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SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATORS ON PERIODIC DISCRETE GRAPHS

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ABSTRACT. We consider Schrödinger operators with periodic potentials on periodic discrete graphs. It is known that the spectrum of the Schrödinger operator consists of an absolutely continuous part (a union of a finite number of non-degenerated bands) plus a finite number of flat bands, i.e., eigenvalues of infinite multiplicity. We present results about spectral properties of the operators: 1) estimates of the Lebesgue measure of the spectrum in terms of geometric parameters of the graph, which become identities for some class of graphs; 2) spectral analysis of the Schrödinger operators on loop graphs, defined in our paper; 3) the existence and positions of maximal number of flat bands for specific graphs.

1. INTRODUCTION

We consider Laplace operators and Schrödinger operators with periodic potentials on $\mathbb{Z}^d$-periodic discrete graphs, $d \geqslant 2$. Schrödinger operators on periodic graphs and their spectra often appear in the applied physical science. There are a lot of papers and books on the spectrum of discrete Laplacians on finite and infinite graphs (see [BK12], [Ch97], [CDS95], [CDGT88], [P12] and references therein). There are results about spectral properties of discrete Schrödinger operators on specific $\mathbb{Z}^d$-periodic graphs. Schrödinger operators with decreasing potentials on the lattice $\mathbb{Z}^d$ are considered by Boutet de Monvel-Sahbani [BS99], Isozaki-Korotyaev [IK12], Rosenblum-Solomjak [RoS09] and see references therein. Ando [A12] considers the inverse spectral theory for the discrete Schrödinger operators with finitely supported potentials on the hexagonal lattice. Gieseker-Knörrer-Trubowitz [GKT93] consider Schrödinger operators with periodic potentials on the lattice $\mathbb{Z}^2$, the simplest example of $\mathbb{Z}^2$-periodic graphs. They study its Bloch variety and its integrated density of states. Korotyaev-Kutsenko [KK10] – [KK10b] study the spectra of the discrete Schrödinger operators on graphene nano-tubes and nano-ribbons in external fields.

1.1. **The definition of Schrödinger operators on periodic graphs.** Let $\Gamma = (V, \mathcal{E})$ be a connected graph, possibly having loops and multiple edges, where V is the set of its vertices and $\mathcal{E}$ is the set of its unoriented edges. The graphs under consideration are embedded into $\mathbb{R}^d$. An edge connecting vertices u and v from V will be denoted as the unordered pair $(u, v)_e \in \mathcal{E}$ and is said to be *incident* to the vertices. Vertices $u, v \in V$ will be called *adjacent* and denoted by $u \sim v$, if $(u, v)_e \in \mathcal{E}$. We define the degree $\varkappa_v = \deg v$ of the vertex $v \in V$ as the number of all its incident edges from $\mathcal{E}$ (here a loop is counted twice). Below we consider locally finite $\mathbb{Z}^d$-periodic graphs Γ, i.e., graphs satisfying the following conditions:

1) *the number of vertices from V in any bounded domain $\subset \mathbb{R}^d$ is finite;*
2) *the degree of each vertex is finite;*

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3) Γ has the periods (a basis) $a_1, \ldots, a_d$ in $\mathbb{R}^d$, such that Γ is invariant under translations through the vectors $a_1, \ldots, a_d$:

$$\Gamma + a_s = \Gamma, \qquad \forall\, s \in \mathbb{N}_d = \{1, \ldots, d\}.$$

In the space $\mathbb{R}^d$ we consider a coordinate system with the origin at some point O. The coordinate axes of this system are directed along the vectors $a_1, \ldots, a_d$. Below the coordinates of all vertices of Γ will be expressed in this coordinate system. From the definition it follows that a $\mathbb{Z}^d$-periodic graph Γ is invariant under translations through any integer vector m in the basis $a_1, \ldots, a_d$:

$$\Gamma + m = \Gamma, \qquad \forall m \in \mathbb{Z}^d.$$

Let $\ell^2(V)$ be the Hilbert space of all square summable functions $f : V \to \mathbb{C}$, equipped with the norm

$$\|f\|^2_{\ell^2(V)} = \sum_{v \in V} |f(v)|^2 < \infty.$$

We define the self-adjoint Laplacian (or the Laplace operator) Δ on $f \in \ell^2(V)$ by

$$(\Delta f)(v) = \sum_{(v,u)_e \in \mathcal{E}} \big(f(v) - f(u)\big), \qquad v \in V. \tag{1.1}$$

We recall basic facts about the spectrum for both finite and periodic graphs (see [Me94], [M91], [M92], [MW89]): *the point 0 belongs to the spectrum $\sigma(\Delta)$ containning in $[0, 2\varkappa_+]$, i.e.,*

$$0 \in \sigma(\Delta) \subset [0, 2\varkappa_+], \qquad \text{where} \qquad \varkappa_+ = \sup_{v \in V} \deg v < \infty. \tag{1.2}$$

We consider the Schrödinger operator H acting on the Hilbert space $\ell^2(V)$ and given by

$$H = \Delta + Q, \tag{1.3}$$

$$(Qf)(v) = Q(v) f(v), \qquad \forall v \in V, \tag{1.4}$$

where we assume that the potential Q is real valued and satisfies

$$Q(v + a_s) = Q(v), \qquad \forall\, (v, s) \in V \times \mathbb{N}_d.$$

1.2. **The definitions of fundamental graphs and edge indices.** In order to define the Floquet-Bloch decomposition (1.11) of Schrödinger operators we need to introduce two oriented edges (u, v) and (v, u) for each unoriented edge $(u, v)_e \in \mathcal{E}$: the oriented edge starting at $u \in V$ and ending at $v \in V$ will be denoted as the ordered pair (u, v). We denote the set of all oriented edges by $\mathcal{A}$.

We define *the fundamental graph* $\Gamma_f = (V_f, \mathcal{E}_f)$ of the periodic graph Γ as a graph on the surface $\mathbb{R}^d/\mathbb{Z}^d$ by

$$\Gamma_f = \Gamma/\mathbb{Z}^d \subset \mathbb{R}^d/\mathbb{Z}^d. \tag{1.5}$$

In the literature the fundamental graph is also called the *quotient graph*. The fundamental graph Γ_f has the vertex set V_f, the set $\mathcal{E}_f$ of unoriented edges and the set $\mathcal{A}_f$ of oriented edges, which are finite. Denote by $v_1, \ldots, v_\nu$ the vertices of V_f, where $\nu < \infty$ is the number of vertices of Γ_f. We identify them with the vertices of V from the set $[0, 1)^d$ by

$$V_f = [0, 1)^d \cap V = \{v_1, v_2, \ldots, v_\nu\}, \tag{1.6}$$

see Fig.1. Due to (1.6) for any $v \in V$ the following unique representation holds true:

$$v = [v] + \tilde{v}, \qquad [v] \in \mathbb{Z}^d, \qquad \tilde{v} \in V_f \subset [0, 1)^d. \tag{1.7}$$

In other words, each vertex v can be represented uniquely as the sum of an integer part $[v] \in \mathbb{Z}^d$ and a fractional part $\tilde{v}$ that is a vertex of the fundamental graph Γ_f. We introduce *an edge index*, which is important to study the spectrum of Schrödinger operators on periodic graphs. For any oriented edge $\mathbf{e} = (u, v) \in \mathcal{A}$ we define **the edge "index"** $\tau(\mathbf{e})$ as the integer vector by

$$\tau(\mathbf{e}) = [v] - [u] \in \mathbb{Z}^d, \tag{1.8}$$

where due to (1.7) we have

$$u = [u] + \tilde{u}, \qquad v = [v] + \tilde{v}, \qquad [u], [v] \in \mathbb{Z}^d, \qquad \tilde{u}, \tilde{v} \in V_f.$$

If $\mathbf{e} = (u, v)$ is an oriented edge of the graph Γ, then by the definition of the fundamental graph there is an oriented edge $\tilde{\mathbf{e}} = (\tilde{u}, \tilde{v})$ on Γ_f. For the edge $\tilde{\mathbf{e}} \in \mathcal{A}_f$ we define the edge index $\tau(\tilde{\mathbf{e}})$ by

$$\tau(\tilde{\mathbf{e}}) = \tau(\mathbf{e}). \tag{1.9}$$

In other words, edge indices of the fundamental graph Γ_f are induced by edge indices of the periodic graph Γ. In a fixed coordinate system the index of the fundamental graph edge is uniquely determined by (1.9), since we have

$$\tau(\mathbf{e} + m) = \tau(\mathbf{e}), \qquad \forall\, (\mathbf{e}, m) \in \mathcal{A} \times \mathbb{Z}^d.$$

But generally speaking, the edge indices depend on the choice of the coordinate origin O and the choice of the basis $a_1, \ldots, a_d$. Edges with nonzero indices will be called **bridges** (see Fig.1). They are important to describe the spectrum of the Schrödinger operator. The bridges provide the connectivity of the periodic graph and the removal of all bridges disconnects the graph into infinitely many connected components. The set of all bridges of the fundamental graph Γ_f we denote by $\mathcal{B}_f$.

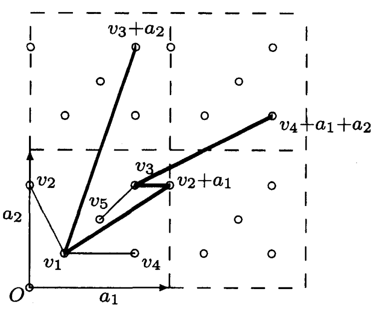

FIGURE 1. A graph Γ with $\nu = 5$; only edges of the fundamental graph Γ_f are shown; the bridges $(v_1, v_2 + a_1)$, $(v_1, v_3 + a_2)$, $(v_3, v_2 + a_1)$, $(v_3, v_4 + a_1 + a_2)$ of Γ_f are marked by bold.

1.3. **Floquet decomposition of Schrödinger operators.** Recall that the fundamental graph $\Gamma_f = (V_f, \mathcal{E}_f)$ has the finite vertex set $V_f = \{v_1, \ldots, v_\nu\} \subset [0, 1)^d$. Due to this notation, we can denote the potential Q on the fundamental graph Γ_f by

$$Q(v_j) = q_j, \qquad j \in \mathbb{N}_\nu = \{1, \ldots, \nu\}. \tag{1.10}$$

The Schrödinger operator $H = \Delta + Q$ on $\ell^2(V)$ has the standard decomposition into a constant fiber direct integral

$$\ell^2(V) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}^{\oplus} \ell^2(V_f)\, d\vartheta, \qquad UHU^{-1} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}^{\oplus} H(\vartheta)d\vartheta, \tag{1.11}$$

for some unitary operator U. Here $\ell^2(V_f) = \mathbb{C}^\nu$ is the fiber space and $H(\vartheta)$ is the Floquet $\nu \times \nu$ (fiber) matrix and $\vartheta \in \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ is the quasimomentum.

Note that the decomposition of discrete Schrödinger operators on periodic graphs into a constant fiber direct integral (1.11) (without an exact form of fiber operators) was discussed by Higuchi-Shirai [HS04], Rabinovich-Roch [RR07], Higuchi-Nomura [HN09]. In particular, they prove that the spectrum of Schrödinger operators consists of an absolutely continuous part and a finite number of flat bands (i.e., eigenvalues with infinite multiplicity). The absolutely continuous spectrum consists of a finite number of intervals (spectral bands) separated by gaps.

Theorem 1.1. *i) The Schrödinger operator $H = \Delta+Q$ acting on $\ell^2(V)$ has the decomposition into a constant fiber direct integral (1.11), where the Floquet (fiber) matrix $H(\vartheta)$ is given by*

$$H(\vartheta) = \Delta(\vartheta) + q, \qquad q = \mathrm{diag}(q_1, \ldots, q_\nu), \qquad \forall\, \vartheta \in \mathbb{T}^d. \tag{1.12}$$

The Floquet matrix $\Delta(\vartheta) = \{\Delta_{jk}(\vartheta)\}_{j,k=1}^{\nu}$ for the Laplacian Δ is given by

$$\Delta_{jk}(\vartheta) = \varkappa_j\delta_{jk} - \begin{cases} \sum\limits_{\mathbf{e}=(v_j,v_k)\in\mathcal{A}_f} e^{i\langle\tau(\mathbf{e}),\vartheta\rangle}, & \text{if } \ (v_j,v_k) \in \mathcal{A}_f \\ 0, & \text{if } \ (v_j,v_k) \notin \mathcal{A}_f \end{cases}, \tag{1.13}$$

where $\varkappa_j$ is the degree of v_j, δ_{jk} is the Kronecker delta and $\langle\cdot\,,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^d$.

ii) Let $H^{(1)}(\vartheta)$ be a Floquet matrix for H defined by (1.12), (1.13) in another coordinate system with an origin O_1. Then the matrices $H^{(1)}(\vartheta)$ and $H(\vartheta)$ are unitarily equivalent for all $\vartheta \in \mathbb{T}^d$.

iii) The entry $\Delta_{jk}(\cdot)$ of the Floquet matrix $\Delta(\cdot) = \{\Delta_{jk}(\cdot)\}_{j,k=1}^{\nu}$ is constant iff there is no bridge $(v_j, v_k) \in \mathcal{A}_f$.

iv) The Floquet matrix $\Delta(\cdot)$ has at least one non-constant entry $\Delta_{jk}(\cdot)$ for some $j \leqslant k$.

Remark. 1) The identity (1.13) for the Floquet (fiber) operator is new. It is important to study spectral properties of Schrödinger operators acting on graphs.

2) Badanin-Korotyaev-Saburova [BKS13] derived different spectral properties of normalized Laplacians on $\mathbb{Z}^2$-periodic graphs.

2. Main results

Theorem 1.1 and standard arguments (see Theorem XIII.85 in [RS78]) describe the spectrum of the Schrödinger operator $H = \Delta + Q$. Each Floquet $\nu \times \nu$ matrix $H(\vartheta), \vartheta \in \mathbb{T}^d$, has ν eigenvalues $\lambda_n(\vartheta)$, $n \in \mathbb{N}_\nu$, which are labeled in increasing order (counting multiplicities) by

$$\lambda_1(\vartheta) \leqslant \lambda_2(\vartheta) \leqslant \ldots \leqslant \lambda_\nu(\vartheta), \qquad \forall\, \vartheta \in \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d. \tag{2.1}$$

Since $H(\vartheta)$ is self-adjoint and analytic in $\vartheta \in \mathbb{T}^d$, each $\lambda_n(\cdot)$, $n \in \mathbb{N}_\nu$, is a real and piecewise analytic function on the torus $\mathbb{T}^d$ and defines *a dispersion relation.* Define the *spectral bands* $\sigma_n(H)$ by

$$\sigma_n(H) = [\lambda_n^-, \lambda_n^+] = \lambda_n(\mathbb{T}^d), \qquad n \in \mathbb{N}_\nu. \tag{2.2}$$

Sy and Sunada [SS92] show that the lower point of the spectrum $\sigma(H)$ of the operator H is $\lambda_1(0)$, i.e., $\lambda_1(0) = \lambda_1^-$. Thus, the spectrum of the operator H on the graph Γ is given by

$$\sigma(H) = \bigcup_{\vartheta\in\mathbb{T}^d} \sigma\big(H(\vartheta)\big) = \bigcup_{n=1}^{\nu} \sigma_n(H). \tag{2.3}$$

Note that if $\lambda_n(\cdot) = C_n = \text{const}$ on some set $\mathscr{B} \subset \mathbb{T}^d$ of positive Lebesgue measure, then the operator H on Γ has the eigenvalue C_n with infinite multiplicity. We call C_n a *flat band.* Each flat band is generated by finitely supported eigenfunction, see [HN09]. Thus, the spectrum of the Schrödinger operator H on the periodic graph Γ has the form

$$\sigma(H) = \sigma_{ac}(H) \cup \sigma_{fb}(H). \tag{2.4}$$

Here $\sigma_{ac}(H)$ is the absolutely continuous spectrum, which is a union of non-degenerated intervals, and $\sigma_{fb}(H)$ is the set of all flat bands (eigenvalues of infinite multiplicity). An open interval between two neighboring non-degenerated spectral bands is called a *spectral gap.*

The eigenvalues of the Floquet matrix $\Delta(\vartheta)$ for the Laplacian Δ will be denoted by $\lambda_n^0(\vartheta)$, $n \in \mathbb{N}_\nu$. The *spectral bands* for the Laplacian $\sigma_n^0 = \sigma_n(\Delta), n \in \mathbb{N}_\nu$, have the form

$$\sigma_n^0 = \sigma_n(\Delta) = [\lambda_n^{0-}, \lambda_n^{0+}] = \lambda_n^0(\mathbb{T}^d). \tag{2.5}$$

Theorem 2.1. *Let the Schrödinger operator $H = \Delta + Q$ act on $\ell^2(V)$. Then*

i) The first spectral band $\sigma_1(H) = [\lambda_1^-, \lambda_1^+]$ is non-degenerated, i.e., $\lambda_1^- < \lambda_1^+$.

ii) The Lebesgue measure $|\sigma(H)|$ of the spectrum of the Schrödinger operator H satisfies

$$|\sigma(H)| \leqslant \sum_{n=1}^{\nu} |\sigma_n(H)| \leqslant 2\beta, \tag{2.6}$$

where β is the number of fundamental graph bridges. Moreover, if in the spectrum $\sigma(H)$ there exist s spectral gaps $\gamma_1(H), \dots, \gamma_s(H)$, then the following estimates hold true:

$$\begin{aligned} &\sum_{n=1}^{s} |\gamma_n(H)| \geqslant \lambda_\nu^+ - \lambda_1^- - 2\beta \geqslant C_0 - 2\beta, \\ &C_0 = \max\{\lambda_\nu^{0+} - q_\bullet\,, q_\bullet - 2\varkappa_+\}, \qquad q_\bullet = \max_n q_n - \min_n q_n. \end{aligned} \tag{2.7}$$

The estimates (2.6) and the first estimate in (2.7) become identities for some classes of graphs, see (2.14).

Remark. 1) The total length of spectral bands depends essentially on the number of bridges on the fundamental graph Γ_f. If we remove the coordinate system, then the number of bridges on Γ_f is changed in general. In order to get the best estimate in (2.6) we have to choose a coordinate system in which the number β is minimal.

2) If $q_\bullet > 2\varkappa_+$ ($q_\bullet$ is large enough), then $C_0 = q_\bullet - 2\varkappa_+$. If $q_\bullet < \lambda_\nu^{0+}$ ($q_\bullet$ is small enough), then $C_0 = \lambda_\nu^{0+} - q_\bullet$.

We consider the Schrödinger operator $H_t = \Delta + tQ$, where the potential Q is "generic" and $t \in \mathbb{R}$ is the coupling constant. We discuss spectral bands of H_t for t large enough.

Theorem 2.2. *Let the Schrödinger operator $H_t = \Delta + tQ$, where the potential Q satisfies $q_j \neq q_k$ for all $j,k \in \mathbb{N}_\nu$, $j \neq k$, and the real coupling constant t is large enough. Without loss of generality we assume that $q_1 < q_2 < \ldots < q_\nu$. Then each eigenvalue $\lambda_n(\vartheta,t)$ of the corresponding Floquet matrix $H_t(\vartheta)$ and each spectral band $\sigma_n(H_t)$, $n \in \mathbb{N}_\nu$, satisfy*

$$\begin{aligned} \lambda_n(\vartheta,t) &= tq_n + \Delta_{nn}(\vartheta) - \frac{1}{t}\sum_{\substack{j=1\\ j\neq n}}^{\nu} \frac{|\Delta_{jn}(\vartheta)|^2}{q_j - q_n} + \frac{O(1)}{t^2}, \\ |\sigma_n(H_t)| &= |\Delta_{nn}(\mathbb{T}^d)| + O(1/t) \end{aligned} \tag{2.8}$$

as $t \to \infty$, uniformly in $\vartheta \in \mathbb{T}^d$. In particular, we have

$$|\sigma(H_t)| = C + O(1/t), \qquad \text{where} \quad C = \sum_{n=1}^{\nu} |\Delta_{nn}(\mathbb{T}^d)| \tag{2.9}$$

and $C > 0$, if there are bridge-loops on Γ_f and $C = 0$ if there are no bridge-loops on Γ_f.

Remark. Asymptotics (2.8) yield that a small change of the potential gives that all spectral bands of the Schrödinger operator H_t become open for t large enough, i.e., the spectrum of H_t is absolutely continuous.

Definition of Loop Graphs. i) A periodic graph Γ is called *a loop graph* if all bridges of some fundamental graph Γ_f are loops. This graph Γ_f is called *a loop fundamental graph.*

ii) A loop graph Γ is called *precise* if $\cos\langle\tau(\mathbf{e}), \vartheta_0\rangle = -1$ for all bridges $\mathbf{e} \in \mathcal{B}_f$ and some $\vartheta_0 \in \mathbb{T}^d$, where $\tau(\mathbf{e}) \in \mathbb{Z}^d$ is the index of a bridge $\mathbf{e}$ of Γ_f. This point ϑ_0 is called *a precise quasimomentum* of the loop graph Γ.

The class of all precise loop graphs is large enough. The simplest example of precise loop graphs is the lattice graph $\mathbb{L}^d = (V, \mathcal{E})$, where the vertex set and the edge set are given by

$$V = \mathbb{Z}^d, \qquad \mathcal{E} = \big\{(m, m+a_1), \ldots, (m, m+a_d), \quad \forall\, m \in \mathbb{Z}^d\big\}, \tag{2.10}$$

and $a_1, \ldots, a_d$ is the standard orthonormal basis. The "minimal" fundamental graph $\mathbb{L}^d_f$ of the lattice $\mathbb{L}^d$ consists of one vertex $v = 0$ and d unoriented edge-loops (v, v). All bridges of $\mathbb{L}^d_f$ are loops and their indices have the form $\pm a_1, \pm a_2, \ldots, \pm a_d$. Thus, for the quasimomentum $\vartheta_0 = (\pi, \ldots, \pi) \in \mathbb{T}^d$ we have $\cos\langle\tau(\mathbf{e}), \vartheta_0\rangle = -1$ for all bridges $\mathbf{e} \in \mathbb{L}^d_f$ and the graph $\mathbb{L}^d$ is a precise loop graph. It is known that the spectrum of the Laplacian Δ on $\mathbb{L}^d$ has the form $\sigma(\Delta) = \sigma_{ac}(\Delta) = [0, 4d]$.

We consider perturbations of loop graphs and precise loop graphs. A simple example of a precise loop graph Γ_* obtained by perturbations of the square lattice $\mathbb{L}^2$ is given in Fig.2*a*.

Proposition 2.3. *i) There exists a loop graph, which is not precise (see Fig.3).*

ii) Let $\Gamma = (\mathcal{E}, V)$ be a loop graph and let $\Gamma' = (\mathcal{E}', V') \subset \mathbb{R}^d$ be any connected finite graph such that its diameter is small enough. We take some point $v \in V$ and some point $v' \in V'$. We joint the graph Γ' with each point from the vertex set $v + \mathbb{Z}^d$, identifying the vertex v' with each vertex of $v + \mathbb{Z}^d$. Then the obtained graph Γ_* is a loop graph. Moreover, if Γ is precise, then Γ_* is also precise and the precise quasimomentum ϑ_0 of Γ is also a precise quasimomentum of Γ_*.

Remark. Applying this procedure to the obtained loop graph Γ_* and to another connected finite graph Γ_1 we obtain a new loop graph Γ_{**} and so on. Thus, from one loop graph we obtain a whole class of loop graphs.

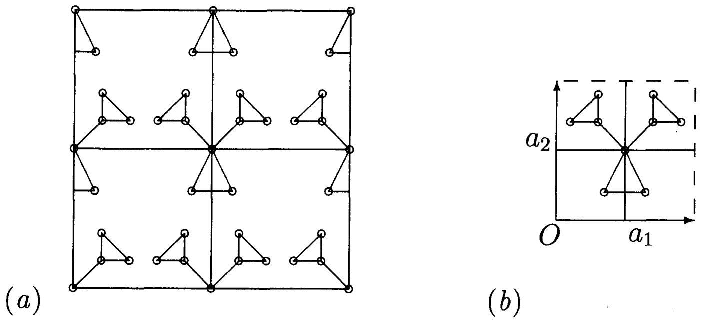

FIGURE 2. a) Precise loop graph Γ_*; b) the fundamental graph Γ_{*f}.

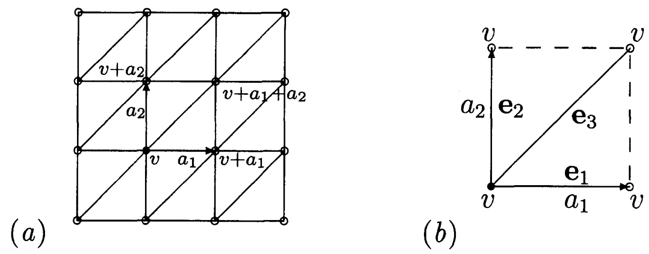

FIGURE 3. a) Triangular lattice $\mathbf{T}$; b) the fundamental graph $\mathbf{T}_f$.

We now describe bands for precise loop periodic graphs.

Theorem 2.4. *i) Let the Schrödinger operator $H = \Delta + Q$ act on a loop graph Γ. Then spectral bands $\sigma_n = \sigma_n(H) = [\lambda_n^-, \lambda_n^+]$ satisfy*

$$\lambda_n^- = \lambda_n(0), \qquad \forall\, n \in \mathbb{N}_\nu. \tag{2.11}$$

ii) Let, in addition, Γ be precise with a precise quasimomentum $\vartheta_0 \in \mathbb{T}^d$. Then

$$\sigma_n = [\lambda_n^-, \lambda_n^+] = [\lambda_n(0), \lambda_n(\vartheta_0)], \qquad \forall n \in \mathbb{N}_\nu, \tag{2.12}$$

$$\sum_{n=1}^{\nu} |\sigma_n| = 2\beta, \tag{2.13}$$

where β is the number of bridge-loops on the loop fundamental graph Γ_f. In particular, if all bridges of Γ_f have the form (v_k, v_k) for some vertex $v_k \in V_f$, then

$$|\sigma(H)| = \sum_{n=1}^{\nu} |\sigma_n| = 2\beta. \tag{2.14}$$

Remark. 1) Due to (2.13), the total length of all spectral bands of the Schrödinger operators $H = \Delta + Q$ on precise loop graphs does not depend on the potential Q.

2) The number of the loop fundamental graph bridges can be any integer, then due to (2.14) the Lebesgue measure $|\sigma(H)|$ of the spectrum of H (on the specific graphs) can be arbitrary large.

3) λ_n^-, $n \in \mathbb{N}_\nu$, are the eigenvalues of the Schrödinger operator $H(0)$ defined by (1.12), (1.13) on the fundamental graph Γ_f. The identities (2.12) are similar to the case of N-periodic Jacobi matrices on the lattice $\mathbb{Z}$ (and for Hill operators). The spectrum of these operators is absolutely continuous and is a union of spectral bands, separated by gaps. The endpoints of the bands are the so-called $2N$-periodic eigenvalues.

No we formulate some results about the possible number of flat bands of Schrödinger operators.

Proposition 2.5. *Let a fundamental graph Γ_f of a periodic graph Γ have $\nu \geqslant 2$ vertices and have no bridge-loops. Then the number of degenerated spectral bands (flat bands) of the Schrödinger operator H on Γ does not exceed $\nu - 2$. Moreover, there exists $\mathbb{Z}^d$-periodic graph Γ (for $d = 2$ see Fig.4), such that the spectrum of the Laplacian Δ on Γ has exactly 2 open separated spectral bands $\sigma_1(\Delta)$ and $\sigma_\nu(\Delta)$ and between them, in the gap, $\nu - 2$ flat bands $\sigma_2(\Delta) = \ldots = \sigma_{\nu-1}(\Delta)$.*

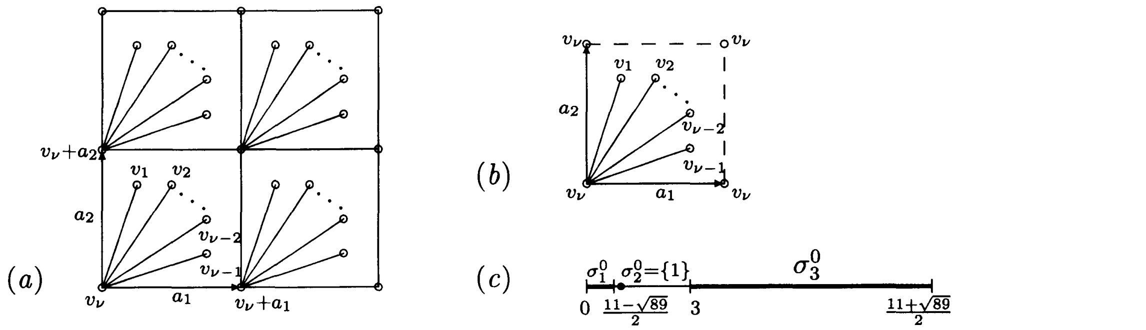

FIGURE 4. *a)* $\mathbb{Z}^2$-periodic graph Γ; *b)* the fundamental graph Γ_f; only 2 unoriented loops in the vertex v_ν are bridges; *c)* the spectrum of the Laplacian ($\nu = 3$).

Remark. There is an open problem: does there exist a $\mathbb{Z}^d$-periodic graph with any $\nu \geqslant 2$ vertices in the fundamental graph such that the spectrum of the Laplacian on Γ has only 1 spectral band and $\nu - 1$ flat bands, counting multiplicity?

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