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Local risk-minimization for Lévy markets

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1 Introduction

The purpose of this paper is to announce the results of our paper ([1]). Locally risk-minimizing (LRM, for short) is a very well-known hedging method for contingent claims in a quadratic way. Theoretical aspects of LRM has been developed to a high degree (see e.g., [5] and [6]). On the other hand, the necessity of researches on its explicit representations has been increasing. From this insight, we aim to obtain explicit representations of LRM by using Malliavin calculus for Lévy processes, based on [7]. Espeacially, we will formulate representations of LRM including Malliavin derivatives of the claim to hedge by using the Clark-Ocone type formula under change of measure (COCM) shown by Suzuki ([8] and [9]). We also derive formulas on representations of LRM for three typical options such as call options, Asian options and lookback options.

2 Preliminaries

2.1 Model description

We begin with preparation of the probabilistic framework and the underlying Lévy process X under which we discuss Malliavin calculus in the sequel. Let $T>0$ be a finite time horizon, $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ a one-dimensional Wiener space on $[0,T]$; and W its coordinate mapping process, that is, a one-dimensional standard Brownian motion with $W_0=0$. Let $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$ be the canonical Lévy space (see [7] and [3]) for a pure jump Lévy process J on $[0,T]$ with Lévy measure ν, that is, for $\omega_J=\{(t_1,z_1),\ldots,(t_n,z_n)\}\in([0,T]\times\mathbb{R}_0)^n$, $J_t(\omega_J)=\sum_{i=1}^n z_i\mathbf{1}_{\{t_i\leq t\}}$ for $t\in[0,T]$, where $\mathbb{R}_0:=\mathbb{R}\setminus\{0\}$. Now, we assume that $\int_{\mathbb{R}_0}z^2\nu(dz)<\infty$; and denote $(\Omega,\mathcal{F},\mathbb{P})=(\Omega_W\times\Omega_J, \mathcal{F}_W\times\mathcal{F}_J, \mathbb{P}_W\times\mathbb{P}_J)$. Let $\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$ be the canonical filtration completed for $\mathbb{P}$. Let X be a square integrable centered Lévy process on $(\Omega,\mathcal{F},\mathbb{P})$ represented as

$$X_t=\sigma W_t+J_t-t\int_{\mathbb{R}_0}z\nu(dz), \tag{2.1}$$

where $\sigma>0$. Now, we denote by N the Poisson random measure defined by $N(t,A):=\sum_{s\leq t}\mathbf{1}_A(\Delta X_s)$, $A\in\mathcal{B}(\mathbb{R}_0)$ and $t\in[0,T]$, where $\Delta X_s:=X_s-X_{s-}$. Thus, we have $J_t=\int_0^t\int_{\mathbb{R}_0}zN(ds,dz)$. In addition, we define its compensated measure as $\widetilde{N}(dt,dz):=N(dt,dz)-\nu(dz)dt$.

We consider, throughout this paper, a financial market being composed of one risk-free asset and one risky asset with finite time horizon T. For simplicity, we assume that the interest rate of the market is given by 0. The fluctuation of the risky asset is assumed to be given by a solution to the following stochastic differential equation (SDE, for short):

$$dS_t = S_{t-}\left[\alpha_t dt + \beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z}\widetilde{N}(dt,dz)\right], \quad S_0 > 0, \tag{2.2}$$

where α, β and γ are predictable processes. Recall that γ is a stochastic process measurable with respect to the σ-algebra generated by $A \times (s,u] \times B$, $A \in \mathcal{F}_s$, $0 \leq s < u \leq T$, $B \in \mathcal{B}(\mathbb{R}_0)$. Now, we assume the following:

Assumption 2.1 *1. (2.2) has a solution S satisfying the so-called structure condition (SC, for short). That is, S is a special semimartingale with the canonical decomposition $S = S_0 + M + A$ such that*

$$\left\| [M]_T^{1/2} + \int_0^T |dA_s| \right\|_{L^2(\mathbb{P})} < \infty, \tag{2.3}$$

where $dM_t = S_{t-}(\beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z}\widetilde{N}(dt,dz))$ and $dA_t = S_{t-}\alpha_t dt$. Moreover, defining a process $\lambda_t := \dfrac{\alpha_t}{S_{t-}(\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz))}$, we can rewrite the canonical decomposition as $S = S_0 + M + \int \lambda d\langle M\rangle$. Thirdly, the mean-variance trade-off process $K_t := \int_0^t \lambda_s^2 d\langle M\rangle_s$ is finite, that is, K_T is finite $\mathbb{P}$-a.s.
2. $\gamma_{t,z} > -1$, (t,z,ω)-a.e., that is, $\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \mathbf{1}_{\{\gamma_{t,z} \leq -1\}} \nu(dz)dt\right] = 0$.

Remark 2.2 *(1) The SC is closely related to the no-arbitrage condition. For more details on the SC, see [5] and [6].*
(2) The process K as well as A is continuous.
(3) (2.3) implies that $\sup_{t\in[0,T]} |S_t| \in L^2(\mathbb{P})$ by Theorem V.2 of Protter [4].
(4) Condition 2 ensures that $S_t > 0$ for any $t \in [0,T]$.

2.2 Locally risk-minimizing

We define locally risk-minimizing (LRM, for short) for a contingent claim $F \in L^2(\mathbb{P})$. The following definition is based on Theorem 1.6 of [6].

Definition 2.3 *(1) Θ_S denotes the space of all $\mathbb{R}$-valued predictable processes ξ satisfying*

$$\mathbb{E}\left[\int_0^T \xi_t^2 d\langle M\rangle_t + \left(\int_0^T |\xi_t dA_t|\right)^2\right] < \infty.$$

(2) An L^2-strategy is given by a pair $\varphi = (\xi, \eta)$, where $\xi \in \Theta_S$ and η is an adapted process such that $V(\varphi) := \xi S + \eta$ is a right continuous process with $\mathbb{E}[V_t^2(\varphi)] < \infty$ for every $t \in [0,T]$. Note that ξ_t (resp. η_t) represents the amount of units of the risky asset (resp. the risk-free asset) an investor holds at time t.
(3) For $F \in L^2(\mathbb{P})$, the process $C^F(\varphi)$ defined by $C_t^F(\varphi) := F\mathbf{1}_{\{t=T\}} + V_t(\varphi) - \int_0^t \xi_s dS_s$ is called the cost process of $\varphi = (\xi, \eta)$ for F.
(4) An L^2-strategy φ is said locally risk-minimizing for F if $V_T(\varphi) = 0$ and $C^F(\varphi)$ is a martingale orthogonal to M, that is, $C^F(\varphi)M$ is a martingale.

The above definition of LRM is a simplified version, since the original one, introduced in [5] and [6], is rather complicated

Now, we focus on a representation of LRM. To this end, we define Föllmer-Schweizer decomposition (FS decomposition, for short).

Definition 2.4 *An $F \in L^2(\mathbb{P})$ admits a Föllmer-Schweizer decomposition if it can be described by*

$$F = F_0 + \int_0^T \xi_t^F dS_t + L_T^F, \tag{2.4}$$

where $F_0 \in \mathbb{R}$, $\xi^F \in \Theta_S$ and L^F is a square-integrable martingale orthogonal to M with $L_0^F = 0$.

Proposition 5.2 of [6] shows the following:

Proposition 2.5 (Proposition 5.2 of [6]) *Under Assumption 2.1, an LRM $\varphi = (\xi, \eta)$ for F exists if and only if F admits an FS decomposition, and its relationship is given by*

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + \int_0^t \xi_s^F dS_s + L_t^F - F1_{\{t=T\}} - \xi_t^F S_t.$$

As a result, it suffices to obtain a representation of ξ^F in (2.4) in order to obtain LRM. Henceforth, we identify ξ^F with LRM. To this end, we consider the process $Z := \mathcal{E}(-\int \lambda dM)$, where $\mathcal{E}(Y)$ represents the stochastic exponential of Y, that is, Z is a solution to the SDE $dZ_t = -\lambda_t Z_{t-} dM_t$. In addition to Assumption 2.1, we suppose the following:

Assumption 2.6 *Z is a positive square integrable martingale; and $Z_T F \in L^2(\mathbb{P})$.*

A martingale measure $\mathbb{P}^* \sim \mathbb{P}$ is called minimal if any square-integrable $\mathbb{P}$-martingale orthogonal to M remains a martingale under $\mathbb{P}^*$. Under Assumption 2.1, it is easy to show that a minimal martingale measure $\mathbb{P}^*$ exists with $d\mathbb{P}^* = Z_T d\mathbb{P}$ if Z is a positive square integrable martingale.

Example 2.7 *We assume the following three conditions:*

1. $\gamma_{t,z} > -1$ *a.e.*
2. $\sup_{t\in[0,T]}(|\alpha_t| + \beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)) < C$ *for some* $C > 0$.
3. $\alpha_t \gamma_{t,z} < \beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)$, (t, z, ω)*-a.e., and there exists an* $\varepsilon > 0$ *such that* $\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz) > \varepsilon$, (t, z, ω)*-a.e.*

Then, all conditions of Assumption 2.1 are satisfied. On the other hand, the above condition 3 guarantees the positivity of Z.

3 Representation results for LRM

In this section, we focus on representations of LRM ξ^F for claim F. First of all, we study it through the martingale representation theorem.

3.1 Approach based on the martingale representation theorem

Throughout this subsection, we assume that Assumptions 2.1 and 2.6. Let $\mathbb{P}^*$ be a minimal martingale measure, that is, $d\mathbb{P}^* = Z_T d\mathbb{P}$ holds. The martingale representation theorem (see, e.g. Proposition 9.4 of [2]) provides

$$Z_T F = \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T g_t^0 dW_t + \int_0^T \int_{\mathbb{R}_0} g_{t,z}^1 \widetilde{N}(dt, dz)$$

for some predictable processes g_t^0 and $g_{t,z}^1$. By the same sort of calculations as the proof of Theorem 4.4 of [8], we have

$$\begin{aligned} F &= \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T \frac{g_t^0 + \mathbb{E}[Z_T F|\mathcal{F}_{t-}]u_t}{Z_{t-}} dW_t^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \frac{g_{t,z}^1 + \mathbb{E}[Z_T F|\mathcal{F}_{t-}]\theta_{t,z}}{Z_{t-}(1-\theta_{t,z})} \widetilde{N}^{\mathbb{P}^*}(dt,dz) \\ &=: \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T h_t^0 dW_t^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} h_{t,z}^1 \widetilde{N}^{\mathbb{P}^*}(dt,dz) \end{aligned}$$

where $u_t := \lambda_t S_{t-}\beta_t$, $\theta_{t,z} := \lambda_t S_{t-}\gamma_{t,z}$, $dW_t^{\mathbb{P}^*} := dW_t + u_t dt$ and $\widetilde{N}^{\mathbb{P}^*}(dt,dz) := \widetilde{N}(dt,dz) + \theta_{t,z}\nu(dz)dt$. Girsanov's theorem implies that $W^{\mathbb{P}^*}$ and $\widetilde{N}^{\mathbb{P}^*}$ are a Brownian motion and the compensated Poisson random measure of N under $\mathbb{P}^*$, respectively. Additionally, we assume that

$$\mathbb{E}\left[\int_0^T \left\{(h_t^0)^2 + \int_{\mathbb{R}_0}(h_{t,z}^1)^2\nu(dz)\right\} dt\right] < \infty. \tag{3.1}$$

Denoting $i_t^0 := h_t^0 - \xi_t S_{t-}\beta_t$, $i_{t,z}^1 := h_{t,z}^1 - \xi_t S_{t-}\gamma_{t,z}$, and

$$\xi_t := \frac{\lambda_t}{\alpha_t}\{h_t^0\beta_t + \int_{\mathbb{R}_0} h_{t,z}^1\gamma_{t,z}\nu(dz)\}, \tag{3.2}$$

we can see

$$i_t^0\beta_t + \int_{\mathbb{R}_0} i_{t,z}^1\gamma_{t,z}\nu(dz) = 0 \tag{3.3}$$

for any $t \in [0,T]$, which implies $i_t^0 u_t + \int_{\mathbb{R}_0} i_{t,z}^1\theta_{t,z}\nu(dz) = 0$. We have then

$$\begin{aligned} F - \mathbb{E}_{\mathbb{P}^*}[F] - \int_0^T \xi_t dS_t &= \int_0^T i_t^0 dW_t^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} i_{t,z}^1 \widetilde{N}^{\mathbb{P}^*}(dt,dz) \\ &= \int_0^T i_t^0 dW_t + \int_0^T \int_{\mathbb{R}_0} i_{t,z}^1 \widetilde{N}(dt,dz). \end{aligned}$$

By the following lemma, together with (3.3), we obtain that $L_t^F := \mathbb{E}[F - \mathbb{E}_{\mathbb{P}^*}[F] - \int_0^T \xi_s dS_s|\mathcal{F}_t]$ is a square integrable martingale orthogonal to M with $L_0^F = 0$. Under Assumptions 2.1 and 2.6, and (3.1), we can show that

$$\mathbb{E}\left[\int_0^T (i_t^0)^2 dt + \int_0^T \int_{\mathbb{R}_0} (i_{t,z}^1)^2\nu(dz)dt\right] < \infty.$$

Consequently, we can conclude the following:

Theorem 3.1 *Assume that Assumptions 2.1, 2.6; and (3.1). We have then $\xi^F = \xi$ defined in (3.2).*

In the above theorem, a representation of LRM ξ^F is obtained under a mild setting. Since the processes h^0 and h^1 appeared in (3.2) are induced by the martingale representation theorem, it is almost impossible to calculate them explicitly. In the rest of this section, we aim to get concrete expressions for h^0 and h^1 by using Malliavin calculus.

3.2 Malliavin calculus

We introduce some definitions and terminologies with respect to Malliavin calculus, in particular, a COCM (under $\mathbb{P}^*$).

We adapt the canonical Lévy space framework undertaken by [7]. Remark that Malliavin calculus is discussed based on the underlying Lévy process X. First of all, we define measures q and Q on $[0,T] \times \mathbb{R}$ as

$$q(E) := \sigma^2 \int_E \delta_0(dz)dt + \int_E z^2\nu(dz)dt,$$

and

$$Q(E) := \sigma \int_E \delta_0(dz)dW_t + \int_E z\widetilde{N}(dt,dz),$$

where $E \in \mathcal{B}([0,T] \times \mathbb{R})$ and δ_0 is the Dirac measure at 0. Denote by $L^2_{T,q,n}$ the set of product measurable, deterministic functions $h : ([0,T] \times \mathbb{R})^n \to \mathbb{R}$ satisfying

$$\|h\|^2_{L^2_{T,q,n}} := \int_{([0,T]\times\mathbb{R})^n} |h((t_1,z_1),\cdots,(t_n,z_n))|^2 q(dt_1,dz_1)\cdots q(t_n,z_n) < \infty.$$

For $n \in \mathbb{N}$ and $h \in L^2_{T,q,n}$, we define

$$I_n(h) := \int_{([0,T]\times\mathbb{R})^n} h((t_1,z_1),\cdots,(t_n,z_n))Q(dt_1,dz_1)\cdots Q(dt_n,dz_n).$$

Under this setting, any $F \in L^2(\mathbb{P})$ has the unique representation $F = \sum_{n=0}^\infty I_n(h_n)$ with functions $h_n \in L^2_{T,q,n}$ that are symmetric in the n pairs $(t_i,z_i), 1 \le i \le n$, and we have $\mathbb{E}[F^2] = \sum_{n=0}^\infty n!\|f_n\|^2_{L^2_{T,q,n}}$. We prepare some notations.

Definition 3.2 *(1) Let $\mathbb{D}^{1,2}$ denote the set of random variables $F \in L^2(\mathbb{P})$ with $F = \sum_{n=0}^\infty I_n(h_n)$ satisfying $\sum_{n=1}^\infty nn!\|h_n\|^2_{L^2_{T,q,n}} < \infty$.*
(2) For any $F \in \mathbb{D}^{1,2}$, $DF : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is defined by

$$D_{t,z}F = \sum_{n=1}^\infty nI_{n-1}(h_n((t,z),\cdot)).$$

(3) $\mathbb{L}_0^{1,2}$ denotes the space of $G : [0,T] \times \Omega \to \mathbb{R}$ satisfying
(a) $G_s \in \mathbb{D}^{1,2}$ for a.e. $s \in [0,T]$,
(b) $E\left[\int_{[0,T]} |G_s|^2 ds\right] < \infty$,
(c) $E\left[\int_{[0,T]\times\mathbb{R}} \int_0^T |D_{t,z}G_s|^2 dsq(dt,dz)\right] < \infty$.
(4) $\mathbb{L}_1^{1,2}$ is defined as the space of $G : [0,T] \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ such that
(a) $G_{s,x} \in \mathbb{D}^{1,2}$ for q-a.e. $(s,x) \in [0,T] \times \mathbb{R}$,
(b) $E\left[\int_{[0,T]\times\mathbb{R}_0} |G_{s,x}|^2 \nu(dx)ds\right] < \infty$,
(a) $E\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G_{s,x}|^2 \nu(dx)dsq(dt,dz)\right] < \infty$.
(5) $\widetilde{\mathbb{L}}_1^{1,2}$ is defined as the space of $G \in \mathbb{L}_1^{1,2}$ such that
(a) $E\left[\left(\int_{[0,T]\times\mathbb{R}_0} |G_{s,x}|\nu(dx)ds\right)^2\right] < \infty$,
(b) $E\left[\int_{[0,T]\times\mathbb{R}} \left(\int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G_{s,x}|\nu(dx)ds\right)^2 q(dt,dz)\right] < \infty$.

Theorem 3.4 below is a Clark-Ocone type formula under $\mathbb{P}^*$, which is concerned about an integral representation of $F \in L^2(\mathbb{P})$. The assumptions needed to see it are given in Assumption 3.3. Note that Assumption 2.1 is unrelated.

Assumption 3.3 *(1) $u, u^2 \in \mathbb{L}_0^{1,2}$; and $2u_s D_{t,z}u_s + z(D_{t,z}u_s)^2 \in L^2(q \times \mathbb{P})$ for a.e. $s \in [0,T]$.*
(2) $\theta + \log(1-\theta) \in \widetilde{\mathbb{L}}_1^{1,2}$, and $\log(1-\theta) \in \mathbb{L}_1^{1,2}$.
(3) For q-a.e. $(s,x) \in [0,T] \times \mathbb{R}_0$, there is an $\varepsilon_{s,x} \in (0,1)$ such that $\theta_{s,x} < 1 - \varepsilon_{s,x}$.
(4) $Z_T\left\{D_{t,0}\log Z_T \mathbf{1}_{\{0\}}(z) + \frac{e^{zD_{t,z}\log Z_T}-1}{z}\mathbf{1}_{\mathbb{R}_0}(z)\right\} \in L^2(q \times \mathbb{P})$.
(5) $F \in \mathbb{D}^{1,2}$; and $Z_T D_{t,z}F + FD_{t,z}Z_T + zD_{t,z}F \cdot D_{t,z}Z_T \in L^2(q \times \mathbb{P})$. (6) $FH^*_{t,z}$, $H^*_{t,z}D_{t,z}F \in L^1(\mathbb{P}^*)$ for q-a.e. $(t,z) \in [0,T] \times \mathbb{R}$, where $H^*_{t,z} := \exp\{zD_{t,z}\log Z_T - \log(1-\theta_{t,z})\}$.

Theorem 3.4 (Theorem 3.4 of [9]) *Under Assumptions 2.6 and 3.3, we have, for any $F \in L^2(\mathbb{P})$,*

$$\begin{aligned} F &= \mathbb{E}_{\mathbb{P}^*}[F] + \sigma \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - F\Big[\int_0^T D_{t,0}u_s dW_s^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}} \widetilde{N}^{\mathbb{P}^*}(ds,dx) \Big] \Big| \mathcal{F}_{t-} \right] dW_t^{\mathbb{P}^*} \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^* - 1) + zH_{t,z}^* D_{t,z}F | \mathcal{F}_{t-}] \widetilde{N}^{\mathbb{P}^*}(dt,dz). \end{aligned}$$

3.3 Main results

Under the above preparations, we calculate h^0 and h^1 by using Theorem 3.4. Together with Theorem 3.1, we obtain the following:

Theorem 3.5 *Under Assumptions 2.1, 2.6 and 3.3, h^0 and h^1 are described as*

$$h_t^0 = \sigma \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - F\left[\int_0^T D_{t,0}u_s dW_s^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}} \widetilde{N}^{\mathbb{P}^*}(ds,dx) \right] \Big| \mathcal{F}_{t-} \right], \tag{3.4}$$

$$h_{t,z}^1 = \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^* - 1) + zH_{t,z}^* D_{t,z}F | \mathcal{F}_{t-}]. \tag{3.5}$$

Moreover, LRM ξ^F is given by substituting (3.4) and (3.5) for h^0 and h^1 in (3.2) respectively, if (3.1) holds.

When we try to calculate LRM concretely through Theorem 3.5, we need to confirm if all the assumptions in Theorem 3.5 are satisfied for a given model. But, it seems to be a hard work. So that, we introduce a simple framework satisfying all the assumptions.

Example 3.6 *We consider the case where α, β, and γ in (2.2) are deterministic functions satisfying the three conditions in Example 2.7. Additionally, we assume that*

$$Z_T F \in L^2(\mathbb{P}), \text{ and condition 5 in Assumption 3.3.} \tag{3.6}$$

Then, all the conditions in Theorem 3.5 are satisfied; and ξ^F is given by

$$\xi_t^F = \frac{\sigma\beta_t \mathbb{E}_{\mathbb{P}^*}[D_{t,0}F|\mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[zD_{t,z}F|\mathcal{F}_{t-}]\gamma_{t,z}\nu(dz)}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)\right)}. \tag{3.7}$$

4 Call options

In this section, we deal with call options as common examples of contingent claims. The payoff of the call option with strike price $K > 0$ is expressed by $(S_T - K)^+$ where $x^+ = x \vee 0$. First of all, we calculate the Malliavin derivatives of $(F-K)^+$ for $F \in \mathbb{D}^{1,2}$ and $K \in \mathbb{R}$.

Theorem 4.1 *For any $F \in \mathbb{D}^{1,2}$, $K \in \mathbb{R}$ and q-a.e. $(t,z) \in [0,T] \times \mathbb{R}$, we have $(F-K)^+ \in \mathbb{D}^{1,2}$ and*

$$D_{t,z}(F-K)^+ = \mathbf{1}_{\{F>K\}} D_{t,0}F \cdot \mathbf{1}_{\{0\}}(z) + \frac{(F + zD_{t,z}F - K)^+ - (F-K)^+}{z} \mathbf{1}_{\mathbb{R}_0}(z).$$

We shall give an explicit representation of LRM for the deterministic coefficients case discussed in Example 3.6. Here after, we consider the case where α, β and γ in (2.2) are deterministic functions satisfying the three conditions in Examples 2.7. Additionally, we assume the following condition:

$$\int_{\mathbb{R}_0} \{\gamma_{t,z}^4 + |\log(1+\gamma_{t,z})|^2\} \nu(dz) < C \text{ for some } C > 0. \tag{4.1}$$

As seen in Example 3.6, this model satisfies all the conditions in Theorem 3.5, if the call option $(S_T - K)^+$ satisfies (3.6). First of all, we calculate the Malliavin derivatives of S_T.

Proposition 4.2 *We have* $S_T \in \mathbb{D}^{1,2}$*; and*

$$D_{t,z}S_T = \frac{S_T\beta_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{S_T\gamma_{t,z}}{z}\mathbf{1}_{\mathbf{R}_0}(z) \tag{4.2}$$

for q-a.e. $(t,z) \in [0,T] \times \mathbb{R}$.

Remark 4.3 *A similar argument with Proposition 4.2, together with Example 3.6, yields*

$$D_{t,z}Z_T = -Z_T\left(\frac{u_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\theta_{t,z}}{z}\mathbf{1}_{\mathbf{R}_0}(z)\right).$$

Moreover, note that Condition (3.6) in Example 3.6 is satisfied.

We can calculate an explicit representation of LRM for call options as follows:

Proposition 4.4 *For any* $K > 0$ *and* $t \in [0,T]$*, we have*

$$\begin{aligned}\xi_t^{(S_T-K)^+} &= \frac{1}{S_{t-}\left(\beta_t^2 + \int_{\mathbf{R}_0}\gamma_{t,z}^2\nu(dz)\right)}\left\{\beta_t^2\mathbb{E}_{\mathbf{P}^*}[\mathbf{1}_{\{S_T>K\}}S_T|\mathcal{F}_{t-}]\right.\\ &\left.+\int_{\mathbf{R}_0}\mathbb{E}_{\mathbf{P}^*}[(S_T(1+\gamma_{t,z})-K)^+ - (S_T-K)^+|\mathcal{F}_{t-}]\gamma_{t,z}\nu(dz)\right\}.\end{aligned} \tag{4.3}$$

5 Asian Options

In this section, we study Asian options, which are options whose payoff is depending on $\frac{1}{T}\int_0^T S_s ds$. By Lemma 3.2 in [3], we first derive the following proposition:

Proposition 5.1 *Besides Assumption 2.1, we assume the following two conditions:*

1. $S_s \in \mathbb{D}^{1,2}$ *for a.e.* $s \in [0,T]$.
2. $\mathbb{E}\left[\int_{[0,T]\times\mathbf{R}}\int_{[0,T]}|D_{t,z}S_s|^2 ds q(dt,dz)\right] < \infty$.

We have then $\frac{1}{T}\int_0^T S_s ds \in \mathbb{D}^{1,2}$ *and* $D_{t,z}\frac{1}{T}\int_0^T S_s ds = \frac{1}{T}\int_0^T D_{t,z}S_s ds$ *for q-a.e.* $(t,z) \in [0,T] \times \mathbb{R}$.

Next, we calculate Malliavin derivatives and LRM of Asian options for the same setting as section 4.

Proposition 5.2 *When* α, β *and* γ *are deterministic functions satisfying the three conditions in Example 2.7 and (4.1), we have* $\frac{1}{T}\int_0^T S_s ds \in \mathbb{D}^{1,2}$ *and*

$$D_{t,z}\frac{1}{T}\int_0^T S_s ds = \frac{1}{T}\left\{\frac{\beta_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\gamma_{t,z}}{z}\mathbf{1}_{\mathbf{R}_0}(z)\right\}\int_t^T S_s ds$$

for q-a.e. $(t,z) \in [0,T] \times \mathbb{R}$.

We illustrate LRM for Asian options with payoff $(\frac{1}{T}\int_0^T S_s ds - K)^+$.

Proposition 5.3 *Under the same setting as Proposition 5.2, we have, for any $K > 0$ and $t \in [0, T]$,*

$$\zeta_t^{(V_0-K)^+} = \frac{1}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)\right)} \left\{ \beta_t^2 \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{V_0>K\}} V_t | \mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}\left[(V_0 + \gamma_{t,z} V_t - K)^+ - (V_0 - K)^+ | \mathcal{F}_{t-}\right] \gamma_{t,z} \nu(dz) \right\},$$

where $V_t = \frac{1}{T}\int_t^T S_s ds$.

6 Lookback Options

We focus on lookback options, that is, options whose payoff depends on the running maximum of the asset price process $M^S := \sup_{t\in[0,T]} S_t$.

In this section, we treat only the case where S_t is given as an exponential Lévy process $S_t = S_0 \exp\{L_t\}$, where $S_0 > 0$ and $L_t = \mu t + X_t$, where X is the underlying Lévy process defined in (2.1), and $\mu \in \mathbb{R}$. Note that $L_t \in \mathbb{D}^{1,2}$ for any $t \in [0, T]$.

First of all, we calculate Malliavin derivatives of M^L.

Theorem 6.1 $M^L \in \mathbb{D}^{1,2}$ *and*

$$D_{t,z} M^L = \mathbf{1}_{\{\tau\geq t\}} \mathbf{1}_{\{0\}}(z) + \frac{\sup_{s\in[0,T]}(L_s + z\mathbf{1}_{\{t\leq s\}}) - M^L}{z} \mathbf{1}_{\mathbb{R}_0}(z) \tag{6.1}$$

where $\tau = \inf\{t \in [0, T] | L_t \vee L_{t-} = M^L\}$.

We next calculate Malliavin derivatives and LRM of lookback options whose payoffs are given as $M^S - S_T$ and $(M^S - K)^+$ for $K > 0$. Here we assume $\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$, $\int_1^\infty (e^z - 1)^4 \nu(dz) < \infty$; and

$$\frac{\left\{\mu + \frac{\sigma^2}{2} + \int_{\mathbb{R}_0}(x - e^x + 1)\nu(dx)\right\}(e^z - 1)}{\sigma^2 + \int_{\mathbb{R}_0}(e^x - 1)^2 \nu(dx)} < 1$$

for any $z \in \mathbb{R}_0$. Remark that $\int_{\mathbb{R}_0}(z - e^z + 1)\nu(dz)$ is well-defined since $e^z - 1 - z \leq (e - 1)z^2$ for any $z \in [-1, 1]$; and the three conditions in Example 2.7 and (4.1) are satisfied under these conditions. Note that Condition (3.6) also holds for both $M^S - S_T$ and $(M^S - K)^+$.

Now, we calculate Malliavin derivatives and LRM for lookback options by using Theorem 4.1 and (3.7).

Proposition 6.2 *(1) We have $M^S \in \mathbb{D}^{1,2}$; and*

$$D_{t,z} M^S = \mathbf{1}_{\{\tau\geq t\}} \mathbf{1}_{\{0\}}(z) + \frac{\sup_{s\in[0,T]}\left(S_s e^{z\mathbf{1}_{\{t\leq s\}}}\right) - M^S}{z} \mathbf{1}_{\mathbb{R}_0}(z).$$

(2)

$$D_{t,z}(M^S - S_T) = (M^S \mathbf{1}_{\{\tau\geq t\}} - S_T)\mathbf{1}_{\{0\}}(z) + \left(\frac{\sup_{s\in[0,T]}\left(S_s e^{z\mathbf{1}_{\{t\leq s\}}}\right) - M^S}{z} - S_T \frac{e^z - 1}{z}\right) \mathbf{1}_{\mathbb{R}_0}(z).$$

(3) For any $K > 0$, we have

$$D_{t,z}(M^S - K)^+ = M^S \mathbf{1}_{\{M^L > \log(K/S_0)\}} \mathbf{1}_{\{\tau\geq t\}} \mathbf{1}_{\{0\}}(z) + \frac{\left(\sup_{s\in[0,T]}\left(S_s e^{z\mathbf{1}_{\{t\leq s\}}}\right) - K\right)^+ - (M^S - K)^+}{z} \mathbf{1}_{\mathbb{R}_0}(z).$$

Corollary 6.3 *For any $K > 0$ and $t \in [0,T]$, we have*

$$\xi_t^{M^S - S_T} = \frac{1}{CS_{t-}} \left\{ \sigma^2 \mathbb{E}_{\mathbf{P}^*}[M^S \mathbf{1}_{\{\tau \geq t\}} - S_T | \mathcal{F}_{t-}] + \int_{\mathbf{R}_0} \mathbb{E}_{\mathbf{P}^*} \left[\sup_{u \in [0,T]} \left(S_u e^{z \mathbf{1}_{\{t \leq u\}}} \right) - M^S - S_T \gamma_z | \mathcal{F}_{t-} \right] \gamma_z \nu(dz) \right\},$$

and

$$\begin{aligned} \xi_t^{(M^S - K)^+} &= \frac{1}{CS_{t-}} \left\{ \sigma^2 \mathbb{E}_{\mathbf{P}^*}[M^S \mathbf{1}_{\{M^L > \log(K/S_0)\}} \mathbf{1}_{\{\tau \geq t\}} | \mathcal{F}_{t-}] \right. \\ &\quad \left. + \int_{\mathbf{R}_0} \mathbb{E}_{\mathbf{P}^*} \left[\left(\sup_{u \in [0,T]} \left(S_u e^{z \mathbf{1}_{\{t \leq u\}}} \right) - K \right)^+ - (M^S - K)^+ | \mathcal{F}_{t-} \right] \gamma_z \nu(dz) \right\}. \end{aligned}$$

where $\gamma_z := e^z - 1$ and $C := \left(\sigma^2 + \int_{\mathbf{R}_0} \gamma_z^2 \nu(dz) \right)$.

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