数理解析研究所講究録
第 1901 巻 2014 年 79-91

On profiles of critical eigenfunctions for linearized problems of bistable reaction diffusion equations[1]

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1 Introduction and main results

Let us consider the boundary value problem

$$\begin{cases} \varepsilon^2 u_{xx}(x) + f(u(x)) = 0, & \text{in } (0,1), \\ u_x(0) = u_x(1) = 0, \end{cases} \tag{1.1}$$

Here ε is a positive parameter. The corresponding parabolic PDE to (1.1) is given by

$$\begin{cases} u_t(x,t) = \varepsilon^2 u_{xx}(x,t) + f(u(x,t)), & (x,t) \in (0,1) \times (0,+\infty), \\ u_x(0,t) = u_x(1,t) = 0, & t \in (0,+\infty), \\ u(x,0) = u_0(x), & x \in (0,1). \end{cases} \tag{1.2}$$

We are interested in the case that the function $f \in C^1$ is *a bistable nonlinearity*:

(A1) f has exactly three zeros $u_- < 0 < u_+$ with $f_u(0) > 0, \quad f_u(u_\pm) < 0$.

As a typical example of f, we refer $f(u) = u - u^3$ with $u_\pm = \pm 1$. Then (1.2) is the simplest model which describes phase transition of materials; both $u_\pm$ and 0 represent two stable states and one unstable state. Under (A1) the reaction-diffusion problem (1.2) defines a gradient system on Sobolev space $H^1(0,1)$, and in particular, the solution $u(x,t)$ exists globally in time and converges to a solution of (1.1) as $t \to +\infty$.

Main interests to (1.1) and (1.2) are to understand the existence and stability of (stationary) solutions of (1.1), and to understand the tranisient and asymptotic dynamics in (1.2). Moreover, it is interesting to consider these problems in a situation that ε is small, in view of *pattern formation*.

We summarize the classical results on (1.1) (see Propositions 2.1-2.3, and also [1], [3] and [5]). Any nontrivial solution of (1.1) is characterized by the number of zeros in the interval $(0,1)$. We say $u_{n,\varepsilon}$ is *the n-mode solution* of (1.1) if $u_{n,\varepsilon}$ satisfies (1.1) and it exactly admit n-zeros $z_1, \cdots, z_\ell, \cdots, z_n$ in the interval $(0,1)$. Here we also assume that

(A2) f_u is decreasing for $u \in (0, u_+)$.

(A3) f is odd; $u_- = -u_+$.

[1]This article is based on a jointwork with Shoji Yotsutani (Ryukoku Univ.).

It is well known that for arbitrary $n \in \mathbf{N}$, exactly two n-mode solutions $\pm u_{n,\varepsilon}$ (with $u_{n,\varepsilon}(0) > 0$) exists if and only if $\varepsilon \in (0, \sqrt{f_u(0)}/(n\pi))$. The n-mode solution $u_{n,\varepsilon}$ that $-u_- < u_{n,\varepsilon}(x) < u_+$ for every $x \in [0,1]$, and it can be extended as a periodic function on $\mathbf{R}$ because of the boundary condition. Moreover, the concept of bifurcation theory is useful to understand the solution structure of (1.1): the two bifurcation curves

$$\mathcal{S}_n^{\pm} := \{(\varepsilon, u) \in \mathbf{R}_+ \times H^1(0,1) \mid (\varepsilon, u) = (\varepsilon, \pm u_{n,\varepsilon}(x)),\ \varepsilon \in (0, \sqrt{f_u(0)}/n\pi)\}$$

appears from the line of unstable solution $u = 0$ at the point $(\varepsilon, u) = ((\sqrt{f_u(0)}/n\pi)^{-1}, 0)$. See Figs. 1 and 2.

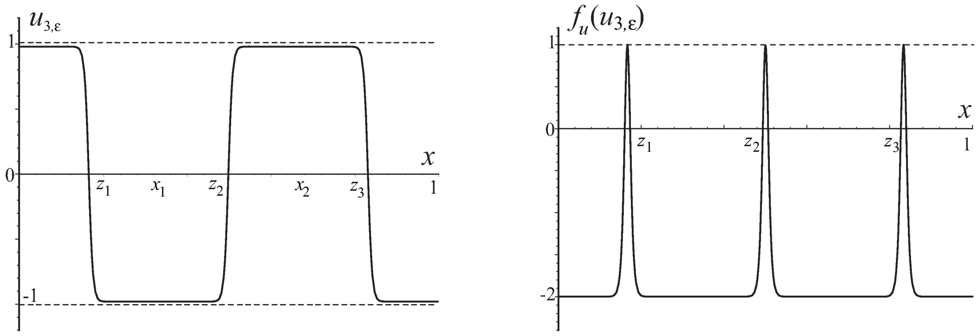

Figure 1: Profiles of $u_{3,\varepsilon}(x)$ and $f_u(u_{3,\varepsilon})$ for $f(u) = u - u^3$ when ε is small.

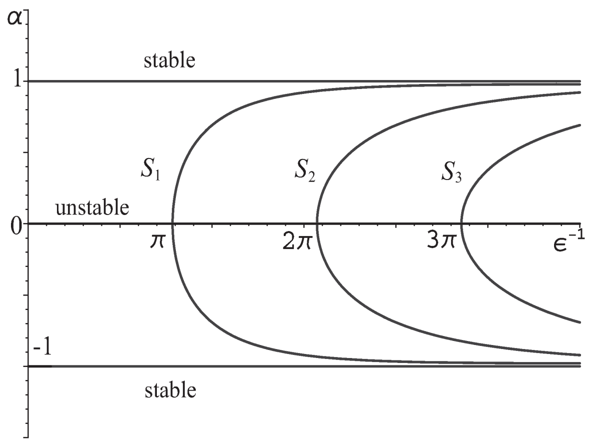

Figure 2: Bifurcation diagram of (1.1) for $f(u) = u - u^3$ in $(\varepsilon^{-1}, \alpha)$-plane, where $\alpha = u(0)$.

Let us introduce

$$z_\ell^n = \frac{2\ell - 1}{2n},\ \ell = 1, \ldots, n, \quad x_\ell^n = \frac{\ell}{n},\ \ell = 0, \ldots, n.$$

It follows from (A3) that the set of zeros to $u_{n,\varepsilon}$ is given by $\{z_\ell^n\}_{\ell=1}^n$ and the period (resp. anti-period) of $u_{n,\varepsilon}$ is $2/n$ (resp. $1/n$). If ε is sufficiently small, then $u_{n,\varepsilon}(x)$ is close to either $u_\pm$ away from zeros and *the transition layers* connecting between $u_\pm$ appears in the neighborhoods of $\{z_\ell^n\}$. The transition layer at $x = z_\ell^n$ is characterized by

$$u_{n,\varepsilon}(x) \sim (-1)^\ell U\Big(\frac{x - z_\ell^n}{\varepsilon}\Big),$$

in a neighborhood of z_ℓ^n, where $U = U(z)$ is the (unique) heteloclinic solution of the rescaled problem

$$\begin{cases} U_{zz}(z) + f(U(z)) = 0 & \text{in } \mathbf{R}, \\ U(-\infty) = u_-, \quad U(+\infty) = u_+, \quad U(0) = 0. \end{cases} \tag{1.3}$$

Now we fix $n \in \mathbf{N}$ arbitrarily and choose $\varepsilon > 0$ small enough (with respect to n). Let us consider the linearized eigenvalue problem associated with $u_{n,\varepsilon}$

$$\begin{cases} \varepsilon^2 \varphi_{xx}(x) + f_u(u_{n,\varepsilon}(x))\varphi(x) + \lambda\varphi(x) = 0 & \text{in } (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0. \end{cases} \tag{1.4}$$

For $j \in \mathbf{N} \cup \{0\}$, we denote by $\lambda_j = \lambda_j^{n,\varepsilon}$ and $\varphi_j(x) = \varphi_j^{n,\varepsilon}(x)$, the $(j+1)$-th eigenvalue and the corresponding eigenfunction. Fig. 3 displays some profiles of eigenfunctions in the case $f(u) = u - u^3$.

It follows from the Sturm-Liouville theory with (A2) and (A3) that $\lambda_0^{n,\varepsilon} < \cdots < \lambda_{n-1}^{n,\varepsilon} < 0 < \lambda_n^{n,\varepsilon} < \cdots < +\infty$ (see e.g., [1]). This fact concludes that $u_{n,\varepsilon}$ is unstable. On the other hand, $u_{n,\varepsilon}$ is known *metastable* in the following sense: for $0 \le j < n$, $\lambda_j^{n,\varepsilon} = O(\exp(-d/\varepsilon))$ with some $d > 0$ as $\varepsilon \to 0$. This fact is closely related with *the super slow dynamics* arrising in (1.2) (see Carr-Pego [2], Fusco-Hale [4]).

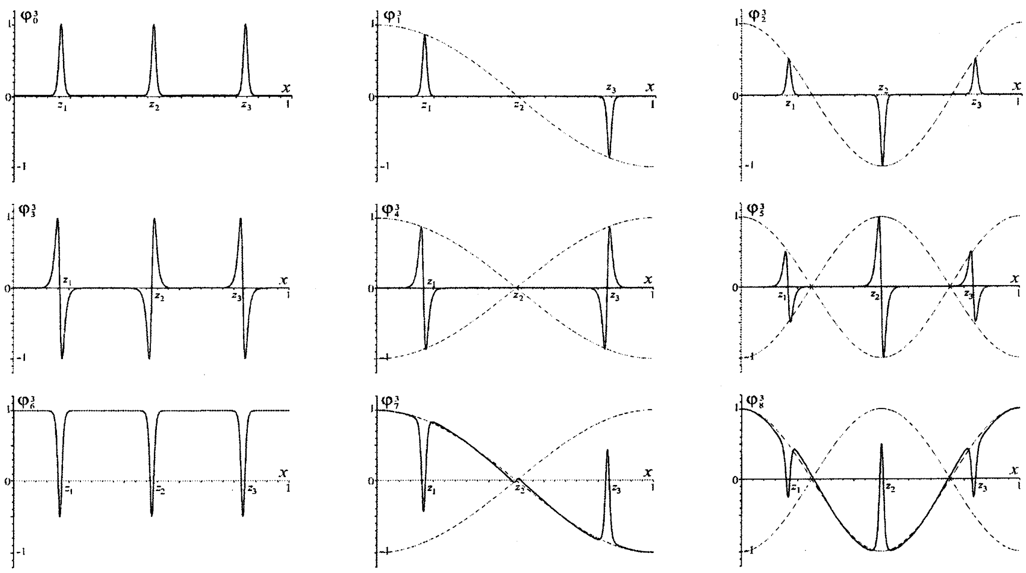

Figure 3: Profiles of $\varphi_j^{3,\varepsilon}$ $(j = 0, \ldots, 8)$ for $f(u) = u - u^3$.

In this article we are interested in the ε-dependence of $\lambda_j^{n,\varepsilon}$ and $\varphi_j^{n,\varepsilon}$ of (1.4) in the situation that n is fixed and ε is given small enough. More precisely, we will investigate the asymptotic profiles $\varphi_j^{n,\varepsilon}$ with $0 \leq j < n$ as $\varepsilon \to 0$. The eigenfunctions and eigenvalues are often called as *critical eigenfunctions* and *critical eigenvalues*, and they play an essential role for the stability of $u_{n,\varepsilon}$. Moreover, the profiles of critical eigenfunctions give us significant information on the dynamical behavior of (1.2) near the stationary solutions, and it has been investigated by the previous works [2] and [4].

The purpose of this article is to show the following conjecture:

"For the case $0 \leq j < n$, $\varphi_j^{n,\varepsilon}$ has spikes in neighborhoods of z_ℓ^n $(\ell = 1, \cdots, n)$, and the height of each spike is proportional to $\cos j\pi z_\ell^n$",

which is proposed by E. Yanagida. In order to justify Yanagida's conjecture, our problems are formulated as follows:

- to characterize the locallized patterns (spike) for the critical eigenfunctions as $\varepsilon \to 0$,
- to give some *"symmetric" property* arising in critical eigenfunctions.

For the principal eigenfunction $\varphi_0^{n,\varepsilon}$, which is chosen as positive function, some fundamental results can be easily obtained (see Proposition 2.5-2.7). In particular, it has exactly n-spikes with the same height at $x = z_\ell^n$.

Fix the height of $\varphi_0^{n,\varepsilon}$ arbitrarily. Our main results are given by the following two theorems.

Theorem 1. *Assume* (A1)-(A3) *and the normalized condition* $\displaystyle\lim_{\varepsilon\to 0}\frac{\varphi_j^{n,\varepsilon}(0)}{\varphi_0^{n,\varepsilon}(0)} = \cos j\pi z_1^n$. *For any δ with $0 < \delta < 1/(4n)$,*

(i) $\displaystyle\lim_{\varepsilon\to 0}\frac{\varphi_j^{n,\varepsilon}(x)}{\varphi_0^{n,\varepsilon}(x)} = \cos j\pi z_1^n$, *uniformly in* $[0, x_1^n - \delta]$,

(ii) $\displaystyle\lim_{\varepsilon\to 0}\frac{\varphi_j^{n,\varepsilon}(x)}{\varphi_0^{n,\varepsilon}(x)} = \cos j\pi z_\ell^n$, *uniformly in* $[x_{\ell-1}^n + \delta, x_\ell^n - \delta]$, $\ell = 2, \ldots, n-1$,

(iii) $\displaystyle\lim_{\varepsilon\to 0}\frac{\varphi_j^{n,\varepsilon}(x)}{\varphi_0^{n,\varepsilon}(x)} = \cos j\pi z_n^n$, *uniformly in* $[x_{n-1}^n + \delta, 1]$.

Theorem 2. *Suppose the assumptions in Theorem 1 holds. Then for* $\ell = 1, 2, \ldots, n-1$,

$$\lim_{\varepsilon\to 0}\frac{\varphi_j^{n,\varepsilon}(x_\ell^n + \varepsilon z)}{\varphi_0^{n,\varepsilon}(x_\ell^n + \varepsilon z)} = c_{j,\ell}^n + d_{j,\ell}^n \tanh\sqrt{-f_u(u_\pm)}z,$$

uniformly in any compact subset in $\mathbf{R}$,

$$c_{j,\ell}^n = \frac{1}{2}(\cos j\pi z_{\ell+1}^n + \cos j\pi z_\ell^n), \quad d_{j,\ell}^n = \frac{1}{2}(\cos j\pi z_{\ell+1}^n - \cos j\pi z_\ell^n).$$

To this end, we remark that in the previous work in [7]-[9] we have obtained more precise result on every eigenfunction in the two cases of typical f: $f(u) = \sin u$ and $f(u) = u - u^3$.

2 Preliminaries

2.1 n-mode solutions

We recall well known results on (1.1) under assumptions (A1)-(A3). Let us introduce

$$F(u) := \int_0^u f(s)ds.$$

Proposition 2.1. *Suppose that $f \in C^1$ satisfies (A1)-(A3). Let $\varepsilon > 0$ and $n \in \mathbf{N}$. The equation*

$$\int_{-\alpha}^{\alpha} \frac{1}{\sqrt{2(F(\alpha) - F(w))}} dw = \frac{1}{n\varepsilon}, \quad \alpha \in (0, u_+) \tag{2.1}$$

has a solution $\alpha = \alpha_{n,\varepsilon}$ if and only if $\varepsilon \in (0, \sqrt{f_u(0)}/n\pi)$. Moreover, $\alpha_{n,\varepsilon}$ is unique, decreasing with respect to ε,

$$\lim_{\varepsilon \to 0} \alpha_{n,\varepsilon} = u_+, \ \text{and} \ \lim_{\varepsilon \to \sqrt{f_u(0)}/n\pi} \alpha_{n,\varepsilon} = 0.$$

Proposition 2.2. *Suppose that $f \in C^1$ satisfies (A1)-(A3). Let $\varepsilon > 0$ and $n \in \mathbf{N}$. If $\varepsilon \in (0, f_u(0)/(n\pi))$, there exists two n-mode solutions $\pm u_{n,\varepsilon}(x)$ of (1.1) satisfying*

$$\int_{u_{n,\varepsilon}(x)}^{\alpha_{n,\varepsilon}} \frac{1}{\sqrt{2(F(\alpha_{n,\varepsilon}) - F(w))}} dw = \frac{x}{n\varepsilon} \quad \text{for } x \in [0, 1/n], \tag{2.2}$$

where $\alpha_{n,\varepsilon}$ is as in Proposition 2.1. Moreover, $u_{n,\varepsilon}$ satisfies the following properties:

(i) *$u_{n,\varepsilon}$ is monotone decreasing in $[0, x_1^n]$,*

(ii) $u_{n,\varepsilon}\left(x + \dfrac{1}{n}\right) = -u_{n,\varepsilon}(x)$ *for $x \in \mathbf{R}$,*

(iii) $u_{n,\varepsilon}(x_\ell^n) = (-1)^\ell \alpha_{n,\varepsilon}$ *and* $(u_{n,\varepsilon})_x(x_\ell^n) = 0$ *for $\ell = 0, 1, \ldots, n$,*

(iv) $u_{n,\varepsilon}\left(\dfrac{1}{n} - x\right) = -u_{n,\varepsilon}(x)$ *for $x \in \mathbf{R}$,*

(v) $u_{n,\varepsilon}(z_\ell^n) = 0$, *for $\ell = 1, \ldots, n$.*

Conversely, any n-mode solutions of (1.1) is given either $u_{n,\varepsilon}(x)$ or $-u_{n,\varepsilon}(x)$.

Proposition 2.3. *Assume (A1)-(A3). Let $U = U(z)$ be the (unique) heteloclinic solution of the rescaled problem (2.5). Then, for each $\ell = 1, \ldots, n$,*

$$\lim_{\varepsilon \to 0} u_{n,\varepsilon}(z_\ell^n + \varepsilon z) = (-1)^\ell U(z),$$

in the topology of $C^2_{loc}(\mathbf{R})$.

Remark 2.1. For a wider class of the bistable f, Propositions 2.1-2.3 can be generalized; (A2) and (A3) are to be replaced in more general assumptions

(A2′) $f_u(u) \leq \dfrac{f(u)}{u}$ for $u \in (0, u_+)$,

(A3′) $\displaystyle\int_{u_-}^{u_+} f(s)ds = 0,$

respectively. Roughly speaking, (A2′) coinsides with the monotonicity of $\alpha_{n,\varepsilon}$ and also the degeneracy of the linearized oparators around $u_{n,\varepsilon}$. The assumption (A3′) proves the existence of $U(z)$ of (2.5); it characterizes the limiting profiles of n-mode solutions as $\varepsilon \to 0$.

2.2 Linearized eigenvalue problem

We rewrite (1.4) as follows:

$$\begin{cases} \varepsilon^2(\varphi_{xx}(x) + (g_{n,\varepsilon}(x) + \lambda)\varphi(x) = 0, & \text{in } (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0, \end{cases} \tag{2.3}$$

where

$$g_{n,\varepsilon}(x) = f_u(u_{n,\varepsilon}(x)).$$

On the other hand, we have

$$\begin{cases} \varepsilon^2(u_{n,\varepsilon})_{xx}(x) + \bar{g}_{n,\varepsilon}(x)u_{n,\varepsilon}(x) = 0, & \text{in } (0,1), \\ (u_{n,\varepsilon})_x(0) = (u_{n,\varepsilon})_x(1) = 0, \end{cases} \tag{2.4}$$

where

$$\bar{g}_{n,\varepsilon}(x) := \frac{f(u_{n,\varepsilon}(x))}{u_{n,\varepsilon}(x)}.$$

By comparing these two problems with the Sturm-Liouville theory and (A2), (A3), it is obtained that $\lambda_0^{n,\varepsilon} < \cdots < \lambda_{n-1}^{n,\varepsilon} < 0 < \lambda_n^{n,\varepsilon} < \cdots < +\infty$ (see e.g., [1]). This fact concludes that $u_{n,\varepsilon}$ is unstable.

We now summarize some fundamental properties of $g_{n,\varepsilon}(x)$:

(G1) $g_{n,\varepsilon}$ is $1/n$-periodic,

(G2) for each $\ell = 1, \ldots, 2n-1$, $g_{n,\varepsilon}$ is even with respect to $x = \dfrac{\ell}{2n}$,

(G3) $g_{n,\varepsilon}$ is monotone increasing for $x \in [0, 1/2n]$,

(G4) for each $\ell = 1, \ldots, n$, $\lim_{\varepsilon\to 0} g_{n,\varepsilon}(x_\ell^n + \varepsilon z) = f_u(U(z))$, in the topology of $C^2_{loc}(\mathbf{R})$,

(G5) $\lim_{\varepsilon\to 0} g_{n,\varepsilon}(\varepsilon z) = f_u(u_\pm)$, in the topology of $C^2_{loc}(\mathbf{R})$,

(G6) for arbitrality fixed $\delta \in (0, 1/(4n))$, there exists $\beta_{n,\varepsilon}$, $\bar{\beta}_{n,\varepsilon} \in \mathbf{R}$ such that $\lim_{\varepsilon\to 0}\beta_{n,\varepsilon} = \lim_{\varepsilon\to 0}\bar{\beta}_{n,\varepsilon} = f_u(u_\pm)$, and for sufficiently small ε, $\beta_{n,\varepsilon} \leq g_{n,\varepsilon}(x) \leq \bar{\beta}_{n,\varepsilon}$ for $x \in [0, \delta]$.

Roughly speaking, (G1)-(G2), (G3) and (G4)-(G6) provide the symmetic, monotone and asymptotic properties of $\varphi_0^{n,\varepsilon}$. In particular, we see from (2.5) that

$$\begin{cases} (U_z)_{zz}(z) + f_u(U(z))U_z(z) = 0 & \text{in } \mathbf{R}, \\ U_z \in H^1(\mathbf{R}), \end{cases} \tag{2.5}$$

and hence, we can expect that $\lambda_0^{n,\varepsilon} \sim 0$ as $\varepsilon \to 0$ and

$$\varphi_0^{n,\varepsilon}(x) \sim U_z\Big(\frac{x - z_\ell^n}{\varepsilon}\Big),$$

in a neighborhood of z_ℓ^n.

However, we would need more precise estimates on $g_{n,\varepsilon}(x)$ than (G6) in order to justify the above observation. Under the assumptions (A1)-(A3) one can derive from Proposition 2.1 that as $\varepsilon \to 0$, $\beta_{n,\varepsilon} = f_u(u_\pm) + O(e^{-d/n\varepsilon})$ and $\bar\beta = f_u(u_\pm) + O(e^{-\bar d/n\varepsilon})$ with some $d, \bar d > 0$. They lead us to the asymptotic estimates of critical eigenvalues, which is obtained in Carr-Pego [2] already stated as in the introduction.

Here we would like to refer the result of Carr and Pego, and prepare the following proposition, which is an easy consequence of the result by [2].

Proposition 2.4. *Assume* (A1)-(A3). *Let* $n \in \mathbf{N}$ *and* $\varepsilon \in (0, \sqrt{f_u(0)}/n\pi)$. *Then,*

(i) $\lim_{\varepsilon \to 0} \lambda_0^{n,\varepsilon} = 0$,

(ii) *for* $0 < j < n$, $\lim_{\varepsilon \to 0} \dfrac{\lambda_j^{n,\varepsilon} - \lambda_0^{n,\varepsilon}}{\varepsilon^2} = 0$.

2.3 Properties of principal eigenfunction

With use of (i) of Proposition 2.4, one can easily obtain some properties of the principal eigenfunctions.

Proposition 2.5. *Assume* (A1)-(A3). *Let* $n \in \mathbf{N}$ *and* $\varepsilon \in (0, \sqrt{f_u(0)}/n\pi)$. *Then,*

(i) $\varphi_0^{n,\varepsilon}$ *is* $1/n$*- periodic,*

(ii) $\varphi_0^{n,\varepsilon}$ *is even w.r.t.* $x = z_\ell^n$,

(iii) $\varphi_0^{n,\varepsilon}$ *is monotone increasing for* $x \in [0, 1/2n]$.

Proof. The claims can be proved by using the properties (G1)-(G3). We omit a detail. □

Proposition 2.6. *Assume* (A1)-(A3). *Let* $n \in \mathbf{N}$ *and* $\varepsilon \in (0, \sqrt{f_u(0)}/n\pi)$. *Then, Under a suitable normalizations on* $\varphi_0^{n,\varepsilon}$,

$$\lim_{\varepsilon \to 0} \varphi_0^{n,\varepsilon}(z_\ell^n + \varepsilon z) = (-1)^{\ell j} U_z(z) \tag{2.6}$$

and

$$\lim_{\varepsilon \to 0} \frac{\varphi_0^{n,\varepsilon}(x_\ell^n + \varepsilon z)}{\varphi_0^{n,\varepsilon}(0)} = \cosh(\sqrt{-f_u(u_\pm)}z) \tag{2.7}$$

in the topology of $C^2_{loc}(\mathbf{R})$.

Proof. The claims can be proved by using the properties (G4), (G5) and (i) of Proposition 2.4. We omit a detail. $\square$

Proposition 2.7. *Assume* (A1)-(A3). *Let* $n \in \mathbf{N}$ *and* $\varepsilon \in (0, \sqrt{f_u(0)}/n\pi)$. *Then for any* $\delta \in (0, 1/(4n))$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^x \frac{\varphi_0^{n,\varepsilon}(0)^2}{\varphi_0^{n,\varepsilon}(\xi)^2} d\xi = \frac{1}{\sqrt{-f_u(u_\pm)}}, \quad \textit{uniformly in } [\delta, z_1^n].$$

Proof. The comparison argument with (G6) and (i) of Proposition 2.4 proves the proposition. We omit a detail. $\square$

3 Symmetric Properties on eigenfunctions

Fix $n \in \mathbf{N}$, $\varepsilon \in (0, \sqrt{f_u(0)}/n\pi)$ and $0 < j < n$. Let us consider the auxially problem:

$$\begin{cases} \varepsilon^2 \tilde{\varphi}_{xx}(x) + (g_{n,\varepsilon}(x) + \lambda)\tilde{\varphi}(x) = 0 & \text{in } (0,1), \\ \tilde{\varphi}(0) = 0, \end{cases} \tag{3.1}$$

where $\tilde{\varphi}_x(0) \neq 0$ is to be chosen in later. When $\lambda = \lambda_j^{n,\varepsilon}$ for some $0 < j < n$, we denote the solution of (3.1) by $\tilde{\varphi}_j^{n,\varepsilon}$.

Set

$$M_j^{n,\varepsilon}(x) := \begin{bmatrix} \varphi_j^{n,\varepsilon}(x) & \tilde{\varphi}_j^{n,\varepsilon}(x) \\ (\varphi_j^{n,\varepsilon})_x(x) & (\tilde{\varphi}_j^{n,\varepsilon})_x(x) \end{bmatrix}.$$

The following proposition on *the monodromy martix* is proved by Floquet theory with (G1) and (G2), and by a standard linear algebra .

Proposition 3.1. *Let* $n \in \mathbf{N}$, $\varepsilon \in (0, \sqrt{f_u(0)}/n\pi)$. *For any* $\varphi_j^{n,\varepsilon}$ *and* $\tilde{\varphi}_j^{n,\varepsilon}$ *of* (2.3) *and* (3.1), *there exists a unique* $\gamma_j^{n,\varepsilon} > 0$ *and a unique* $\kappa_j^{n,\varepsilon} > 0$ *such that*

$$M_j^{n,\varepsilon}\left(\frac{1}{n}\right) M_j^{n,\varepsilon}(0)^{-1} = \begin{bmatrix} \cos\frac{j\pi}{n} & \frac{1}{\gamma_j^{n,\varepsilon}} \sin\frac{j\pi}{n} \\ -\gamma_j^{n,\varepsilon} \sin\frac{j\pi}{n} & \cos\frac{j\pi}{n} \end{bmatrix},$$

$$M_j^{n,\varepsilon}\left(\frac{1}{2n}\right) M_j^{n,\varepsilon}(0)^{-1} = \begin{bmatrix} \kappa_j^{n,\varepsilon} \cos\frac{j\pi}{2n} & \frac{\kappa_j^{n,\varepsilon}}{\gamma_j^{n,\varepsilon}} \sin\frac{j\pi}{2n} \\ -\frac{\gamma_j^{n,\varepsilon}}{\kappa_j^{n,\varepsilon}} \sin\frac{j\pi}{2n} & \frac{1}{\kappa_j^{n,\varepsilon}} \cos\frac{j\pi}{2n} \end{bmatrix}.$$

Remark 3.1. In the proof of Proposition 3.1, (G2) plays an essential role (see [6]). In absense of (G2), the representation of $M_j^{n,\varepsilon}$ seems to be complicated.

Moreover, under a suitable normalizing condition on $\varphi_j^{n,\varepsilon}$ and $\tilde{\varphi}_j^{n,\varepsilon}$, fine symmetric properties are obtained.

Corollary 1. *Suppose that* $\gamma_j^{n,\varepsilon}$ *and* $\kappa_j^{n,\varepsilon}$ *be positive numbers given in Proposition* 3.1. *Under the normalizing condition*

$$(\tilde{\varphi}_j^{n,\varepsilon})_x(0) = \gamma_j^{n,\varepsilon}\varphi_j^{n,\varepsilon}(0)$$

the following (i)-(iv) *hold* :

(i) $M_j^{n,\varepsilon}\left(\frac{1}{n} - x\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} M_j^{n,\varepsilon}(x) \begin{bmatrix} \cos\frac{j\pi}{n} & \sin\frac{j\pi}{n} \\ \sin\frac{j\pi}{n} & -\cos\frac{j\pi}{n} \end{bmatrix}$,

(ii) $M_j^{n,\varepsilon}\left(x + \frac{1}{n}\right) = M_j^{n,\varepsilon}(x) \begin{bmatrix} \cos\frac{j\pi}{n} & \sin\frac{j\pi}{n} \\ -\sin\frac{j\pi}{n} & \cos\frac{j\pi}{n} \end{bmatrix}$,

(iii) *for* $\ell = 0, 1, \dots, n$,

$$M_j^{n,\varepsilon}(x_\ell^n) = \varphi_j^{n,\varepsilon}(0) \begin{bmatrix} \cos j\pi x_\ell^n & \sin j\pi x_\ell^n \\ -\gamma_j^{n,\varepsilon}\sin j\pi x_\ell^n & \gamma_j^{n,\varepsilon}\cos j\pi x_\ell^n \end{bmatrix}.$$

Let $\varphi_j^{n,\varepsilon}$ and $\tilde{\varphi}_j^{n,\varepsilon}$ be as in Proposition 3.2. Then,

$$\frac{d}{dx}\left(\frac{\tilde{\varphi}_j^{n,\varepsilon}}{\varphi_j^{n,\varepsilon}}\right) = \frac{1}{(\varphi_j^{n,\varepsilon})^2} W(\varphi_j^{n,\varepsilon}, \tilde{\varphi}_j^{n,\varepsilon}) = \frac{\gamma_j^{n,\varepsilon}\varphi_j^{n,\varepsilon}(0)^2}{\varphi_j^{n,\varepsilon}(x)^2},$$

where W denotes the Wronskian. It implies that

$$\frac{\tilde{\varphi}_j^{n,\varepsilon}(x)}{\varphi_j^{n,\varepsilon}(x)} = \int_0^x \frac{\gamma_j^{n,\varepsilon}\varphi_j^{n,\varepsilon}(0)^2}{\varphi_j^{n,\varepsilon}(\xi)^2} d\xi$$

for sufficiently small $x \in [0, z_1^n]$ and putting $x = 1/(2n)$ we see

$$\tan\frac{j\pi}{2n} = \int_0^{\frac{1}{2n}} \frac{\gamma_j^{n,\varepsilon}\varphi_j^{n,\varepsilon}(0)^2}{\varphi_j^{n,\varepsilon}(\xi)^2} d\xi. \tag{3.2}$$

4 Outline of proofs

4.1 Quotient of eigenfunctions

Fix $n \in \mathbf{N}$, $\varepsilon \in (0, \sqrt{f_u(0)}/(n\pi))$. Suppose $0 < j < n$. For the eigenfunction $\varphi_j^{n,\varepsilon}$ of (2.3) and the eigenfunction $\tilde{\varphi}_j^{n,\varepsilon}$ of the auxially problem (3.1), set

$$\psi_j^{n,\varepsilon}(x) := \frac{\varphi_j^{n,\varepsilon}(x)}{\varphi_0^{n,\varepsilon}(x)} \quad \text{and} \quad \tilde{\psi}_j^{n,\varepsilon}(x) := \frac{\tilde{\varphi}_j^{n,\varepsilon}(x)}{\varphi_0^{n,\varepsilon}(x)} \tag{4.1}$$

respectively, A direct calculation shows us that $\psi_j^{n,\varepsilon}$ and also $\tilde{\psi}_j^{n,\varepsilon}$ satisfy the following linear differential equation

$$(\psi_j^{n,\varepsilon})_{xx} + 2\frac{(\varphi_0^{n,\varepsilon}(x))_x}{\varphi_0^{n,\varepsilon}(x)}(\psi_j^{n,\varepsilon})_x + \frac{\lambda_j^{n,\varepsilon} - \lambda_0^{n,\varepsilon}}{\varepsilon^2}\psi_j^{n,\varepsilon} = 0 \tag{4.2}$$

(in a subinterval in $[0,1]$) with

$$(\psi_j^{n,\varepsilon})_x(0) = (\psi_j^{n,\varepsilon})_x(1) = 0 \quad \text{and} \quad \tilde{\psi}_j^{n,\varepsilon}(0) = \tilde{\psi}_j^{n,\varepsilon}(1) = 0$$

Without loss of generality, we can always assume that

$$\psi_j^{n,\varepsilon}(0) = \cos j\pi z_\ell^n.$$

In addition, for each $\varphi_j^{n,\varepsilon}$, $\tilde{\varphi}_j^{n,\varepsilon}$ should be chosen to satisfy the normalized condition

$$(\tilde{\varphi}_j^{n,\varepsilon})_x(0) = \gamma_j^{n,\varepsilon}\varphi_j^{n,\varepsilon}(0)$$

as in Proposition 3.1; so that

$$(\tilde{\psi}_j^{n,\varepsilon})_x(0) = \frac{(\tilde{\varphi}_j^{n,\varepsilon})_x(0)}{\varphi_{j^*}^{n,\varepsilon}(0)} = \gamma_j^{n,\varepsilon}\psi_j^{n,\varepsilon}(0).$$

Proposition 4.1. *Let $n \in \mathbf{N}$, and let $0 < j < n$ be fixed. Suppose that $\psi_j^{n,\varepsilon}$ and $\tilde{\psi}_j^{n,\varepsilon}$ are the functions defined in* (4.1). *Then,*

$$\lim_{\varepsilon\to 0}\psi_j^{n,\varepsilon}(x) = \cos j\pi z_1^n \quad \textit{uniformly in } [0, z_1^n].$$

Proof. We first remark that $\varphi_n^{n,\varepsilon}(x) > 0$ for $x \in [0, z_1^n)$ because of (G2). Hence it follows from Sturm comparison theorem and $0 < \lambda_j^{n,\varepsilon} - \lambda_0^{n,\varepsilon} < \lambda_n^{n,\varepsilon} - \lambda_0^{n,\varepsilon}$ that $\psi_j^{n,\varepsilon}$ is positive and bounded:

$$\cos j\pi z_1^n \frac{\varphi_n^{n,\varepsilon}(x)}{\varphi_0^{n,\varepsilon}(x)} \le \psi_j^{n,\varepsilon}(x) \le \cos j\pi z_1^n \quad \text{for } x \in [0, z_1^n].$$

Now we rewrite (4.2) as follows:

$$\left[\varphi_0^{n,\varepsilon}(x)^2(\psi_j^{n,\varepsilon})_x\right]_x + \frac{\lambda_j^{n,\varepsilon} - \lambda_0^{n,\varepsilon}}{\varepsilon^2}\varphi_0^{n,\varepsilon}(x)^2\psi_j^{n,\varepsilon} = 0.$$

By integrating the above equation from 0 to x we have

$$(\psi_j^{n,\varepsilon})_x(x) + \frac{\lambda_j^{n,\varepsilon} - \lambda_0^{n,\varepsilon}}{\varepsilon^2}\int_0^x \left(\frac{\varphi_0^{n,\varepsilon}(\xi)}{\varphi_0^{n,\varepsilon}(x)}\right)^2 \psi_j^{n,\varepsilon}(\xi)d\xi = 0. \tag{4.3}$$

By using (iii) of Theorem 2, we obtain

$$|(\psi_j^{n,\varepsilon})_x(x)| \le \frac{\lambda_j^{n,\varepsilon} - \lambda_0^{n,\varepsilon}}{\varepsilon^2} \cdot z_1^n \cos j\pi z_1^n.$$

Therefore, by (i) of Theorems 2

$$\lim_{\varepsilon\to 0}(\psi_j^{n,\varepsilon})_x(x) = 0 \quad \text{uniformly in } [0, z_1^n],$$

and we obtain

$$\lim_{\varepsilon\to 0}\psi_j^{n,\varepsilon}(x) = \cos j\pi z_1^n \quad \text{uniformly in } [0, z_1^n].$$

□

Corollary 2. *Let $n \in \mathbf{N}$, and let $0 < j < n$ be fixed. Suppose $\gamma_j^{n,\varepsilon}$ be the number as in Proposition 3.2. Then,*

$$\lim_{\varepsilon \to 0} \varepsilon \gamma_j^{n,\varepsilon} = \sqrt{-f_u(u_\pm)} \tan \frac{j\pi}{2n}.$$

Proof. We see that for each $x \in [0, z_1^n]$,

$$\frac{\tilde{\psi}_j^{n,\varepsilon}(x)}{\psi_j^{n,\varepsilon}(x)} = \varepsilon \gamma_j^{n,\varepsilon} \int_0^x \frac{\cos^2 j\pi z_1^n}{\psi_j^{n,\varepsilon}(\xi)^2} \cdot \frac{1}{\varepsilon} \frac{\varphi_0^{n,\varepsilon}(0)^2}{\varphi_0^{n,\varepsilon}(\xi)^2} d\xi. \tag{4.4}$$

Set $x = z_1^n = 1/(2n)$ in (4.4). It follows from (3.2) with the mean value theorem that

$$\tan \frac{j\pi}{2n} = \varepsilon \gamma_j^{n,\varepsilon} \cdot \frac{\cos^2 j\pi z_1^n}{\psi_j^{n,\varepsilon}(\xi_1)^2} \cdot \frac{1}{\varepsilon} \int_0^{1/2n} \frac{\varphi_0^{n,\varepsilon}(0)^2}{\varphi_0^{n,\varepsilon}(\xi)^2} d\xi,$$

where $\xi_1 \in (0, z_1^n)$.

By letting $\varepsilon \to 0$ with Propositions 4.1 and 5.1, we obtain a desired result. □

Proposition 4.2. *Let $n \in \mathbf{N}$, $0 < j < n$, $\psi_j^{n,\varepsilon}$ and $\tilde{\psi}_j^{n,\varepsilon}$ as in Proposition 5.1. Then, for any δ with $0 < \delta < 1/(4n)$,*

$$\lim_{\varepsilon \to 0} \tilde{\psi}_j^{n,\varepsilon}(x) = \sin j\pi z_1^n \quad \textit{uniformly in } [\delta, z_1^n].$$

Proof. The proof is a modification of the proof of Corollary 2. We omit a detail. □

4.2 Proof of Theorem 1

Proof of Theorem 1. Now we extend the convergence result in Proposition 5.1 to $x \in [0, x_1^n - \delta]$. Suppose $x \in [z_1^n, x_1^n - \delta]$. By (i) of Proposition 3.2 we have

$$\psi_j^{n,\varepsilon}(x) + i\tilde{\psi}_j^{n,\varepsilon}(x) = \left(\psi_j^{n,\varepsilon}\left(\frac{1}{n} - x\right) - i\tilde{\psi}_j^{n,\varepsilon}\left(\frac{1}{n} - x\right)\right) e^{ij\pi x_1^n}.$$

Hence it follows from Proposition 4.1 that

$$\lim_{\varepsilon \to 0} \left(\psi_j^{n,\varepsilon}(x) + i\tilde{\psi}_j^{n,\varepsilon}(x)\right) = e^{-ij\pi/(2n)} e^{ij\pi/n} = e^{ij\pi z_1^n}. \tag{4.5}$$

uniformly in $[z_1^n, x_1^n - \delta]$. By combining Proposition 4.1 with (4.5), we obtain

$$\lim_{\varepsilon \to 0} \psi_j^{n,\varepsilon}(x; \varepsilon) = \cos j\pi z_1^n \quad \text{uniformly in } [0, x_1^n - \delta] \tag{4.6}$$

and

$$\lim_{\varepsilon \to 0} \tilde{\psi}_j^{n,\varepsilon}(x; \varepsilon) = \sin j\pi z_1^n \quad \text{uniformly in } [\delta, x_1^n - \delta]. \tag{4.7}$$

Therefore (i) of Theorem 1 follows from (4.6).

We note tha (iii) of Theorem 1 can be proved in similarly.

Finally, we suppose $x \in [x_{\ell-1}^n + \delta, x_\ell^n - \delta]$ for some $\ell = 2, \dots, n$. It follows from (ii) of Proposition 3.2 that

$$\psi_j^{n,\varepsilon}(x) + i\tilde{\psi}_j^{n,\varepsilon}(x) = \left(\psi_j^{n,\varepsilon}\left(x - \frac{\ell - 1}{n}\right) + i\tilde{\psi}_j^{n,\varepsilon}\left(x - \frac{\ell - 1}{n}\right)\right) e^{ij\pi(\ell-1)/n}.$$

Therefore (4.6) and (4.7) leads us to

$$\lim_{\varepsilon\to 0}\left(\psi_j^{n,\varepsilon}(x;\varepsilon)+i\tilde{\psi}_j^{n,\varepsilon}(x;\varepsilon)\right)=e^{ij\pi/(2n)}e^{ij\pi(\ell-1)/n}=e^{ij\pi z_\ell^n}.$$

uniformly in $[x_{\ell-1}^n+\delta, x_\ell^n-\delta]$. In particular, we have

$$\lim_{\varepsilon\to 0}\psi_j^{n,\varepsilon}(x;\varepsilon)=\cos j\pi z_\ell^n$$

uniformly in $[x_{\ell-1}^n+\delta, x_\ell^n-\delta]$ and it proves (ii) of Theorem 1.

Thus it completes the proof of Theorem 1. □

4.3 Proofs of Theorem 2

Proof of Theorem 2. For each $j=1,\dots,n-1$ and $\ell=1,\dots,n-1$ and

$$\Psi_j^{n,\varepsilon}(z;x_\ell^n):=\psi_j^{n,\varepsilon}(x_\ell^n+\varepsilon z)=\frac{\varphi_j^{n,\varepsilon}(x_\ell^n+\varepsilon z)}{\varphi_0^{n,\varepsilon}(x_\ell^n+\varepsilon z)}.$$

Similarly as (4.2) we can see that $\Psi_j^{n,\varepsilon}(\cdot;x_\ell^n)$ satisfy the differential equation

$$(\Psi_j^{n,\varepsilon})_{zz}+2\frac{\varepsilon(\varphi_0^{n,\varepsilon})_x(x_\ell^n+\varepsilon z)}{\varphi_0^{n,\varepsilon}(x_\ell^n+\varepsilon z)}(\Psi_j^{n,\varepsilon})_z+(\lambda_j^{n,\varepsilon}-\lambda_0^{n,\varepsilon})\Psi_j^{n,\varepsilon}=0. \tag{4.8}$$

By (iii) of Proposition 3.2 and Theorem 1 the initial condition on $\Phi_j^{n,\varepsilon}(\cdot;x_\ell^n)$ at $z=0$ is given by

$$\Phi_j^{n,\varepsilon}(0;x_\ell^n)=\cos j\pi z_1^n\cos j\pi x_\ell^n=\frac{1}{2}(\cos j\pi z_{\ell+1}^n+\cos j\pi z_\ell^n)$$

and

$$(\Phi_{j,\ell}^{n,\varepsilon})_z(0;x_\ell^n)=\frac{\varepsilon\gamma_j^{n,\varepsilon}}{\tan j\pi z_1^n}\cdot\frac{1}{2}(\cos j\pi z_{\ell+1}^n-\cos j\pi z_\ell^n).$$

Hence it follows from (ii) of Proposition 2.4, and Corollary 2 that

$$\lim_{\varepsilon\to 0}\Psi_j^{n,\varepsilon}(z;x_\ell^n)=\Psi_j^{n,0}(z;x_\ell^n)$$

uniformly in any compact subset in $\mathbf{R}$, where $\Psi_j^{n,0}(z;x_\ell^n)$ is the solution of

$$(\Psi_j^{n,0})_{zz}+2\sqrt{-f_u(u_\pm)}\tanh\sqrt{-f_u(u_\pm)}z(\Psi_j^{n,0})_z=0 \tag{4.9}$$

with the initial condition

$$\Psi_j^{n,0}(0;x_\ell^n)=\frac{1}{2}(\cos j\pi z_{\ell+1}^n+\cos j\pi z_\ell^n),$$

$$(\Psi_j^{n,0})_z(0;x_\ell^n)=\frac{\sqrt{-f_u(u_\pm)}}{2}(\cos j\pi z_{\ell+1}^n-\cos j\pi z_\ell^n).$$

By solving (4.9) with the initial condition above, we are led to

$$\Psi_{j,\ell}^{n,0}(z)=\frac{\cos z_{\ell+1}^n+\cos z_\ell^n}{2}+\frac{\cos z_{\ell+1}^n-\cos z_\ell^n}{2}\tanh\sqrt{-f_u(u_\pm)}z.$$

Thus it completes the proof. □

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