $f''(t) = \dfrac{1}{f(t)}(f'(t)^4 - f'(t)^2)$ *for* $t > 0$.

(iii) $\dfrac{1}{2}f(t) \le f'(t)t \le f(t)$ *for all* $t \ge 0$.

(iv) $\displaystyle\lim_{s\to 0} \frac{f(s)}{s} = 1$.

Using the function $f(t)$, we consider the following semilinear elliptic problem, which we call the *dual problem*:

$$-\Delta v + \lambda f(v)f'(v) = g(f(v))f'(v) \text{ in } \mathbb{R}^N. \tag{2.2}$$

Then we have the following relation between (1.1) and (2.2).

Proposition 2.2 [1]. *$u \in X \cap C^2(\mathbb{R}^N)$ is a positive radial solution of* (1.1) *if and only if $v = f^{-1}(u) \in H^1 \cap C^2(\mathbb{R}^N)$ is a positive radial solution of* (2.2).

Proposition 2.2 tells us that if (2.2) has a unique positive radial solution $\tilde{w}$, then $w = f(\tilde{w})$ is a unique positive radial solution of (1.1). Thus we have only to study the uniqueness of the positive radial solution of the semilinear problem (2.2).

In order to study the non-degeneracy of the unique positive radial solution, we need more detailed correspondence between (1.1) and (2.2).

To state the result, let $\tilde{L} : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be a linearized operator around $\tilde{w}$ of (2.2), which is defined by

$$\begin{aligned} \tilde{L}(\psi) := & -\Delta\psi + \lambda \left(f'(\tilde{w})^2 + f(\tilde{w})f''(\tilde{w})\right)\psi \\ & - \Big(g'\left(f(\tilde{w})\right)f'(\tilde{w})^2 + g(f(\tilde{w}))f''(\tilde{w})\Big)\psi. \end{aligned} \tag{2.3}$$

Then we have the following.

Lemma 2.3. *Suppose that $w \in H^1 \cap C^2(\mathbb{R}^N)$ is a positive solution of* (1.1) *and put $\tilde{w} = f^{-1}(w)$. Let L and $\tilde{L} : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be the linearized operators defined by* (1.4) *and* (2.3) *respectively. Finally for $\phi \in H^2(\mathbb{R}^N)$, we put $\psi = \sqrt{1 + 2\kappa w^2}\phi$. Then it follows that*

$$\tilde{L}(\psi) = \frac{1}{\sqrt{1 + 2\kappa w^2}} L(\phi). \tag{2.4}$$

Proof. By direct computations, we have

$$\nabla\psi = \frac{2\kappa w\phi}{\sqrt{1 + 2\kappa w^2}}\nabla w + \sqrt{1 + 2\kappa w^2}\nabla\phi,$$

and

$$\begin{aligned} \Delta\psi = & \nabla\left(\frac{2\kappa w\phi}{\sqrt{1 + 2\kappa w^2}}\right) \cdot \nabla w + \frac{2\kappa w\phi}{\sqrt{1 + 2\kappa w^2}}\Delta w \\ & + \nabla\left(\sqrt{1 + 2\kappa w^2}\right) \cdot \nabla\phi + \sqrt{1 + 2\kappa w^2}\Delta\phi \\ = & \sqrt{1 + 2\kappa w^2}\Delta\phi + \frac{4\kappa w}{\sqrt{1 + 2\kappa w^2}}\nabla w \cdot \nabla\phi + \frac{2\kappa|\nabla w|^2}{\left(\sqrt{1 + 2\kappa w^2}\right)^3}\phi + \frac{2\kappa w\Delta w}{\sqrt{1 + 2\kappa w^2}}\phi. \end{aligned}$$

Next by Lemma 2.1 (ii) and from (2.1), we get

$$\left(f'(\tilde{w})^2 + f(\tilde{w})f''(\tilde{w})\right)\psi = f'(\tilde{w})^4\psi = \frac{1}{\left(\sqrt{1 + 2\kappa w^2}\right)^3}\phi,$$

and

$$\Big(g'(f(\tilde{w}))f'(\tilde{w})^2 + g(f(\tilde{w}))f''(\tilde{w})\Big)\psi$$
$$= g'(f(\tilde{w}))f'(\tilde{w})^2\psi + g(f(\tilde{w}))\frac{f'(\tilde{w})^4 - f'(\tilde{w})^2}{f(\tilde{w})}\psi$$
$$= \frac{g'(f(\tilde{w}))}{\sqrt{1+2\kappa w^2}}\phi - \frac{2\kappa w}{\left(\sqrt{1+2\kappa w^2}\right)^3}g(f(\tilde{w}))\phi.$$

Thus from (1.1), (1.4) and (2.3), we obtain

$$\begin{aligned}
\tilde{L}(\psi) &= -\Delta\psi + \lambda(f'^2 + ff'')\psi - \left(g'(f(\tilde{w}))f'^2 + g(f(\tilde{w}))f''\right)\psi \\
&= -\sqrt{1+2\kappa w^2}\Delta\phi - \frac{4\kappa w}{\sqrt{1+2\kappa w^2}}\nabla w\cdot\nabla\phi - \frac{2\kappa|\nabla w|^2}{\left(\sqrt{1+2\kappa w^2}\right)^3}\phi \\
&\quad - \frac{2\kappa w\Delta w}{\sqrt{1+2\kappa w^2}}\phi + \frac{\lambda}{\left(\sqrt{1+2\kappa w^2}\right)^3}\phi - \frac{g'(w)}{\sqrt{1+2\kappa w^2}}\phi + \frac{2\kappa w}{\left(\sqrt{1+2\kappa w^2}\right)^3}g(w)\phi \\
&= \frac{1}{\sqrt{1+2\kappa w^2}}\Bigg(-(1+2\kappa w^2)\Delta\phi - 4\kappa w\nabla w\cdot\nabla\phi - \frac{2\kappa|\nabla w|^2}{1+2\kappa w^2}\phi \\
&\qquad -2\kappa w\Delta w\phi + \frac{\lambda}{1+2\kappa w^2}\phi - g'(w)\phi + \frac{2\kappa w}{1+2\kappa w^2}g(w)\phi\Bigg) \\
&= \frac{1}{\sqrt{1+2\kappa w^2}}L(\phi) \\
&\quad + \frac{2\kappa w}{\left(\sqrt{1+2\kappa w^2}\right)^3}\left(\Delta w - \lambda w + 2\kappa w|\nabla w|^2 + 2\kappa w^2\Delta w + g(w)\right)\phi \\
&= \frac{1}{\sqrt{1+2\kappa w^2}}L(\phi).
\end{aligned}$$

This completes the proof. ∎

By Lemma 2.3, we obtain the following result on the linearized operators.

Proposition 2.4. *Suppose that $w \in H^1 \cap C^2(\mathbb{R}^N)$ is a positive solution of* (1.1) *and put $\tilde{w} = f^{-1}(w)$. Then*

(i) *$\phi \in \mathrm{Ker}(L)$ if and only if $\psi = \sqrt{1+2\kappa w^2}\phi \in \mathrm{Ker}(\tilde{L})$.*

(ii) *w is non-degenerate if and only if $\tilde{w}$ is non-degenerate.*

(iii) $\mathrm{Ker}(L) = \mathrm{span}\left\{\frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_N}\right\}$ *if and only if* $\mathrm{Ker}(\tilde{L}) = \mathrm{span}\left\{\frac{\partial \tilde{w}}{\partial x_1}, \cdots, \frac{\partial \tilde{w}}{\partial x_N}\right\}$

Proof. (i) From (2.4), it follows that

$$\tilde{L}(\psi) = 0 \Leftrightarrow L(\phi) = 0.$$

Thus the claim holds.

(ii) The claim follows from (i).

(iii) We assume that $\mathrm{Ker}(L) = \mathrm{span}\left\{\frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_N}\right\}$. Suppose by contradiction that $\mathrm{span}\left\{\frac{\partial \tilde{w}}{\partial x_1}, \cdots, \frac{\partial \tilde{w}}{\partial x_N}\right\} \neq \mathrm{Ker}(\tilde{L})$. Since $\frac{\partial \tilde{w}}{\partial x_i} \in \mathrm{Ker}(\tilde{L})$ for $i = 1, \cdots, N$, it follows that

$$\mathrm{span}\left\{\frac{\partial \tilde{w}}{\partial x_1}, \cdots, \frac{\partial \tilde{w}}{\partial x_N}\right\} \subsetneq \mathrm{Ker}(\tilde{L}).$$

Thus there exists $\psi \not\equiv 0$ such that

$$\psi \in \mathrm{Ker}(\tilde{L}) \setminus \mathrm{span}\left\{\frac{\partial \tilde{w}}{\partial x_1}, \cdots, \frac{\partial \tilde{w}}{\partial x_N}\right\}.$$

Since $\psi \in \mathrm{Ker}(\tilde{L})$, we have $\tilde{L}(\psi) = 0$. Putting $\psi = \sqrt{1 + 2\kappa w^2}\phi$, we obtain $L(\phi) = 0$ by Lemma 2.3. Then by the assumption $\mathrm{Ker}(L) = \mathrm{span}\left\{\frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_N}\right\}$, there exist $c_1, \cdots, c_N$ such that

$$\phi = c_1 \frac{\partial w}{\partial x_1} + \cdots + c_N \frac{\partial w}{\partial x_N}.$$

Now since $w = f(\tilde{w})$, it follows that

$$\frac{\partial w}{\partial x_i} = f'(\tilde{w}) \frac{\partial \tilde{w}}{\partial x_i} = \frac{1}{\sqrt{1 + 2\kappa w^2}} \frac{\partial \tilde{w}}{\partial x_i} \text{ for } i = 1, \cdots, N.$$

Thus we have

$$\psi = c_1 \frac{\partial \tilde{w}}{\partial x_1} + \cdots + c_N \frac{\partial \tilde{w}}{\partial x_N} \in \mathrm{span}\left\{\frac{\partial \tilde{w}}{\partial x_1}, \cdots, \frac{\partial \tilde{w}}{\partial x_N}\right\}.$$

This is a contradiction and hence $\mathrm{Ker}(\tilde{L}) = \mathrm{span}\left\{\frac{\partial \tilde{w}}{\partial x_1}, \cdots, \frac{\partial \tilde{w}}{\partial x_N}\right\}$.

We can show the converse in a similar way. ∎

By Proposition 2.4, we have only to study the non-degeneracy of the unique positive radial solution of the semilinear problem (2.2).

3. Uniqueness of the positive radial solution

In this section, we study the uniqueness of the positive radial solutions (2.2). For simplicity, we put

$$h(s) = g(f(s))f'(s) - \lambda f(s) f'(s) \text{ for } s \geq 0. \tag{3.1}$$

We distinguish the cases $N \geq 3$ and $N = 2$.

3.1. Uniqueness for $N \geq 3$

In this case, we suppose that $g(s) = s^p$, $1 < p < \frac{3N+2}{N-2}$. We apply the following uniqueness result due to [**17**].

Proposition 3.1 [**17**]. *Suppose that there exists $b > 0$ such that*

(i) *h is continuous on $(0, \infty)$, $h(s) \leq 0$ on $(0, b]$ and $h(s) > 0$ for $s > b$.*

(ii) *$g \in C^1(b, \infty)$ and $\frac{d}{ds}\Big(\frac{sh'(s)}{h(s)}\Big) \leq 0$ on (b, ∞).*

Then the semilinear problem:

$$-\Delta u = h(u) \text{ in } \mathbb{R}^N, \ u > 0, \ u \to 0 \text{ as } |x| \to \infty, \ u(0) = \max u(x)$$

has at most one positive radial solution.

Now we can see that h defined in (3.1) is of the class $C^1[0, \infty)$ and

$$h(s) = 0 \Longleftrightarrow f^{p-1}(s) = \lambda \Longleftrightarrow s = f^{-1}(\lambda^{\frac{1}{p-1}}).$$

We put $b := f^{-1}(\lambda^{\frac{1}{p-1}})$. Since $(s-b)g(s) = (s-b)ff'(f^{p-1} - \lambda)$, we can see (i) of Proposition 3.1 holds. From (2.1), we can also observe that

$$f'(b) = \frac{1}{\sqrt{1 + 2\kappa\lambda^{\frac{2}{p-1}}}}.$$

Since $f'(s) \to 0$ as $s \to \infty$, this implies

$$b \to \infty \text{ if and only if } \kappa\lambda^{\frac{2}{p-1}} \to \infty. \tag{3.2}$$

Lemma 3.2 [**1**]. *There exists $c_0 = c_0(p) > 0$ such that if $\kappa\lambda^{\frac{2}{p-1}} \geq c_0$, then h satisfies (ii) of Proposition 3.1.*

3.2. Uniqueness for $N = 2$

In this case, we suppose that $g(s) = e^s - 1$. We apply the following uniqueness result due to Pucci-Serrin [**15, 16**].

Proposition 3.3 ([15, 16]). *Suppose that the function $h(s)$ satisfies the following assumptions:*

(i) *h is continuous on $[0, \infty)$ and $h(0) = 0$.*

(ii) *h is continuously differentiable on $(0, \infty)$.*

(iii) *There exists $s_0 > 0$ such that $h(s_0) = 0$ and*

$$\begin{cases} h(s) < 0 & \text{for } \ 0 < s < s_0, \\ h(s) > 0 & \text{for } \ s_0 < s < \infty. \end{cases}$$

(iv) $\dfrac{d}{ds}\left(\dfrac{H(s)}{h(s)}\right) \geq 0$ *for* $s > 0,\ s \neq s_0$. *Here* $H(s) = \int_0^s h(t)\,dt$.

Then the semilinear problem:

$$-\Delta u = h(u) \text{ in } \mathbb{R}^2,\ u > 0,\ u(x) \to 0 \text{ as } |x| \to \infty,\ u(0) = \max u(x)$$

has at most one positive radial solution.

Now we can see that the function $h(s)$ defined in (3.1) satisfies (i) and (ii). Moreover since $f'(s) \neq 0$ for all $s > 0$, there exists a unique $s_0 > 0$ such that

$$h(s_0) = (e^{f(s_0)} - 1 - \lambda f(s_0))f'(s_0) = 0,$$

$$h(s) < 0 \text{ for } 0 < s < s_0 \text{ and } h(s) > 0 \text{ for } s_0 < s < \infty.$$

Thus it remains to show that (iv) holds.

Lemma 3.4 [3]. *There exists* $\lambda^* > 0$ *independent of* $\kappa > 0$ *such that for any* $\lambda > \lambda^*$ *and* $\kappa > 0$, *it follow that*

$$\frac{d}{ds}\left(\frac{H(s)}{h(s)}\right) \geq 0 \text{ for all } s > 0, s \neq s_0.$$

By Theorem 1.1, Propositions 3.1, 3.3 and Lemmas 3.2, 3.4, we obtain the uniqueness result.

Proposition 3.5.

(i) *Suppose* $N \geq 3$, $g(s) = s^p$ *and* $1 < p < \frac{3N+2}{N-2}$. *There exists* $c_0 = c_0(p) > 0$ *such that if* $\kappa\lambda^{\frac{2}{p-1}} \geq c_0$, *then* (2.2) *has a unique positive radial solution.*

(ii) *Suppose* $N = 2$, $\kappa > 0$ *and* $g(s) = e^s - 1 - s$. *There exists* $\lambda^* > 0$ *independent of* κ *such that if* $\lambda \geq \lambda^*$, *then* (2.2) *has a unique positive radial solution.*

4. Non-degeneracy of the unique positive radial solution

In this section, we show that the unique positive radial solution of (2.2) is non-degenerate. We argue as in [9]. To this aim, we study the structure of radial solutions of the following ODE:

$$\begin{cases} v'' + \dfrac{N-1}{r}v' + \hat{g}(v) = 0, & r \in (0,\infty), \\ v(0) = d > 0. \end{cases} \tag{4.1}$$

Here we denote $' = \frac{d}{dr}$ and

$$\hat{g}(s) = g(f(s))_+ f'(s) - (\lambda - 1) f(s) f'(s). \tag{4.2}$$

Then we can see that for each $d > 0$, (4.1) has a solution $v(r, d)$.

As in [**11**], we classify each $d > 0$ as follows:

$N = \{d > 0\,; \text{there exists } r_0 = r_0(d) \in (0, \infty) \text{ such that } v(r_0, d) = 0\}$.

$G = \{d > 0\,; v(r, d) > 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} v(r, d) = 0\}$.

$P = \{d > 0\,; v(r, d) > 0 \text{ for all } r > 0 \text{ but } \liminf_{r\to\infty} v(r, d) > 0\}$.

First we prove the following properties on N.

Lemma 4.1. *N satisfies the following properties:*

(i) *There exists $\hat{d} > 0$ such that $v(r, \hat{d})$ has a finite zero. Especially it follows that $N \neq \emptyset$.*

(ii) *N is an open set.*

(iii) *For $d \in N$, it follows that $v(r, d) \to -\infty$ as $r \to \infty$.*

Proof. (i) Let $R > 0$ be arbitrarily given. We consider the auxiliary problem:

$$\begin{cases} -\Delta v = \hat{g}(v) & \text{in } B_R(0). \\ v > 0 & \text{in } B_R(0). \\ v = 0 & \text{on } \partial B_R(0). \end{cases} \tag{4.3}$$

Then we can show that (4.3) has a positive radial solution $v_R(x)$. Putting $\hat{d} = v_R(0)$, we obtain $v(R, \hat{d}) = 0$ for a solution of (4.1).

(ii) The claim follows from the continuous dependence on the initial value. (see [**11**] Lemma 13, P. 253.)

(iii) For $d > 0$, let $r_0 = r_0(d) > 0$ be the first zero of $v(r) = v(r, d)$. Then we have $v'(r_0) < 0$.

Suppose that there exists $r_1 > r_0$ such that $v(r_1) < 0$ and $v'(r_1) = 0$. Then from Lemma 2.1 (i), (4.1) and (4.2), we have

$$v''(r_1) = -\hat{g}(v(r_1)) = (\lambda - 1) f(v(r_1)) f'(v(r_1)) < 0.$$

Thus $v(r)$ can not take a negative local minimum for $r > r_0$. This implies that $v(r)$ does not converge to zero as $r \to \infty$ and $v(r)$ does not oscillate at infinity.

Next we suppose by contradiction that there exists $c < 0$ such that $v(r) \to c < 0$ as $r \to \infty$. Then we have $v'(r) \to 0$ as $r \to \infty$. Since $\hat{g}(s) > 0$ for $s < 0$, it follows from (4.1) that $v''(r) < M < 0$ for sufficiently large r and some $M < 0$. This contradicts to the fact $v(r) \to c < 0$ as $r \to \infty$. Thus we obtain $v(r) \to -\infty$ as $r \to \infty$. ∎

Next we show the following result on P.

Lemma 4.2. *P satisfies the following properties:*

(i) *Let $s_1 > 0$ be a unique zero of $\hat{G}(s)$, where $\hat{G}(s) = \int_0^s \hat{g}(t)\,dt$. Then for any $d \leq s_1$, it follows that $d \in P$. Especially we have $(0, s_1] \subset P$.*

(ii) *P is an open set.*

Proof. (i) We define the energy E by

$$E(r) = E(v(r,d)) := \frac{1}{2}(v'(r))^2 + \hat{G}(v(r)), \tag{4.4}$$

Then from (4.1), we have

$$E'(r) = -\frac{N-1}{r}(v'(r))^2 < 0.$$

Now we take $d \leq s_1$. Then it follows from $v(0) = d$ and $v'(0) = 0$ that $E(0) = \hat{G}(d)$. Since $\hat{G}(s) \leq 0$ for $0 \leq s \leq s_1$, we get

$$E(r) < E(0) \leq 0 \text{ for all } r > 0. \tag{4.5}$$

Next we prove that $s_1 \notin N \cup G$. First we show that $v(r, s_1)$ does not have a finite zero. To this aim, suppose by contradiction that $v(r_0) = 0$ for some $r_0 > 0$. Then from $\hat{G}(0) = 0$ and (4.4), it follows that $E(r_0) = \frac{1}{2}(v'(r_0))^2 > 0$. This contradicts to (4.5).

Finally we show that $v(r, s_1)$ does not converges to zero as $r \to \infty$. If $v(r) \to 0$ as $r \to \infty$, then $v(r)$ decays exponentially up to the first derivative. Thus it follows that $E(r) \to 0$ as $r \to \infty$. This is a contradiction.

(ii) By the continuous dependence of the initial value, the conclusion holds. ∎

Now by Proposition 3.5, we know that the positive radial solution of (2.2) is unique. This implies that there exists $d^* > 0$ such that $G = \{d^*\}$. Moreover by the proof of Lemma 4.2, we can see that $s_1 < d^*$. Since N ans P are open, we obtain the following structure.

Proposition 4.3. *There exists a unique $d^* > 0$ such that*

$$N = (d^*, \infty), \ G = \{d^*\} \text{ and } P = (0, d^*).$$

In order to prove the non-degeneracy, we define the Pohozaev value P by

$$P(r) = P(r; v(r,d)) := \frac{r^N}{2}(v'(r))^2 + r^N \hat{G}(v(r)).$$

Then from (4.1), we obtain the Pohozaev type identity:

$$\frac{d}{dr}P(r) = -\frac{N-2}{2}r^{N-1}(v'(r))^2 + Nr^{N-1}\hat{G}(v(r)). \tag{4.6}$$

Moreover we have the following.

Lemma 4.4. *It follows that*

$$\lim_{r\to\infty} P(r; v(r,d)) = \begin{cases} 0 & \textit{for } d = d^* \\ +\infty & \textit{for } d > d^*. \end{cases}$$

Proof. If $d = d^*$, then $v(r, d^*)$ and $v'(r, d^*)$ decay exponentially as $r \to \infty$. Thus we can see that the claim holds.

For $d > d^*$, we have $v(r, d) \to -\infty$ as $r \to \infty$ by Lemma 4.1 (iii) and Proposition 4.3. From (4.2), it follows that $\hat{G}(s) = \frac{\lambda-1}{2} f(s)^2$ for $s < 0$ and hence $\hat{G}(s) \to +\infty$ as $s \to -\infty$. Thus we have $P(r; v(r, d)) \to +\infty$ for $d > d^*$. ∎

Next we consider the linearized equation of (4.1):

$$\begin{cases} \phi'' + \dfrac{N-1}{r}\phi' + \hat{g}'(v)\phi = 0, \quad r \in (0, \infty). \\ \phi(0) = 1, \phi'(0) = 0. \end{cases} \tag{4.7}$$

Since $\frac{\partial v}{\partial d}(r, d^*)$ satisfies (4.7), $\frac{\partial v}{\partial d}$ can be written by a constant multiple of ϕ. Moreover we have the following.

Proposition 4.5. $\dfrac{\partial v}{\partial d}(r, d^*)$ *does not belong to* $H^1(\mathbb{R}^N)$.

Proof. Suppose by contradiction that $\frac{\partial v}{\partial d}(r, d^*) \in H^1(\mathbb{R}^N)$.

Now from (4.6), we have

$$P(r; v(r,d)) = -\frac{N-2}{2}\int_0^r s^{N-1}(v'(s,d))^2\, ds + N\int_0^r s^{N-1}\hat{G}(v(s,d))\, ds.$$

Differentiating it with respect to d, we get

$$\begin{aligned} \frac{\partial}{\partial d}P(r; v(r,d)) &= -(N-2)\int_0^r s^{N-1}v'\Big(\frac{\partial v}{\partial d}\Big)'\, ds + N\int_0^r s^{N-1}\hat{g}(v)\frac{\partial v}{\partial d}\, ds \\ &= \Big[-(N-2)s^{N-1}v'(s,d)\frac{\partial v}{\partial d}(s,d)\Big]_0^r \\ &\quad + (N-2)\int_0^r \Big((N-1)s^{N-2}v' + s^{N-1}v''\Big)\frac{\partial v}{\partial d}\, ds \\ &\quad + N\int_0^r s^{N-1}\hat{g}(v)\frac{\partial v}{\partial d}\, ds. \end{aligned}$$

From (4.1) and $v'(0) = 0$, it follows that

$$\frac{\partial}{\partial d}P(r; v(r,d)) = -(N-2)r^{N-1}v'(r,d)\frac{\partial v}{\partial d}(r,d) + 2\int_0^r s^{N-1}\hat{g}(v)\frac{\partial v}{\partial d}\, ds.$$

Especially taking $d = d^*$, we obtain

$$\frac{\partial}{\partial d}P(r; v(r,d))\Big|_{d=d^*} = -(N-2)r^{N-1}v'(r,d^*)\frac{\partial v}{\partial d}(r,d^*) + 2\int_0^r s^{N-1}\hat{g}(v)\frac{\partial v}{\partial d}(s,d^*)\,ds. \tag{4.8}$$

Moreover from (4.1) and (4.7), we also have

$$\begin{aligned}\left(r^N v'\left(\frac{\partial v}{\partial d}\right)' + r^N\hat{g}(v)\frac{\partial v}{\partial d}\right)' &= r^N\left(\frac{\partial v}{\partial d}\right)'\left(v'' + \frac{N-1}{r}v' + \hat{g}(v)\right)\\ &\quad + r^N v'\left(\left(\frac{\partial v}{\partial d}\right)'' + \frac{N-1}{r}\left(\frac{\partial v}{\partial d}\right)' + \hat{g}'(v)\frac{\partial v}{\partial d}\right)\\ &\quad - (N-2)r^{N-1}v'\left(\frac{\partial v}{\partial d}\right)' + Nr^{N-1}\hat{g}(v)\frac{\partial v}{\partial d}\\ &= -(N-2)r^{N-1}v'\left(\frac{\partial v}{\partial d}\right)' + Nr^{N-1}\hat{g}(v)\frac{\partial v}{\partial d}.\end{aligned}$$

Thus we obtain

$$\frac{\partial}{\partial d}P(r; v(r,d))\Big|_{d=d^*} = r^N v'\left(\frac{\partial v}{\partial d}\right)' + r^N\hat{g}(v)\frac{\partial v}{\partial d}. \tag{4.9}$$

Next by the assumption, it follows that $r^{\frac{N-1}{2}}\frac{\partial v}{\partial d}$, $r^{\frac{N-1}{2}}(\frac{\partial v}{\partial d})' \in L^2(0,\infty)$. Since $v(r,d^*)$ and $v'(r,d^*)$ decay exponentially as $r \to \infty$, we have from (4.9) that

$$\lim_{r\to\infty}\frac{\partial}{\partial d}P(r; v(r,d))\Big|_{d=d^*} = 0. \tag{4.10}$$

Next let ϕ be a solution of (4.7). We claim that ϕ has a definite sign near infinity. First we observe that $\hat{g}'(0) = -(\lambda - 1) < 0$. Since $v(r,d^*)$ decays exponentially as $r \to \infty$, there exists $r_1 > 0$ such that $\hat{g}'(v(r,d^*)) < 0$ for $r > r_1$.

Next we suppose that there exists $r_2 > r_1$ such that $\phi(r_1) > 0$ and $\phi'(r_1) = 0$. Then from (4.7), we have

$$\phi''(r_1) = -\frac{N-1}{r_1}\phi'(r_1) - \hat{g}'(v)\phi(r_1) > 0.$$

This means that ϕ can not take a positive local maximum for $r > r_1$. Similarly we can see that ϕ can not take a negative local minimum. Thus ϕ has a constant sign for $r > r_1$. Hence it follows that either $\frac{\partial v}{\partial d}(r,d^*) > 0$ or $\frac{\partial v}{\partial d}(r,d^*) < 0$ for $r > r_1$.

If $\frac{\partial v}{\partial d}(r,d^*) > 0$ for $r > r_1$, then $v(r,d)$ is increasing with respect to d near d^*. Since $v(r,d^*) > 0$, it follows that $v(r,d) > 0$ for $d > d^*$ and $r > r_1$. By Lemma 4.1 (iii) and Proposition 4.3, this is a contradiction.

Finally suppose that $\frac{\partial v}{\partial d}(r, d^*) < 0$ for $r > r_1$. Now from (4.8) and (4.10) and by the exponential decay of v', we have

$$0 = \lim_{r\to\infty} \frac{\partial}{\partial d} P(r; v)\Big|_{d=d^*} = 2\int_0^\infty s^{N-1}\hat{g}(v)\frac{\partial v}{\partial d}\, ds.$$

On the other hand since $\hat{g}(v) < 0$ and $\frac{\partial v}{\partial d} < 0$ for $r > r_1$, we also have

$$2\int_r^\infty s^{N-1}\hat{g}(v)\frac{\partial v}{\partial d}\, ds > 0.$$

Thus from $v' < 0$ and $\frac{\partial v}{\partial d} < 0$, it follows that

$$\frac{\partial}{\partial d} P(r; v)\Big|_{d=d^*} = -(N-2)r^{N-1}v'\frac{\partial v}{\partial d} + 2\int_0^r s^{N-1}\hat{g}(v)\frac{\partial v}{\partial d}\, ds < 0 \text{ for } r > r_1.$$

This implies that $P(r; v(r, d))$ is decreasing with respect to d near d^*. Thus for $r > r_1$ and $d > d^*$, we obtain

$$P(r; v(r, d^*)) > P(r; v(r, d)).$$

However by Lemma 4.4, we know that $P(r; v(r, d^*)) \to 0$ and $P(r; v(r, d)) \to +\infty$ for $d > d^*$ as $r \to \infty$. This is a contradiction. ∎

Proposition 4.5 implies that the unique positive radial solution $\tilde{w}$ of (2.2) is non-degenerate in $H^1_{rad}(\mathbb{R}^N)$. Finally we show the following result on the linearized operator $\tilde{L} = -\Delta + g'(\tilde{w})$ of (2.2).

Proposition 4.6. *The kernel of $\tilde{L}$ is given by*

$$\mathrm{Ker}(\tilde{L}) = \mathrm{span}\left\{\frac{\partial \tilde{w}}{\partial x_1}, \cdots, \frac{\partial \tilde{w}}{\partial x_N}\right\}.$$

Proof. First we observe that $\mathrm{span}\{\frac{\partial \tilde{w}}{\partial x_1}, \cdots, \frac{\partial \tilde{w}}{\partial x_N}\} \subset \mathrm{Ker}(\tilde{L})$. In fact, since $\tilde{w}$ is a solution of (2.2), $\frac{\partial \tilde{w}}{\partial x_i}$ satisfies

$$-\Delta\left(\frac{\partial \tilde{w}}{\partial x_i}\right) + g'(\tilde{w})\frac{\partial \tilde{w}}{\partial x_i} = 0 \text{ in } \mathbb{R}^N,\ i = 1, \cdots, N.$$

Moreover by the elliptic regularity theory, we can see that $\frac{\partial \tilde{w}}{\partial x_i} \in H^2(\mathbb{R}^N)$. Thus it follows that $\mathrm{span}\{\frac{\partial \tilde{w}}{\partial x_1}, \cdots, \frac{\partial \tilde{w}}{\partial x_N}\} \subset \mathrm{Ker}(\tilde{L})$.

To complete the proof, it suffices to show that $\dim \mathrm{Ker}(\tilde{L}) \le N$. To this aim, we apply the argument in [**13**, **18**]. Suppose that $\phi \in \mathrm{Ker}(\tilde{L})$, that is, $\phi \in H^2(\mathbb{R}^N)$ and it satisfies

$$-\Delta\phi + g'(\tilde{w})\phi = 0 \text{ in } \mathbb{R}^N.$$

Then by the elliptic regularity theory, it follows that $\phi \in C^2(\mathbb{R}^N)$.

Now let μ_i and $\psi_i(\theta)$ with $\theta \in S^{N-1}$ be the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on S^{N-1}, that is,

$$-\Delta_\theta \psi_i = \mu_i \psi_i.$$

Then it follows that

$$0 = \mu_0 < \mu_1 = \cdots = \mu_N = (N-1) < \mu_{N+1} \cdots$$

and $\{\psi_i\}$ forms an orthonormal basis of $L^2(S^{N-1})$.

For $\phi \in \mathrm{Ker}(\tilde{L})$, we define

$$\phi_i(r) := \int_{S^{N-1}} \phi(r,\theta)\psi_i(\theta)\,d\theta.$$

Then we have

$$\phi_i'' + \frac{N-1}{r}\phi_i' + \left(g'(\tilde{w}) - \frac{\mu_i}{r^2}\right)\phi_i = 0, \ \phi_i'(0) = 0. \tag{4.11}$$

Moreover $\phi \in \mathrm{Ker}(\tilde{L})$ can be written as follows.

$$\phi(x) = \phi(r,\theta) = \sum_{i=0}^{\infty} \phi_i(r)\psi_i(\theta). \tag{4.12}$$

When $i = 0$, we have from $\mu_0 = 0$ that

$$\phi_0'' + \frac{N-1}{r}\phi_0' + g'(\tilde{w})\phi_0 = 0.$$

Then by Proposition 4.5, it follows that $\phi_0 \equiv 0$.

Next we show that $\phi_i \equiv 0$ for $i \geq N+1$. If $\phi_i \not\equiv 0$, then $\phi_i(0) \neq 0$ by the uniqueness of the ODE (4.11). Thus we may assume that $\phi_i(0) > 0$. Let $r_i \in (0,\infty]$ be such that $\phi_i(r) > 0$ on $[0, r_i)$ and $\phi_i(r_i) = 0$.

First we suppose that $r_i < \infty$. Multiplying (4.11) by $r^{N-1}\tilde{w}'$ and integrating it over $[0, r_i]$, we get

$$\int_0^{r_i} r^{N-1}\tilde{w}'\phi_i'' + (N-1)r^{N-2}\tilde{w}'\phi_i' + r^{N-1}g'(\tilde{w})\tilde{w}'\phi_i - \mu_i r^{N-3}\tilde{w}'\phi_i\,dr = 0.$$

By the integration by parts, it follows that

$$r_i^{N-1}\tilde{w}'(r_i)\phi_i'(r_i) - \int_0^{r_i} r^{N-1}\tilde{w}''\phi_i'\,dr + \int_0^{r_i} r^{N-1}g'(\tilde{w})\tilde{w}'\phi_i - \mu_i r^{N-3}\tilde{w}'\phi_i\,dr = 0.$$

By the integration by parts again and combined with $\phi(r_i) = 0$, we obtain

$$r_i^{N-1}\tilde{w}'(r_i)\phi_i'(r_i) + \int_0^{r_i} (r^{N-1}\tilde{w}''' + (N-1)r^{N-2}\tilde{w}'' + r^{N-1}g'(\tilde{w})\tilde{w}')\phi_i\, dr$$
$$- \int_0^{r_i} \mu_i r^{N-3}\tilde{w}'\phi_i\, dr = 0.$$

Moreover since $\tilde{w}$ satisfies (4.1), we have

$$\tilde{w}''' + \frac{N-1}{r}\tilde{w}'' - \frac{N-1}{r^2}\tilde{w}' + g'(\tilde{w})\tilde{w}' = 0.$$

Thus we obtain

$$r_i^{N-1}w'(r_i)\phi_i'(r_i) + (N-1-\mu_i)\int_0^{r_i} r^{N-3}\tilde{w}'\phi_i\, dr = 0.$$

Since $\tilde{w}'(r_i) < 0$ and $\phi_i'(r_i) < 0$, it follows that

$$(N-1-\mu_i)\int_0^{r_i} r^{N-3}\tilde{w}'\phi_i\, dr < 0.$$

On the other hand since $\phi_i(r) > 0$ on $(0, r_i)$ and $\mu_i > N-1$ for $i \geq N+1$, we also have

$$0 < (N-1-\mu_i)\int_0^{r_i} r^{N-3}\tilde{w}'\phi_i\, dr.$$

This is a contradiction.

Next suppose that $r_i = +\infty$. Since $\tilde{w}'(r)$ and $\tilde{w}''(r)$ decay exponentially as $r \to \infty$, we have

$$(N-1-\mu_i)\int_0^{\infty} r^{N-3}\tilde{w}'\phi_i\, dr = 0.$$

This implies again that $\phi_i \equiv 0$ for $i \geq N+1$.

Now since $\phi_0 \equiv 0$ and $\phi_i \equiv 0$ for $i \geq N+1$, we have from (4.12) that

$$\phi(x) = \phi(r,\theta) = \sum_{i=1}^{N} c_i\phi_i(r)\phi_i(\theta).$$

This means that dim Ker$(\tilde{L}) \leq N$ and hence Ker$(\tilde{L}) = \text{span}\left\{\frac{\partial \tilde{w}}{\partial x_1}, \cdots, \frac{\partial \tilde{w}}{\partial x_N}\right\}$. ∎

5. Concluding remarks and open questions

In this note, we review recent results on the uniqueness and the non-degeneracy of positive radial solutions of (1.1).

When $N \geq 3$, the exponent $\frac{3N+2}{N-2}$ appears naturally by applying the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ to u^2. Moreover we can see that $p = \frac{3N+2}{N-2}$ is actually the critical exponent for the existence of nontrivial solutions. (See [**1**] for the detail.) As we have shown in Theorems 1.3-1.4, the uniqueness holds for $1 < p < \frac{3N+2}{N-2}$. This implies that p can be H^1-supercritical.

On the other hand when $N = 2$, we have shown the uniqueness only for the case $g(s) = e^s - 1$. By applying the Trudinger-Moser inequality to u^2, we can see that $g(s)$ may have a faster growth like $g(s) \sim e^{c_0 s^4}$ for some $c_0 > 0$. (See [**14**] for the detail.) Thus it is natural to ask "Can we show the uniqueness for the case $g(s) \sim e^{c_0 s^4}$?" Unfortunately, we have no result even if $g(s) = e^{s^2}$.

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