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THE STABILITY AND THE RATE OF CONVERGENCE TO STATIONARY SOLUTIONS OF THE TWO-DIMENSIONAL NAVIER-STOKES EXTERIOR PROBLEM

MASAO YAMAZAKI (山崎 昌男)

FACULTY OF SCIENCE AND ENGINEERING, WASEDA UNIVERSITY
(早稲田大学 理工学術院)

ABSTRACT. This paper is concerned with the stability of stationary solutions of the two-dimensional Navier-Stokes exterior problem. The stationary solutions are assumed to be small and enjoy certain pointwise decay conditions. If the decay condition is critical, the domains and solutions are assumed to satisfy some symmetry condition as well. Under an initial perturbation in the solenoidal L^2-space, with the same symmetry if the decay order of the stationary solution is critical, the solution of the nonstationary equation tends to the stationary solution in the solenoidal L^2-class. Also given are the decay orders of the perturbation in other function spaces.

1. INTRODUCTION.

Let Ω be an exterior domain with $C^{3+\gamma}$-boundary Γ with some $\gamma > 0$. We consider the nonstationary Navier-Stokes equation on Ω with time-independent external force $f(x)$:

(1.1)
$$\frac{\partial u}{\partial t}u(x,t) - \Delta u(x,t) + \big(u(x,t)\cdot\nabla\big)u(x,t) + \nabla\tilde{p}(x,t) = f(x) \quad \text{in } \Omega,$$

$$\nabla\cdot u(x,t) = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$u(x,t) = a(x) \quad \text{on } \Gamma, \tag{1.3}$$

$$u(x,t) \to 0 \quad \text{as } |x| \to \infty, \tag{1.4}$$

$$u(x,t) = u_0(x) \text{ in } \Omega, \tag{1.5}$$

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where $a(x)$ satisfies the outflow condition

$$\int_\Gamma a(x)\cdot n(x)\,ds(x)=0. \tag{1.6}$$

We are concerned with the asymptotic stability of the stationary solutions of this equation. Suppose that $\big(w(x),\pi(x)\big)$ is a stationary solution of the system (1.1)–(1.5): Namely,

$$\begin{aligned}
&(1.7) && -\Delta w(x)+\big(w(x)\cdot\nabla\big)w(x)+\nabla\pi(x)=f(x) \text{ in } \Omega,\\
&(1.8) && \nabla\cdot w(x)=0, \quad \text{in } \Omega,\\
&(1.9) && w(x)=a(x) \text{ on } \Gamma,\\
&(1.10) && w(x)\to 0 \quad \text{as } |x|\to\infty.
\end{aligned}$$

Putting $v(x,t)=u(x,t)-w(x)$, $p(x,t)=\tilde{p}(x,t)-\pi(x)$ and $v_0(x)=u_0(x)-w(x)$, we can rewrite the system (1.1)–(1.5) into the system

$$\begin{aligned}
&(1.11) && \frac{\partial v}{\partial t}(x,t)-\Delta v(x,t)+\big(w(x)\cdot\nabla\big)v(x,t)\\
& && +\big(v(x,t)\cdot\nabla\big)w(x)+\big(v(x,t)\cdot\nabla\big)v(x,t)+\nabla p(x,t)=0 \quad \text{in } \Omega,\\
&(1.12) && \nabla\cdot v(x,t)=0 \quad \text{in } \Omega,\\
&(1.13) && v(x,t)=0 \quad \text{on } \Gamma,\\
&(1.14) && v(x,t)\to 0 \quad \text{as } |x|\to\infty,\\
&(1.15) && v(x,0)=v_0(x) \text{ in } \Omega.
\end{aligned}$$

We next introduce the Helmholtz decomposition. For $q\in(1,\infty)$, there exists a projection operator P_q in $\big(L^q(\Omega)\big)^2$ onto the space

$$\operatorname{Im}P_q=L^q_\sigma(\Omega)=\left\{u\in\big(L^q(\Omega)\big)^2\,\middle|\,\nabla\cdot u=0 \text{ in } \Omega,\ n\cdot u=0 \text{ on } \Gamma\right\}$$

such that

$$\operatorname{Ker}P_q=G^q(\Omega)=\left\{\nabla f\in\big(L^q(\Omega)\big)^2\,\middle|\,f\in L^q_{\mathrm{loc}}(\Omega)\right\}.$$

Since we have $P_q\equiv P_r$ on $\big(L^q(\Omega)\cap L^r(\Omega)\big)^2$, we abbreviate P_q by P in the sequel.

Applying the projection P to the system (1.11)–(1.14) and putting $A=-P\Delta$, we obtain the abstract differential equation

$$\frac{\partial v}{\partial t}(t)+Av(t)+P\big[\big(w\cdot\nabla\big)v(t)+\big(v(t)\cdot\nabla\big)w+\big(v(t)\cdot\nabla\big)v(t)\big]=0. \tag{1.16}$$

This equation together with the initial condition (1.15) is formally equivalent to the following integral equation:

$$\begin{aligned}
&(1.17)\quad v(t)=\exp(-tA)v_0\\
&-\int_0^t \exp\big(-(t-\tau)A\big)+P\big[\big(w\cdot\nabla\big)v(\tau)+\big(v(\tau)\cdot\nabla\big)w+\big(v(\tau)\cdot\nabla\big)v(\tau)\big]\,d\tau.
\end{aligned}$$

In order to consider the time-local unique solvability of (1.17), we introduce classes of functions. For $s \in (2,\infty)$ and $T \in (0,\infty]$, put

$$\mathscr{Y}(s,T) = \Big\{u(t)\,\Big|\, t^{1/2-1/s}u(t) \in BC\big((0,T),L^s_\sigma(\Omega)\big),\quad t^{1/2}u(t) \in BC\Big((0,T),\big(H^1_0(\Omega)\big)^2\Big)\Big\}$$

equipped with the norm

$$\|u\|_{\mathscr{Y}(s,T)} = \sup_{0<t<T}\Big\{t^{1/2-1/s}\|u(t)\|_s + t^{1/2}\|\nabla u(t)\|_2\Big\}.$$

Then the class $\mathscr{Y}(s,T)$ becomes a Banach space, and

$$\mathscr{Y}_0(s,T) = \Big\{u(t) \in \mathscr{Y}(s,T)\,\Big|\, \lim_{t\to+0} t^{1/2-1/s}u(t) = 0 \text{ in } L^s_\sigma(\Omega),\quad \lim_{t\to+0} t^{1/2}u(t) = 0 \text{ in } \big(H^1_0(\Omega)\big)^2\Big\}.$$

is a closed subspace of $\mathscr{Y}(s,T)$. Then we have the following theorem on the existence of time-local solutions:

Theorem 1.1. *Suppose that $s > 4$ and that Ω is an exterior domain. Suppose moreover that $\big(w(x),\pi(x)\big)$ is a solution of the system* (1.7)–(1.10) *such that $w(x) \in \big(L^s(\Omega)\cap \dot{H}^1(\Omega)\big)^2$. Then, for every initial perturbation $v_0(x) \in L^2_\sigma(\Omega)$, there exists a positive number T_0 such that the integral equation* (1.17) *admits a solution $v(t)$ on $(0,T_0)$ in the class $\mathscr{Y}_0(s,T_0)$ such that $v(t)$ converges to v_0 in $L^2_\sigma(\Omega)$ as $t \to +0$. This solution belongs to the class*

$$C\big([0,T_0),L^2_\sigma(\Omega)\big)\cap C^1\big((0,T_0),L^2_\sigma(\Omega)\big)\cap C\Big((0,T_0),\big(H^2(\Omega)\big)^2\Big),$$

and is a solution of the abstract differential equation (1.16). *Furthermore, if v_0 belongs to the space $\big(H^1(\Omega)\big)^2$ as well, then the number T_0 is estimated from below by s, $\|w\|_s$, $\|\nabla w\|_2$, $\|v_0\|_2$ and $\|\nabla v_0\|_2$.*

We also have the theorem for the uniqueness as follows:

Theorem 1.2. *Suppose that s, Ω, $\big(w(x),\pi(x)\big)$ and $v_0(x)$ are the same as in Theorem 1.1. Suppose that T_1, $T_2 \in (0,\infty]$ and that the functions $v_j(t) \in \mathscr{Y}(s,T_j)\cap C\big([0,T_j),L^2_\sigma(\Omega)\big)$ are solutions of* (1.17) *on $(0,T_j)$ and satisfies $v_j(0) = v_0$ for $j = 1,2$. Then we have $v_1(t) \equiv v_2(t)$ on $[0,T_3)$, where $T_3 = \min\{T_1,T_2\}$.*

In order to state the main result on the asymptotic stability of the aforementioned stationary solution $\big(w(x),\pi(x)\big)$ under initial perturbation $v_0(x) \in L^2_\sigma(\Omega)$. we put

$$\mathscr{X}(b) = \{w(x) \in C(\Omega)\mid \|w\|_{\mathscr{X}(b)} = \sup_{x\in\Omega}(1+|x|)^b|w(x)| < \infty\}$$

for a positive number b, and assume that one of the following conditions holds:

(C) The exterior domain Ω is invariant under the mappings

$$(x_1,x_2) \mapsto (-x_1,x_2), \quad (x_1,x_2) \mapsto (x_1,-x_2),$$

and $\big(w(x),\pi(x)\big)$ satisfies the symmetry conditions

$$\text{(U4)} \qquad \begin{cases} f_1(-x_1,x_2) = -f_1(x_1,x_2), & f_1(x_1,-x_2) = f_1(x_1,x_2), \\ f_2(-x_1,x_2) = f_2(x_1,x_2), & f_2(x_1,-x_2) = -f_2(x_1,x_2), \end{cases}$$

and $\pi(-x_1,x_2) = \pi(x_1,-x_2) = \pi(x_1,x_2)$. Furthermore, $w \in \big(\mathscr{X}(1) \cap \dot{H}^1(\Omega)\big)^2$ such that $\|w\|_{\mathscr{X}(1)}$ and $\|\nabla w\|_2$ are sufficiently small, and $v_0(x)$ satisfy (U4).

(S) $w \in \big(\mathscr{X}(b) \cap \dot{H}^1(\Omega)\big)^2$ with some $b > 1$ such that $\|w\|_{\mathscr{X}(b)}$ and $\|\nabla w\|_2$ are sufficiently small.

Remark 1.1. If $\big(w(x),\pi(x)\big)$ satisfies $w \in \big(\mathscr{X}(b)\big)^2$ with some $b \geq 1$, then $w(x) \in \big(L^s(\Omega)\big)^2$ holds for every $s \in (2,\infty]$.

Remark 1.2. If the condition (C) holds, then Theorem 1.2 implies that $v(\cdot,t)$ satisfies the condition (U4) for every t.

Remark 1.3. For the existence of the stationary solution satisfying the condition (S), the boundary value $a(x)$ must satisfy the condition (1.6).

Remark 1.4. The existence of stationary solutions satisfying the above conditions are already proved in [9] under more restrictive symmetry conditions on the domain and the external forces.

Then our main result is the following:

Theorem 1.3. *Under the assumption* (C) *or* (S), *there uniquely exists a solution* $v(t) \in BC\big([0,\infty), L^2_\sigma(\Omega)\big)$ *of the integral equation* (1.17) *such that* $v(0) = v_0$ *and that* $t^{1/2}v(t) \in BC\Big((0,\infty), \big(H^1_0(\Omega)\big)^2\Big)$. *Furthermore, the function* $\|v(t)\|_2$ *is monotone-decreasing with respect to t, and* $v(t)$ *enjoys the decay properties*

$$\|v(t)\|_q = o(t^{1/q-1/2}) \qquad \text{as } t \to \infty \text{ for } q \in [2,\infty), \tag{1.18}$$

$$\|\nabla v(t)\|_2 = o(t^{-1/2}) \qquad \text{as } t \to \infty, \tag{1.19}$$

$$\|v(t)\|_\infty = o\left(t^{-1/2}\sqrt{\log t}\right) \quad \text{as } t \to \infty. \tag{1.20}$$

Remark 1.5. It follows from the assumption that the solution $v(t)$ enjoys the assumption of Theorem 1.2, from which the uniqueness follows.

Remark 1.6. This theorem asserts that the stationary solution $w(x)$ satisfying the condition (S) is the global attractor in $L^2_\sigma(\Omega)$.

As will be seen later, this note is an abridged version of Galdi and Yamazaki [6] and Yamazaki [10]. However, I believe that it will be worthwhile to provide a unified note of the separate papers on the same problem with the same essential tools.

2. Outline of the proof of Theorems 1.1 and 1.2.

In this section we give a sketch of the proof of the theorems above. Detail is given in [6]. To this end we first review the L^q-L^r estimates given by Borchers and Varnhorn [2] and Dan and Shibata [3, 4].

Theorem 2.1. *For the semigroup* $\exp(-tA)$ *we have the following assertions:*

(i) *Assume that* $1 < q < \infty$, $q \le r \le \infty$ *and* $\alpha \ge 0$. *Then there exists a constant* C *such that, for every* $u \in L^q_\sigma(\Omega)$ *we have* $\|A^\alpha \exp(-tA)u\|_r \le Ct^{-\alpha-1/q+1/r}\|u\|_q$.

(ii) *Assume that* $1 < q \le r \le 2$. *Then there exists a constant* C *such that, for every* $u \in L^q_\sigma(\Omega)$ *we have* $\|\nabla \exp(-tA)u\|_r \le Ct^{-1/2-1/q+1/r}\|u\|_q$.

From this theorems we can prove the following lemmata.

Lemma 2.2. *Suppose that* $2 < s < \infty$. *Then there exists a positive constant* C *such that the function* $u(t) = \exp(-tA)u_0$ *belongs to* $\mathscr{Y}_0(s,1)$ *and the estimate* $\|u\|_{\mathscr{Y}(s,1)} \le C\|u_0\|_2$ *holds for every* $u_0 \in L^2_\sigma(\Omega)$. *Moreover, we have* $u(t) \in BC([0,1), L^2_\sigma(\Omega))$ *with* $u(0) = u_0$. *Furthermore, if* $u_0 \in L^2_\sigma(\Omega) \cap \left(H^{1-2/s}(\Omega)\right)^2$, *the inequality*

$$\|u\|_{\mathscr{Y}(s,T)} \le C\|u_0\|_{H^{1-2/s}}T^{1/2-1/s} \tag{2.1}$$

holds for every $T \in (0,1]$.

Lemma 2.3. *Let* q *and* s *satisfy* $1 < q < 2 < s < \infty$. *Then there exists a positive constant* C *such that the following assertions hold.*

(i) *Suppose that* $u(t) \in C((0,T), L^q_\sigma(\Omega))$ *with some* $T \in (0,1]$, *satisfies the estimate* $B = \sup_{0 \le t < T} t^{3/2-1/q}\|u(t)\|_q < \infty$. *Then the function* $v(t)$ *defined by the formula* $v(t) = \int_0^t \exp(-(t-\tau)A)u(\tau)\,d\tau$ *belongs to* $\mathscr{Y}(s,T)$, *and the estimate* $\|v\|_{\mathscr{Y}(s,T)} \le CB$ *holds. Moreover, we have* $v(t) \in BC((0,T), L^2_\sigma(\Omega))$. *Furthermore, for every* $\alpha < 1 - 1/q$ *and every* $\delta \in (0,T)$, *the function* $v(t)$ *is Hölder continuous of order* α *with values in* $(H^1_0(\Omega))^2$ *on* (δ, T).

(ii) *If we assume in addition that* $\lim_{t \to +0} t^{3/2-1/q}\|u(t)\|_q = 0$, *then we have* $v \in \mathscr{Y}_0(s,T)$, *and* $v(t)$ *converges to* 0 *in* $L^2_\sigma(\Omega)$ *as* $t \to +0$.

Lemma 2.4. *Suppose that* $2 < s < \infty$. *Then we have the following assertions:*

(i) *There exists a positive constant* C *such that the following assertion holds. Let* T *be a positive number such that* $T \le 1$. *Suppose that*

$w(x) \in \left(L^s(\Omega) \cap \dot{H}^1(\Omega)\right)^2$ and that $u(t)$, $v(t) \in \mathscr{Y}(s,T)$. Put

$$S_w[v,u](t) = P\left[(w\cdot\nabla)v(t) + \left(v(t)\cdot\nabla\right)w + \left(u(t)\cdot\nabla\right)v(t)\right].$$

Then we have $t^{1-1/s}S_w[v,u](t) \in BC\left([0,T), L^{2s/(2+s)}_\sigma(\Omega)\right)$ with

$$\sup_{0<t<T} t^{1-1/s}\|S_w[v,u](t)\|_{2s/(2+s)} \leq C\left(T^{1/2-1/s}(\|w\|_s + \|\nabla w\|_2) + \|u\|_{\mathscr{Y}(s,T)}\right)\|v\|_{\mathscr{Y}(s,T)}. \tag{2.2}$$

(ii) *Suppose that $u(t)$ and $v(t)$ are Hölder continuous with values in $\left(H^1_0(\Omega)\right)^2$ on (δ,T) for some $\delta \in (0,T)$ in addition to the assumption in Assertion* (i). *Then $S_w[v,u](t)$ is Hölder continuous with values in $L^{2s/(2+s)}_\sigma(\Omega)$ on (δ,T).*

(iii) *Suppose that $\lim_{t\to+0} t^{1/2-1/s}\|u(t)\|_s = 0$ or $\lim_{t\to+0} t^{1/2}\|\nabla v(t)\|_2 = 0$ holds in addition to the assumption in Assertion* (i). *Then we have $\lim_{t\to+0} t^{1-1/s}\|S_w[v,u](t)\|_{2s/(2+s)} = 0$.*

The following corollary follows immediately from the lemmata above.

Corollary 2.5. *Suppose that $s>2$, there exists a constant C such that the following assertion holds. Suppose that $0<T\leq 1$, and let $w(x)$, $u(t)$ and $v(t)$ be the same as in Lemma 2.4. Put*

$$T_w[v,u](t) = -\int_0^t \exp(-(t-\tau)A)S_w[v,u](\tau)\,d\tau.$$

Then we have $T_w[v,u](t) \in \mathscr{Y}(s,T)$, and we have the estimate

$$\|T_w[v,u]\|_{\mathscr{Y}(s,T)} \leq C\left(T^{1/2-1/s}(\|w\|_s + \|\nabla w\|_2) + \|u\|_{\mathscr{Y}(s,T)}\right)\|v\|_{\mathscr{Y}(s,T)}.$$

Furthermore, if $\lim_{t\to+0} t^{1/2-1/s}\|u(t)\|_s = 0$ or $\lim_{t\to+0} t^{1/2}\|\nabla v(t)\|_2 = 0$ holds, then we have $T_w[v,u](t) \in Y_0(s,T)$ and $T_w[v,u](t) \to 0$ in $L^2_\sigma(\Omega)$ as $t\to+0$. In particular, if $u \in \mathscr{Y}_0(s,T)$ or $v \in \mathscr{Y}_0(s,T)$, then $T_w[u,v] \in \mathscr{Y}_0(s,T)$.

Proof of Theorem 1.1. Put $\tilde{v}_0(t) = \exp(-tA)v_0$ for $v_0 \in L^2_\sigma(\Omega)$. Then Lemma 2.2 implies $\tilde{v}_0 \in \mathscr{Y}_0(s,\infty)$. Next, for every $T'_0 \in (0,1]$, consider the mapping U from $\mathscr{Y}_0(s,T'_0)$ into itself defined by $U[v](t) = \tilde{v}_0(t) + T_w[v,v](t)$. Then Lemma 2.2 and Corollary 2.5 imply that the estimate

$$\|U[v]\|_{\mathscr{Y}(s,T'_0)} \leq \|\tilde{v}_0\|_{\mathscr{Y}(s,T'_0)} + C{T'_0}^{1/2-1/s}\left(\|w\|_s + \|\nabla w\|_2\right)\|v\|_{\mathscr{Y}(s,T'_0)} + C\|v\|_{\mathscr{Y}(s,T'_0)}^{\,2}$$

holds with a constant $C \geq 1$ independent of w, $\tilde{v}_0$, v and $T'_0 \in (0,1]$. If the inequality

$$\|\tilde{v}_0\|_{\mathscr{Y}(s,T'_0)} < \frac{1}{16C} \tag{2.3}$$

holds with some $T_0' \in (0,1]$, put

$$T_0 = \min\left\{ T_0', \left(\frac{1}{2C(\|w\|_s + \|\nabla w\|_2)} \right)^{2s/(s-2)} \right\}.$$

Then the quadratic equation $x = \|\tilde{v}_0\|_{\mathscr{Y}(s,T_0)} + x/2 + Cx^2$ has two distinct real roots. Let α be the smaller one. Then, if $v \in \mathscr{Y}_0(s,T_0)$ satisfies $\|v\|_{\mathscr{Y}(s,T_0)} \le \alpha$, it follows that

$$\|U[v]\|_{\mathscr{Y}(s,T_0)} \le \|\tilde{v}_0\|_{\mathscr{Y}(s,T_0)} + CT_0^{1/2-1/s}(\|w\|_s + \|\nabla w\|_2)\alpha + C\alpha^2 \le \alpha.$$

Hence, if the inequality (2.3) holds with some $T_0' \in (0,1]$, the mapping U maps the closed ball in $\mathscr{Y}_0(s,T_0)$ of center 0 and radius α into itself.

We next show that the constant T_0' which satisfies (2.3) exists for every $v_0 \in L^2_\sigma(\Omega)$. There exists a constant C' such that, for every $T > 0$, $v_0 \in L^2_\sigma(\Omega)$ and $v_1 \in L^2_\sigma(\Omega) \cap (H^1(\Omega))^2$, we have the estimate

$$\begin{aligned} \|\tilde{v}_0\|_{\mathscr{Y}(s,T)} &\le \|\exp(-tA)v_1\|_{\mathscr{Y}(s,T)} + \|\exp(-tA)(v_0 - v_1)\|_{\mathscr{Y}(s,T)} \\ &\le C'T^{1/2-1/s}\|v_1\|_{H^1(\Omega)} + C'\|v_0 - v_1\|_2. \end{aligned}$$

Choose v_1 so that $\|v_0 - v_1\|_2 < 1/32CC'$, and then choose $T_0' \in (0,1]$ for v_1 above by $T_0' = \min\left\{1, (1/32CC'\|v_1\|_{H^1(\Omega)})^{2s/(s-2)}\right\}$.

If $v_0 \in L^2_\sigma(\Omega) \cap (H^1(\Omega))^2$, we have $\|\tilde{v}_0\|_{\mathscr{Y}(s,T)} \le C'T^{1/2-1/s}\|v_0\|_{H^1(\Omega)}$. In this case we put $T_0' = \min\left\{1, (1/64CC'\|v_0\|_{H^1(\Omega)})^{2s/(s-2)}\right\}$. Then we have (2.3) in both cases, and in the latter case we can choose T_0' by the values of s, $\|v_0\|_2$ and $\|\nabla v_0\|_2$. Hence we can choose T_0 by the values of s, $\|v_0\|_2$, $\|\nabla v_0\|_2$, $\|w\|_s$ and $\|\nabla w\|_2$.

Next, let $v(t), \tilde{v}(t) \in \mathscr{Y}_0(s,T_0)$ such that $\|v\|_{\mathscr{Y}(s,T_0)}, \|\tilde{v}\|_{\mathscr{Y}(s,T_0)} \le \alpha$. Then we have

$$\begin{aligned} &U[\tilde{v}](t) - U[v](t) = T_w[\tilde{v},\tilde{v}](t) - T_w[v,v](t) \\ &= \int_0^t \exp(-(t-\tau)A)\big(S_w(v,v)(\tau) - S_w(\tilde{v},\tilde{v})(\tau)\big)\,d\tau \\ &= \int_0^t \exp(-(t-\tau)A)P\Big[(w\cdot\nabla)v(\tau) + (v(\tau)\cdot\nabla)w + (v(\tau)\cdot\nabla)v(\tau) \\ &\qquad - (w\cdot\nabla)\tilde{v}(\tau) - (\tilde{v}(\tau)\cdot\nabla)w - (\tilde{v}(\tau)\cdot\nabla)\tilde{v}(\tau)\Big]\,d\tau \\ &= \int_0^t \exp(-(t-\tau)A)P\Big[(w\cdot\nabla)(v(\tau)-\tilde{v}(\tau)) + ((v(\tau)-\tilde{v}(\tau))\cdot\nabla)w \\ &\qquad + (\tilde{v}(\tau)\cdot\nabla)(v(\tau)-\tilde{v}(\tau)) + ((v(\tau)-\tilde{v}(\tau))\cdot\nabla)v(\tau)\Big]\,d\tau \\ &= T_w[\tilde{v}-v,\tilde{v}](t) + T_0[v,\tilde{v}-v] \end{aligned}$$

for every $t \in (0, T_0)$. Hence Corollary 2.5 implies that

(2.4)

$$\begin{aligned} &\|U[\tilde{v}] - U[v]\|_{\mathscr{Y}(s,T_0)} \\ &\le C\left(T_0^{1/2-1/s}(\|w\|_s + \|\nabla w\|_2) + \|\tilde{v}\|_{\mathscr{Y}(s,T_0)} + \|v\|_{\mathscr{Y}(s,T_0)}\right) \|\tilde{v} - v\|_{\mathscr{Y}(s,T_0)} \\ &\le \left(\frac{1}{2} + 2C\alpha\right) \|\tilde{v} - v\|_{\mathscr{Y}(s,T_0)} . \end{aligned}$$

In view of the definition of α, we have $\frac{1}{2} + 2C\alpha = 1 - \frac{\|\tilde{v}_0\|_{\mathscr{Y}(s,T_0)}}{\alpha} < 1$. Hence (2.4) implies that the mapping U is a contraction mapping from the closed ball in $\mathscr{Y}_0(s, T_0)$ of center 0 and radius α into itself, and therefore it has a unique fixed point $v(t)$ in this ball. If $v_0 \in L^2_\sigma(\Omega) \cap (H^1(\Omega))^2$, the number T_0' is determined by s, $\|v_0\|_2$, $\|\nabla v_0\|_2$, $\|w\|_s$ and $\|\nabla w\|_2$. □

Proof of Theorem 1.2. We first remark that we may assume that $v_1(t) \in \mathscr{Y}_0(s, T_1)$. Indeed, let $y_1(t)$ and $y_2(t)$ the functions satisfying the assumption of this theorem defined on $[0, T']$ and $[0, T_2]$ respectively. Let $v(t) \in \mathscr{Y}_0(s, T_0)$ be the solution constructed in Theorem 1.1. Applying this theorem to $v_1(t) = v(t)$ and $v_2(t) = y_1(t)$, we have $v_1(t) \equiv y_1(t)$ on $(0, \min\{T_0, T'\})$. Hence, putting

$$v_1(t) = \begin{cases} v(t) & \text{if } T' \le T_0, \\ y_1(t) & \text{if } T_0 \le T' \end{cases}$$

we see that $v_1(t) \in \mathscr{Y}_0(s, T_1)$, where $T_1 = \max\{T_0, T'\}$. Then it suffices to show the identity $v_1(t) \equiv v_2(t)$ on the interval $[0, T_4]$ for every $T_4 \in (0, T_3)$. From the assumption we see $v_1(t) \in \mathscr{Y}_0(s, T_1)$. Put $\tilde{v}(t) = v_2(t) - v_1(t)$. Then we have $\tilde{v}(t) = T_w[v_2, v_2](t) - T_w[v_1, v_1](t)$, and hence

$$\begin{aligned} (2.5)\quad \tilde{v}(t) = -\int_0^t &\exp(-(t-\tau)A) \\ &P\left[\left((w + v_2(t)) \cdot \nabla\right) \tilde{v}(t) + \left(\tilde{v}(t) \cdot \nabla\right) \left(w + v_1(t)\right)\right] d\tau \end{aligned}$$

for every $t \in (0, T_4]$. Hence Lemmata 2.3 and 2.4 imply that there exists a constant C such that the estimate

$$\begin{aligned} (2.6)\quad \|\tilde{v}\|_{\mathscr{Y}(s,T)} \le C\Big(&T^{1/2-1/s}\|w\|_s + T^{1/2}\|\nabla w\|_2 \\ &+ \sup_{0<\tau\le T} \tau^{1/2-1/s}\|v_2(\tau)\|_s + \sup_{0<\tau\le T} \tau^{1/2}\|\nabla v_1(\tau)\|_2\Big) \|\tilde{v}\|_{\mathscr{Y}(s,T)} \end{aligned}$$

holds for every $T \in (0, T_4]$. Then, in the same calculation as in the proof of Theorem 1.1, we can find a positive constant T_5 such that

$$T_5^{1/2-1/s}\|w\|_s + T_5^{1/2}\|\nabla w\|_2 + \sup_{0<\tau\leq T_5} \tau^{1/2-1/s}\|v_2(\tau)\|_s + \sup_{0<\tau\leq T_5} \tau^{1/2}\|\nabla v_1(\tau)\|_2 \leq \frac{1}{2C},$$

with the same constant C as in (2.6). Then (2.6) implies that $\|\tilde{v}\|_{\mathscr{Y}(s,T_5)} = 0$, which implies that $\tilde{v}(t) \equiv 0$ on $[0, T_5]$.

For a positive number δ determined later and a nonnegative integer n, consider the condition

$$\tilde{v}(t) \equiv 0 \text{ holds on } [0, T_5 + n\delta]. \tag{2.7}$$

Suppose that (2.7) holds with some n, which we have already seen that we have already verified for $n = 0$. Then the identity (2.5) can be rewritten as

$$\tilde{v}(t) = -\int_{T_5+n\delta}^{t} \exp(-(t-\tau)A) P\left[\left(\left(w+v_2(\tau)\right)\cdot\nabla\right)\tilde{v}(\tau) + \left(\tilde{v}(\tau)\cdot\nabla\right)\left(w+v_1(\tau)\right)\right] d\tau$$

for $t \in (T_5 + n\delta, T_4]$. Then Lemmata 2.3 and 2.4 imply that there exists a constant C independent of v, w and n such that the estimate

$$\begin{aligned} &\|\tilde{v}(t)\|_s + \|\nabla\tilde{v}(t)\|_2 \\ &\leq C\frac{2s}{s-2}(t - T_5 - n\delta)^{(s-2)/2s} \sup_{T_5+n\delta\leq\tau\leq t} \left(\|\tilde{v}(\tau)\|_s + \|\nabla\tilde{v}(\tau)\|_2\right) \\ &\quad \left(\|w\|_s + T_5^{1/s-1/2}\|v_2\|_{\mathscr{Y}(s,T_3)} + \|\nabla w\|_2 + T_5^{-1/2}\|\nabla v_1\|_{\mathscr{Y}(s,T_3)}\right) \end{aligned}$$

holds for $t \in [T_5 + n\delta, T_5 + n\delta + 1]$. Suppose that $T_6 \in (T_5 + n\delta, T_5 + n\delta + 1]$. Taking the supremum with respect to $t \in [T_5 + n\delta, T_6]$, we have

$$\begin{aligned} &\sup_{T_5+n\delta\leq t\leq T_6} \left(\|\tilde{v}(t)\|_s + \|\nabla\tilde{v}(t)\|_2\right)\left(1 - C\frac{2s}{s-2}(T_6 - T_5 - n\delta)^{(s-2)/2s} \times \right. \\ &\left.\left(\|w\|_s + T_5^{1/s-1/2}\|v_2\|_{\mathscr{Y}(s,T_3)} + \|\nabla w\|_2 + T_5^{-1/2}\|\nabla v_1\|_{\mathscr{Y}(s,T_3)}\right)\right) \leq 0. \end{aligned}$$

Now choose $\delta \in (0, 1]$ so small that it satisfies

$$C\frac{2s}{s-2}\delta^{(s-2)/2s} \left(\|w\|_s + T_5^{1/s-1/2}\|v_2\|_{\mathscr{Y}(s,T_3)} + \|\nabla w\|_2 + T_5^{-1/2}\|\nabla v_1\|_{\mathscr{Y}(s,T_3)}\right) \leq \frac{1}{2},$$

and put $T_6 = \min\{T_5 + (n+1)\delta, T_4\}$. Then we have $\tilde{v}(t) \equiv 0$ for $0 \leq t \leq T_6$. If $T_6 = T_4$, we conclude that $\tilde{v}(t) \equiv 0$ for $0 \leq t \leq T_4$. Otherwise we have (2.7) with n replaced by $n+1$. Repeating the argument above, we can arrive $T_6 = T_4$ in finite steps. This completes the proof. □

3. Outline of the proof of Theorem 1.3.

In order to obtain the decay rate of $\|v(t)\|_q$ and $\|\nabla v(t)\|_2$, we follow the method by Kato [7]. However, this calculation requires the smallness of the initial value. Hence, to prove the result for large initial value, another method is needed to prove the global solvability and weak decay property. For this purpose we employ the energy inequality.

We first recall Hardy's inequality as follows:

Lemma 3.1. *Suppose that U is an exterior domain. Then there exists a constant C such that, for every $u(x) \in \dot{H}_0^1(U)$,*

$$\int_U \frac{|u(x)|^2}{|x|^2\left(1+\left|\log|x|\right|\right)^2}\,dx \le C\|\nabla u\|_2{}^2.$$

If U enjoys some symmetry property, we have the following improved version, whose proof is found in Galdi [5].

Lemma 3.2. *Suppose that U is an exterior domain satisfying* (D4). *Then there exists a constant C such that, for every $u(x) \in \dot{H}_0^1(U)$ satisfying* (U4), *we have*

$$\int_U \frac{|u(x)|^2}{|x|^2}\,dx \le C\|\nabla u\|_2{}^2.$$

We now start the proof of Theorem 1.3. The proof consists of four steps as follows:

(i) Global solvability together with the boundedness (a priori estimate)
(ii) Decay of $\|\nabla v(t)\|_2$ ($\|\nabla v(t)\|_2$ cannot grow so rapidly)
(iii) Decay of $\|v(t)\|_2$ (Slowness of energy dispersion)
(iv) Decay rate of $\|v(t)\|_q$ and $\|\nabla v(t)\|_2$ (L^q-L^r estimate for the perturbed semigroup)

Detailed proof of Step (i)–Step (iii) is given in [6], and that of Step (iv) is given in [10].

Step (i): Under the assumption of Theorem 1.3 we have the following lemma, which implies the boundedness of $\|v(t)\|_2$.

Lemma 3.3. *We have the inequality*

$$\frac{d}{dt}\|v(t)\|_2{}^2 \le \left(C\|w\|_{\mathscr{X}(b)} - 1\right)\|\nabla v(t)\|_2{}^2.$$

Proof. Taking the inner product with $v(t)$ with the equality (1.16) and integrating by parts, we obtain the equality

$$\frac{d}{dt}\|v(t)\|_2{}^2 + \|\nabla v(t)\|_2{}^2 - \left(v(t)\otimes w\right)\nabla v(t) = 0. \tag{3.1}$$

Employing Lemma 3.1 under Assumption (S) and Lemma 3.2 under Assumption (C), we can estimate

$$\|v(t)\otimes w\|_2 \le C\|w\|_{\mathscr{X}(b)}\|\nabla v(t)\|_2. \tag{3.2}$$

Substituting this estimate into (3.1) we obtain the conclusion. □

Lemma 3.3 implies the required estimates $\|v(t)\|_2 \leq \|v(s)\|_2$ for s, t with $0 \leq s < t < \infty$ and

$$\int_0^\infty \|\nabla v(t)\|_2{}^2 \, dt < \infty. \tag{3.3}$$

In the same way we have the an estimate for a higher order derivative, which we admit for the moment.

Lemma 3.4. *We have the inequality*

$$\frac{d}{dt}\|\nabla v(t)\|_2{}^2 \leq C' \left(\|w\|_{\mathscr{X}(b)} + \|\nabla u\|_2\right)^4 \|v(t)\|_2{}^2.$$

If $\|w\|_{\mathscr{X}(b)} < 1/2C'$, put $R = 2C'\left(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2\right)^4$. Then Lemmata 3.3 and 3.4 imply

$$\frac{d}{dt}\left(R\|v(t)\|_2{}^4 + \|\nabla v(t)\|_2{}^4\right) \leq -R\|\nabla v(t)\|_2{}^2\|v(t)\|_2{}^2 \leq 0.$$

This estimate ensures the boundedness of $\|\nabla v(t)\|_2$, and hence Theorem 1.1 implies that the solution become a time-global one.

Proof of Lemma 3.4: We have the equality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|_2{}^2 &= \left(\frac{dv}{dt}(t), Av(t)\right)\\ &= \left(Av(t) - P\left[(v(t)\cdot\nabla)w + (w\cdot\nabla)v(t) + (v(t)\cdot\nabla)v(t)\right], Av(t)\right)\\ &= -\|-\Delta v(t)\|_2{}^2 + I_1 + I_2 + I_3,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned} I_1 &= \left((v(t)\cdot\nabla)w, Av(t)\right),\\ I_2 &= \left((w\cdot\nabla)v(t), Av(t)\right),\\ I_3 &= \left((v(t)\cdot\nabla)v(t), Av(t)\right).\end{aligned}$$

By direct calculation we have $I_3 = 0$. Next, in view of the interpolation relation $(L^2, \dot{H}^2)_{1/2,1} = \dot{B}^1_{2,1} \subset L^\infty$, we can estimate

$$\begin{aligned} |I_1| &\leq C\|v(t)\|_2^{1/2}\|\Delta v(t)\|_2^{3/2}\|\nabla w\|_2,\\ |I_2| &\leq C\|v(t)\|_2^{1/2}\|\Delta v(t)\|_2^{3/2}\|w\|_{\mathscr{X}(1)}.\end{aligned}$$

Substituting these estimates into (3.4) we obtain the conclusion. □

Step (ii): We can prove the following lemma, which implies that $\|\nabla v(t)\|_2$ cannot grow so rapidly.

Lemma 3.5. *For s and t such that $1 \leq t-1 \leq s \leq t$, we have the estimate*

$$\begin{aligned}\|\nabla v(s)\|_2 &\geq \|\nabla v(t)\|_2\\ &\quad - C(t-s)^{1/3}\left(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2 + \sup_{t\geq 1}\|v(t)\|_2 + \sup_{t\geq 1}\|\nabla v(t)\|_2\right)^2.\end{aligned}$$

Admitting this lemma for the moment, we can derive $\|\nabla v(t)\|_2 \to 0$ as $t \to \infty$ from (3.3). In view of this fact and the boundedness of $\|v(t)\|_2$, the Gagliardo-Nirenberg inequality implies that $\|v(t)\|_q \to 0$ as $t \to \infty$ for every $q \in (2, \infty)$.

Proof of Lemma 3.5: We have $v(t) = \exp\big(-(t-s)A\big)v(s) + \tilde{v}$, where

(3.5)

$$\tilde{v} = -\int_s^t \exp\big(-(t-\tau)A\big)P\big[\big(v(\tau)\cdot\nabla\big)w + (w\cdot\nabla)v(\tau) + \big(v(\tau)\cdot\nabla\big)v(\tau)\big]\,d\tau.$$

Put

$$g_1(\tau) = P\big[\big(v(\tau)\cdot\nabla\big)w + \big(v(\tau)\cdot\nabla\big)v(\tau)\big] \text{ and } g_2(\tau) = P(w\cdot\nabla)v(\tau).$$

Then we have the estimates

$$\|g_1(\tau)\|_{3/2} \le C\left(\|\nabla w\|_2 + \sup_{t\ge1}\|\nabla v(t)\|_2\right)\sup_{t\ge1}\|v(t)\|_2{}^{1/3}\sup_{t\ge1}\|\nabla v(t)\|_2{}^{2/3}$$

and

$$\|g_2(\tau)\|_2 \le C\|w\|_{\mathscr{X}(1)}\sup_{t\ge1}\|\nabla v(t)\|_2.$$

Substituting these estimates into (3.5) we have

$$\begin{aligned}
\|\nabla\tilde{v}\|_2 \le &\int_s^t C(t-\tau)^{-2/3}d\tau\left(\|\nabla w\|_2 + \sup_{t\ge1}\|\nabla v(t)\|_2\right)\\
&\sup_{t\ge1}\|v(t)\|_2{}^{1/3}\sup_{t\ge1}\|\nabla v(t)\|_2{}^{2/3}\\
&+\int_s^t C(t-\tau)^{-1/2}d\tau\|w\|_{\mathscr{X}(1)}\sup_{t\ge1}\|\nabla v(t)\|_2\\
\le\ & C(t-s)^{1/3}\left(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2 + \sup_{t\ge1}\|v(t)\|_2 + \sup_{t\ge1}\|\nabla v(t)\|_2\right)^2.
\end{aligned}$$

Integrating this inequality on the interval $[s,t]$ we obtain the conclusion. This completes the proof of Lemma 3.5. □

Step (iii): We show an estimate which dominates the increase of the energy far from the origin. Let $\chi(x)$ be a smooth function on $\mathbb{R}$ such that $0 \le \chi(x) \le 1$, $\chi(x) \equiv 0$ on $[0,1]$ and $\chi(x) \equiv 1$ on $[2,\infty)$. Then we have the following lemma.

Lemma 3.6. *We have the estimate*

$$\frac{d}{dt}\left\|\chi\left(\frac{|x|}{R}\right)v(t)\right\|_2^2 \le C\big(\|w\|_{\mathscr{X}(b)} + \|v_0\|_2\big)\|\nabla v(\cdot,t)\|_2{}^2 \tag{3.6}$$

with a constant C independent of $R > 0$.

Admitting this lemma for the moment, we complete the proof of Step (iii). Suppose that $s < t$. Integrating (3.6) on the interval $[s,t]$, we obtain

$$\int_{|x|\geq 2R} |v(x,t)|^2\,dx \leq \int_{|x|\geq R} |v(x,s)|^2\,dx + C(\|w\|_{\mathscr{X}(b)} + \|v_0\|_2)\int_s^t \|\nabla v(\tau)\|_2{}^2\,d\tau.$$

For every fixed $\varepsilon > 0$, choose s so large that

$$\int_s^\infty \|\nabla v(\tau,s)\|_2{}^2\,d\tau < \frac{\varepsilon}{4C(\|w\|_{\mathscr{X}(b)} + \|v_0\|_2)}.$$

For this s, choose $R > 0$ so large that $\int_{|x|\geq R} |v(x,s)|^2\,dx < \frac{\varepsilon}{4}$. Then we have

$$\int_{|x|\geq 2R} |v(x,t)|^2\,dx < \frac{\varepsilon}{2} \quad \text{for every } t \geq s. \tag{3.7}$$

On the other hand, it follows from the fact $\|v(t)\|_q \to 0$ as $t \to \infty$ for $q > 2$ that there exists a constant $T \geq s$ such that

$$\int_{|x|\leq 2R} |v(x,t)|^2\,dx < \frac{\varepsilon}{2} \quad \text{for every } t \geq T. \tag{3.8}$$

Then the required asymptotic stability follows from (3.7) and (3.8). □

Proof of Lemma 3.6: In the same way as in the proof of Lemma 3.3, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\chi\left(\frac{|x|}{R}\right)v(x,t)\right\|_2^2 \\
&= \left(\frac{d}{dt}\left(\chi\left(\frac{|x|}{R}\right)v(x,t)\right), \chi\left(\frac{|x|}{R}v(t,x)\right)\right) \\
&= \left(\chi\left(\frac{|x|}{R}\right)\left(-\Delta v(x,t) + P[(w(x)\cdot\nabla)v(x,t)\right.\right. \\
&\qquad \left.\left. + (v(x,t)\cdot\nabla)w(x) + (v(x,t)\cdot\nabla)v(x,t)]\right), \chi\left(\frac{|x|}{R}\right)v(x,t)\right) \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{3.9}$$

where

$$I_1 = \left(-\Delta v(x,t), \chi\left(\frac{|x|}{R}\right)^2 v(x,t)\right),$$

$$I_2 = \left((v(x,t)\cdot\nabla)v(x,t), P\chi\left(\frac{|x|}{R}\right)^2 v(x,t)\right),$$

$$I_3 = \left((w(x)\cdot\nabla)v(x,t), P\chi\left(\frac{|x|}{R}\right)^2 v(x,t)\right),$$

$$I_4 = \left(\left(v(x,t)\cdot\nabla\right)w(x), P\chi\left(\frac{|x|}{R}\right)^2 v(x,t) \right).$$

We first estimate I_1. Since $\nabla v(t,x) \in L^2(\Omega)$ and $v(t,x) = 0$ on $\partial\Omega$, integration by parts yields

$$\begin{aligned} I_1 &= -\|\nabla v(\cdot,t)\|_2{}^2 + \left(\nabla v(x,t), \left(\nabla\left(\chi\left(\frac{|x|}{R}\right)^2\right)\right) v(x,t) \right) \\ &= -\|\nabla v(\cdot,t)\|_2{}^2 + \frac{1}{R}\left(\nabla v(x,t), 2(\nabla\chi)\left(\frac{|x|}{R}\right)\chi\left(\frac{|x|}{R}\right) v(x,t) \right). \end{aligned}$$

It follows that

$$I_1 \le \frac{C}{R}\|\nabla v(\cdot,t)\|_2 \left\| (\nabla\chi)\left(\frac{|x|}{R}\right)\left(\frac{|x|}{R}\right) v(x,t) \right\|_2. \tag{3.10}$$

Since $v(x,t) = 0$ on $\partial\Omega$, we can apply the Poincaré inequality to obtain the estimate

$$\begin{aligned} &\left\| (\nabla\chi)\left(\frac{|x|}{R}\right)\chi\left(\frac{|x|}{R}\right) v(x,t) \right\|_2 \\ &\qquad \le C\left(\int_{\{x\in\Omega \,|\, |x|\le 2R\}} |\nabla v(x,t)|^2\,dx \right)^{1/2} \le CR\|\nabla v\|_2. \end{aligned} \tag{3.11}$$

Substituting this estimate into (3.10) we conclude

$$I_1 \le C\|\nabla v(\cdot,t)\|_2{}^2. \tag{3.12}$$

We next estimate the term I_2 as follows:

$$\begin{aligned} I_2 &\le \|\nabla v(\cdot,t)\|_2 \|v(\cdot,t)\|_4{}^2 \\ &\le C\|\nabla v(\cdot,t)\|_2{}^2 \|v(\cdot,t)\|_2 \le C\|v(\cdot,T)\|_2 \|\nabla v(\cdot,t)\|_2{}^2 \end{aligned} \tag{3.13}$$

for $t \ge T$ in view of the Gagliardo-Nirenberg inequality.

In view of (3.2), the term I_3 can be estimated as

$$I_3 \le C\|w\|_{\mathscr{X}(b)} \|\nabla v(\cdot,t)\|_2{}^2. \tag{3.14}$$

Finally, in order to estimate I_4 we recall the construction of the Helmholtz decomposition in exterior domains by Miyakawa. We have

$$P\chi\left(\frac{|x|}{R}\right)^2 v(x,t) = \chi\left(\frac{|x|}{R}\right)^2 v(x,t) + \nabla q_1(x,t) + \nabla q_2(x,t),$$

where $q_1(x,t)$ is the solution in $\mathbb{R}^2$ of the equation

$$-\Delta q_1(x,t) = \operatorname{div}\left(\chi\left(\frac{|x|}{R}\right)^2 v(x,t)\right) = \frac{1}{R}(\nabla\chi^2)\left(\frac{|x|}{R}\right)\cdot v(x,t) \tag{3.15}$$

and $q_2(x,t)$ is the solution of the Neumann problem

$$\begin{cases} -\Delta q_2(x,t) = 0 & \text{in } \Omega, \\ (n\cdot\nabla)q_2(x,t) = -(n\cdot\nabla)\left(\chi\left(\frac{|x|}{R}\right)v(x,t)+q_1(x,t)\right) = -(n\cdot\nabla)q_1(x,t) & \\ & on\partial\Omega. \end{cases}$$

Then, integrating by parts, we have

$$I_4 = \left(v(x,t)\otimes w(x), -\nabla\left(\chi\left(\frac{|x|}{R}\right)^2 v(x,t)\right) - \nabla^2 q_1(x,t) - \nabla^2 q_2(x,t)\right).$$

It follows that

$$(3.16)\quad I_4 \le C\|w\|_{\mathscr{X}(b)}\|\nabla v\|_2\left(\left\|\chi\left(\frac{|x|}{R}\right)^2 \nabla v(x,t)\right\|_2 + \right.$$

$$\left. +\frac{2}{R}\left\|(\nabla\chi)\left(\frac{|x|}{R}\right)\chi\left(\frac{|x|}{R}\right)v(x,t)\right\|_2 + \|\nabla^2 q_1(\cdot,t)\|_2 + \|\nabla^2 q_2(\cdot,t)\|_2\right).$$

Then the L^2-boundedness of the Riesz transforms implies

(3.17)

$$\|\nabla^2 q_1(\cdot,t)\|_2 \le \frac{C}{R}\left\|(\nabla\chi)\left(\frac{|x|}{R}\right)\chi\left(\frac{|x|}{R}\right)\cdot v(x,t)\right\|_2 \le C\|\nabla v(\cdot,t)\|_2.$$

We next have

$$\|\nabla^2 q_2(\cdot,t)\|_2 \le C\|(n\cdot\nabla)q_2(\cdot,t)\|_{H^{1/2}(\partial\Omega)} = C\|(n\cdot\nabla)q_1(\cdot,t)\|_{H^{1/2}(\partial\Omega)} \le C\|\nabla^2 q_1(\cdot,t)\|_2$$

It follows from (3.17) that

$$(3.18)\qquad \|\nabla^2 q_2(\cdot,t)\|_2 \le C\|\nabla v(\cdot,t)\|_2.$$

Substituting (3.11), (3.17) and (3.18) into (3.16) we obtain

$$(3.19)\qquad I_4 \le C\|w\|_{\mathscr{X}(b)}\|\nabla v\|_2^2.$$

Substituting (3.12), (3.13), (3.14) and (3.19) into (3.9) we conclude that

$$\frac{d}{dt}\left\|\chi\left(\frac{|x|}{R}\right)v(x,t)\right\|_2^2 \le C\left(\|w\|_{\mathscr{X}(b)} + \|v(T)\|_2\right)\|\nabla v(\cdot,t)\|_2^2.$$

Now (3.6) follows from the monotonicity of $\|v(t)\|_2$. □

We now recall the estimate of coerciveness of the Stokes operator.

Lemma 3.7. *We have the following assertions:*

(i) *For* $v\in D(A^{1/2}) = L^2_\sigma(\Omega)\cap\left(H^1_0(\Omega)\right)^2$, *we have* $\|\nabla v\|_2 = \left\|A^{1/2}v\right\|_2$.

(ii) *For* $v \in D(A) = L^2_\sigma(\Omega)\cap\left(H^1_0(\Omega)\cap H^2(\Omega)\right)^2$, *there exists a constant* C *such that we have the estimate* $\|\nabla^2 v\|_2 \le C\left(\|Av\|_2 + \left\|A^{1/2}v\right\|_2\right)$.

We next recall the resolvent estimates of the Stokes operator by Borchers and Varnhorn [2] and Dan and Shibata [3, 4], from which estimates Theorem 2.1 follows.

Proposition 3.8. *Put* $D = \{\zeta \in \mathbb{C} \mid \zeta \neq 0, |\arg \zeta| \leq 3\pi/4\}$. *Then we have the following assertions:*

(i) *For every* q *and* r *such that* $1 < q \leq r \leq \infty$, *there exists a positive constant* $C_{q,r}$ *such that, for every* $\zeta \in D$, *the operator* $(\zeta + A)^{-1}$ *is a bounded operator from* $L^q_\sigma(\Omega)$ *to* $\left(L^r(\Omega)\right)^2$ *satisfying the estimate* $\|(\zeta + A)^{-1}u\|_r \leq C_{q,r}|\zeta|^{-1+1/q-1/r}\|u\|_q$ *for every* $u \in L^q_\sigma(\Omega)$. *In particular, if* $q \leq r < \infty$, *we have* $(\zeta + A)^{-1}u \in L^r_\sigma(\Omega)$.

(ii) *For every* q *and* r *such that* $1 < q \leq r \leq 2$, *there exists a positive constant* $C_{q,r}$ *such that, for every* $\zeta \in D$, *the operator* $\nabla(\zeta + A)^{-1}$ *is a bounded operator from* $L^q_\sigma(\Omega)$ *to* $\left(L^r(\Omega)\right)^4$ *satisfying the estimate* $\|\nabla(\zeta + A)^{-1}u\|_r \leq C_{q,r}|\zeta|^{-1/2+1/q-1/r}\|u\|_q$ *for every* $u \in L^q_\sigma(\Omega)$.

This proposition and Lemma 3.7 yield the following proposition.

Proposition 3.9. *We have the following assertions:*

(i) *Suppose that* $1 < q \leq 2$. *Then there exists a constant* C'_q *such that, for every* $u \in L^q_\sigma(\Omega)$ *and every* $t > 0$, *the function* $\exp(-tA)u$ *belongs to the space* $\left(H^1_0(\Omega) \cap H^2(\Omega)\right)^2$, *and satisfies the estimate*

$$\|\nabla^2 \exp(-tA)u\|_2 \leq C'_{q,s} t^{-1/q}(1 + t^{-1/2})\|u\|_q.$$

(ii) *There exists a constant* C''_s *such that, for every* $u \in L^2_\sigma(\Omega) \cap \left(H^1_0(\Omega)\right)^2$, *the function* $\exp(-tA)u$ *satisfies the estimate*

$$\|\nabla^2 \exp(-tA)u\|_2 \leq C''_s(1 + t^{-1/2})\|\nabla u\|_2.$$

This proposition immediately implies the following corollary.

Corollary 3.10. *Suppose that* $1 \leq s < 3/2$. *Then we have the following assertions:*

(i) *Suppose that* $1 < q \leq 2$. *Then there exists a constant* $C'_{q,s}$ *such that, for every* $u \in L^q_\sigma(\Omega)$ *and every* $t > 0$, *the function* $\exp(-tA)u$ *belongs to the space* $\left(H^s_0(\Omega)\right)^2$, *and satisfies the estimate*

$$\|\exp(-tA)u\|_{\dot{H}^s} \leq C'_{q,s} t^{-1/q}(1 + t^{(s-1)/2})\|u\|_q.$$

(ii) *There exists a constant* C'' *such that, for every* $u \in L^2_\sigma(\Omega) \cap \left(H^1_0(\Omega)\right)^2$, *the function* $\exp(-tA)u$ *satisfies the estimate*

$$\|\exp(-tA)u\|_{\dot{H}^s} \leq C''_s(1 + t^{-(s-1)/2})\|\nabla u\|_2.$$

We now introduce a perturbation of the operator A, and show some properties. Suppose that w satisfies $w \in (\mathscr{X}(b))^2$ with some $b \geq 1$ and $\nabla w \in (L^2(\Omega))^4$, and put $B[u] = P\{(w \cdot \nabla)u + (u \cdot \nabla)w\}$. Then, for every

$$u \in D(A) = L^q_\sigma(\Omega) \cap (H^1_{q,0}(\Omega) \cap H^2_q(\Omega))^2$$

with $1 < q \leq 2$, we have $\nabla u \in (L^q(\Omega))^4$, which implies $(w \cdot \nabla)u \in (L^q(\Omega))^2$. We moreover have $u \in L^{2q/(2-q)}_\sigma(\Omega)$ if $1 < q < 2$ and $u \in (L^\infty(\Omega))^2$ if $q = 2$, which imply $(u \cdot \nabla)w \in (L^q(\Omega))^2$ in both cases. Hence the operator $L_w[u] = Au + B[u]$ is well-defined on $u \in D(A)$.

In the sequel we obtain the resolvent estimate of this operator. For this purpose Borchers and Miyakawa [1] expanded the resolvent into Neumann series. Kozono and Yamazaki [8] extended the range of boundedness by estimating the Neumann series by using fractional powers of the resolvent. However, we cannot employ this method straightforward due to the strong limitation of the range of coerciveness. We get around this difficulty by obtaining the estimate for the fractional power $(\zeta + A)^{-1/2}$ defined by the spectral decomposition of A on $L^2_\sigma(\Omega)$ and estimate the operator $(\zeta + A)^{-1/2} B (\zeta + A)^{-1/2}$ by duality argument.

Let $\mu(\lambda)$ denote the spectral measure associated with the operator A on $L^2_\sigma(\Omega)$. Then, for $\zeta \in D$, we can write

$$(\zeta + A)^{-1} = \int_0^\infty \frac{1}{\zeta + \lambda} d\mu(\lambda), \quad (\zeta + A)^{-1/2} = \int_0^\infty \frac{1}{\sqrt{\zeta + \lambda}} d\mu(\lambda).$$

Then the operator $(\zeta + A)^{-1/2}$ is holomorphic in the interior of D with values in bounded linear operators on $L^2_\sigma(\Omega)$. Here we note that $\zeta \in D$ implies $\zeta + \lambda \in D$ for every $\lambda \geq 0$, and hence the branch of $\sqrt{\zeta + \lambda}$ is well-defined. It is easy to see that $\left\{(\zeta + A)^{-1/2}\right\}^2 = (\zeta + A)^{-1}$. For the operator $(\zeta + A)^{-1/2}$ we can prove the following lemmas by spectral decomposition.

Lemma 3.11. *For every q and r satisfying $1 < q \leq 2 \leq r < \infty$, there exist constants C_q and C_r such that, for every $\zeta \in D$ we have the estimates*

$$\left\|(\zeta + A)^{-1/2} u\right\|_2 \leq C_q |\zeta|^{-1+1/q} \|u\|_q \text{ for every } u \in L^2_\sigma(\Omega) \cap L^q_\sigma(\Omega),$$
$$\left\|(\zeta + A)^{-1/2} u\right\|_r \leq C_r |\zeta|^{-1/r} \|u\|_2 \quad \text{for every } u \in L^2_\sigma(\Omega).$$

Lemma 3.12. *There exists a constant C_2 such that, for every $\zeta \in D$ and every $u \in L^2_\sigma(\Omega)$, we have the estimate $\left\|\nabla(\zeta + A)^{-1/2} u\right\|_2 \leq C_2 \|u\|_2$.*

From these lemmas we can prove the following estimate.

Lemma 3.13. *Suppose that $w \in (\mathscr{X}(b))^2$ with some $b \geq 1$ and $\nabla w \in (L^2(\Omega))^4$. Suppose also that $\zeta \in \mathbb{C} \setminus \{0\}$ satisfies $|\arg \zeta| \leq 3\pi/4$. Then* *the operator* $(\zeta+A)^{-1/2}B(\zeta+A)^{-1/2}$ *is bounded in* $L^2_\sigma(\Omega)$, *and it satisfies the estimate*

$$\left\|(\zeta+A)^{-1/2}B\left[(\zeta+A)^{-1/2}u\right]\right\|_2 \le C\|w\|_{\mathscr{X}(b)}\|u\|_2,$$

where C *is a constant depending only on* Ω.

Proof. Suppose that $\varphi \in C^\infty_{0,\sigma}(\Omega)$. In view of the equalities $\nabla \cdot w = 0$ and $\nabla \cdot (\zeta+A)^{-1/2}u = 0$, we have

(3.20)
$$\begin{aligned}&\left|\left(\varphi,(\zeta+A)^{-1/2}P\left\{(w\cdot\nabla)(\zeta+A)^{-1/2}u+\left(((\zeta+A)^{-1/2}u\cdot\nabla\right)w\right\}\right)\right|\\ &=\left|-\left(\nabla(\zeta+A)^{-1/2}\varphi, w(\zeta+A)^{-1/2}u\right)\right|\\ &\le\left\|\nabla(\zeta+A)^{-1/2}\varphi\right\|_2\left\|w(\zeta+A)^{-1/2}u\right\|_2.\end{aligned}$$

In view of the fact $(\zeta+A)^{-1/2}u \in D(A^{1/2})$, Lemma 3.12 and (3.2) imply

(3.21)
$$\left\|w(\zeta+A)^{-1/2}u\right\|_2 \le C\|w\|_{\mathscr{X}(b)}\left\|\nabla(\zeta+A)^{-1/2}u\right\|_2 \le C\|w\|_{\mathscr{X}(b)}\|u\|_2,$$

where the constant C depends only on Ω. Since $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^2_\sigma(\Omega)$, we obtain the conclusion by substituting Lemma 3.12 and the inequality (3.21) into (3.20). □

For the operator L_w we have the following proposition.

Proposition 3.14. *For every* q, r *such that* $1 < q \le 2 \le r < \infty$, *there exist positive numbers* A *and* $A_{q,r}$ *such that, for every* $w \in (\mathscr{X}(b))^2$ *satisfying* $\nabla w \in (L^2(\Omega))^4$ *and* $\|w\|_{\mathscr{X}(b)} \le A$, *we have the estimates*

$$\|(\zeta+L_w)^{-1}u\|_r \le A_{q,r}|\zeta|^{-1+1/q-1/r}\|u\|_q,$$
$$\|\nabla(\zeta+L_w)^{-1}u\|_2 \le A_{q,2}|\zeta|^{-1+1/q}\|u\|_q$$

for every $u \in L^q_\sigma(\Omega)$ *and every* $\zeta \in D$.

Proof. Suppose that $\|w\|_{\mathscr{X}(b)} \le 1/2C$. Then Lemma 3.13 implies that the operator T defined by

$$T=\sum_{j=0}^{\infty}\left\{-(\zeta+A)^{-1/2}B(\zeta+A)^{-1/2}\right\}^j$$

is bounded on $L^2_\sigma(\Omega)$ uniformly in $\zeta \in D$ and satisfies

$$(3.22)\qquad (\zeta+A)^{-1/2}T(\zeta+A)^{-1/2}=(\zeta+A+B)^{-1}=(\zeta+L_w)^{-1}.$$

For q and r as in the assumption, Lemmata 3.11 and 3.12 imply $\|(\zeta+A)^{-1}u\|_2 \le C_q|\zeta|^{-1+1/q}\|u\|_q$ for $u \in L^q_\sigma(\Omega)\cap L^2_\sigma(\Omega)$, and $\|(\zeta+A)^{-1}u\|_r \le C_r|\zeta|^{-1/r}\|u\|_2$, $\|\nabla(\zeta+A)^{-1}u\|_2 \le C_2\|u\|_2$ for $u \in L^2_\sigma(\Omega)$. Hence the required estimates follow from these estimates. □

Since we can obtain a semigroup by integrating the resolvent of the generator on an appropriate contour in the complex plane, we can deduce the next theorem from the proposition above.

Theorem 3.15. *Let w be the same as in Proposition 3.14. Then the operator $-L_w$ generates a bounded analytic C^0-semigroup $\exp(-tL_w)$ on $L^2_\sigma(\Omega)$, and for every q and r such that $1 < q \leq 2 \leq r < \infty$, there exists a constant $B_{q,r}$ such that, for every $u \in L^q_\sigma(\Omega)$ and $t > 0$, we have the estimates*

$$\|\exp(-tL_w)u\|_r \leq B_{q,r}t^{-1/q+1/r}\|u\|_q, \quad \|\nabla \exp(-tL_w)u\|_2 \leq B_{q,2}t^{-1/q}\|u\|_q.$$

We now proceed to Step (iv). The conclusions of Step (ii) and Step (iii) imply that, for every $\varepsilon > 0$, there exists a positive number T_0 such that, for every $t \geq T_0$ we have $\|v(t)\|_2 < \varepsilon$, $\|v(t)\|_4 < \varepsilon$ and $\|\nabla v(t)\|_2 < \varepsilon$.

Next, for T_1 such that $T_0 < T_1 < \infty$, we put

$$\alpha(T_1) = \sup_{T_0 \leq t \leq T_1} \max\left\{(t-T_0)^{1/4}\|v(t)\|_4, (t-T_0)^{1/2}\|\nabla v(t)\|_2\right\}.$$

Then the function $\alpha(T_1)$ is continuous and monotone-increasing. For $t \in [T_0, T_1]$, we can write

$$v(t) = \exp\big(-(t-T_0)L_w\big)v(T_0) + \int_{T_0}^{t} \exp\big(-(t-\tau)L_w\big)P\left[\big(v(\tau)\cdot\nabla\big)v(\tau)\right] d\tau.$$

From this we can estimate

$$\begin{aligned}\|v(t)\|_4 &\leq B_{2,4}(t-T_0)^{-1/4}\|v(T_0)\|_2 \\ &\quad + C_{4/3}\int_{T_0}^{t} B_{4/3,4}(t-\tau)^{-1/2}\|v(\tau)\|_4\|\nabla v(\tau)\|_2\, d\tau \\ &\leq B_{2,4}(t-T_0)^{-1/4}\varepsilon + C_{4/3}\alpha(t)^2\int_{T_0}^{t} B_{4/3,4}(t-\tau)^{-1/2}\tau^{-3/4}\, d\tau\end{aligned}$$

where $C_{4/3}$ denotes the operator norm of the projection P from $\big(L^{4/3}(\Omega)\big)^2$ to $L^{4/3}_\sigma(\Omega)$. This implies

$$(t-T_0)^{1/4}\|v(t)\|_4 \leq B_{2,4}\varepsilon + C_{4/3}B_{4/3.4}B\left(\frac{1}{2},\frac{1}{4}\right)\alpha(T_1)^2. \tag{3.23}$$

In the same way, from the estimate

$$\|\nabla v(t)\|_2 \leq B_{2,2}(t-T_0)^{-1/2}\varepsilon + C_{4/3}\alpha(t)^2\int_{T_0}^{t} B_{4/3,2}(t-\tau)^{-3/4}\tau^{-3/4}\, d\tau,$$

it follows that

$$(t-T_0)^{1/2}\|\nabla v(t)\|_2 \leq B_{2,2}\varepsilon + C_{4/3}B_{4/3.2}B\left(\frac{1}{4},\frac{1}{4}\right)\alpha(T_1)^2. \tag{3.24}$$

Hence, putting

$$C_1 = \max\left\{C_{4/3}B_{4/3.4}B\left(\frac{1}{2},\frac{1}{4}\right), C_{4/3}B_{4/3.2}B\left(\frac{1}{4},\frac{1}{4}\right)\right\},$$

$$C_2 = \max\{B_{2,4}, B_{2,2}, 1\}$$

and taking the maximum of (3.23) and (3.24), we see that

$$\max\left\{(t-T_0)^{1/4}\|v(t)\|_4, (t-T_0)^{1/2}\|\nabla v(t)\|_2\right\} \le C_1\alpha(T_1)^2 + C_2\varepsilon.$$

Taking the supremum for $t \in [T_0, T_1]$, we see that $\alpha(T_1)$ satisfies

$$\alpha(T_1) \le C_1\alpha(T_1)^2 + C_2\varepsilon. \tag{3.25}$$

We suppose that $\varepsilon < 1/4C_1C_2$. Then there exists two distinct roots of the equation $C_1X^2 - X + C_2\varepsilon = 0$. Let $f(\varepsilon)$ denote the smaller one. Then we have $\varepsilon < f(\varepsilon)$, and the intermediate theorem implies that we have $\alpha(T_1) < f(\varepsilon)$ if $T_1 > T_0$ is sufficiently close to T_0. It follows that

$$\|\nabla v(T_1)\|_2 \le f(\varepsilon)(T_1 - T_0)^{-1/2} \le \sqrt{2} f(\varepsilon) T_1^{-1/2}$$

for every $T_1 \ge 2T_0$. On the other hand, we have $\|v(T_1)\|_2 \le \varepsilon \le f(\varepsilon)$. Hence the Gagliardo-Nirenberg inequality implies that the estimate $\|v(T_1)\|_q \le C_q f(\varepsilon) T_1^{-1/2+1/q}$ holds for every $T_1 > 2T_0$ and $q \in [2,\infty)$. Since we have $f(\varepsilon) \to +0$ as $\varepsilon \to +0$, we conclude (1.18) and (1.19).

It remains only to show (1.20). First, since $v(t) \in H_0^1(\Omega)$ and since $H_0^1(\Omega)$ can be regarded as a closed subset of $H_0^1(\mathbb{R}^2)$, we have

$$\begin{aligned}\|v(t)\|_{\dot B^0_{\infty,2}(\Omega)} &\le \|v(t)\|_{\dot B^0_{\infty,2}(\mathbb{R}^2)}\\ &\le C\|\nabla v(t)\|_{L^2(\mathbb{R}^2)} = C\|\nabla v(t)\|_{L^2(\Omega)} = o(t^{-1/2}),\end{aligned}$$

where $\dot B^0_{\infty,2}(\mathbb{R}^2)$ denotes the homogeneous Besov space on $\mathbb{R}^2$. Then, for every fixed $\varepsilon \in (0,1]$, we can choose $T \ge 2$ so large that

$$\sup_{t\ge T-1}\max\left\{t^{3/8}\|v(t)\|_8, t^{1/2}\|\nabla v(t)\|_2, t^{1/2}\|v(t)\|_{\dot B^0_{\infty,2}}\right\} \le \varepsilon.$$

Suppose that $t \ge T$. Then, for every $\tau \in [t-1, t]$, we have $\tau \ge t-1 \ge t/2$.

We next recall the Littlewood-Paley decomposition. Let $\chi(s)$ be a monotone-decreasing C^∞-function on $(-1,\infty)$ such that $\chi(s) \equiv 1$ on $(-1,1]$ and $\chi(s) \equiv 0$ on $[2,\infty)$. Next, for $\xi \in \mathbb{R}^n$, put $\Phi(\xi) = \chi(|\xi|)$ and $\varphi_j(\xi) = \chi(2^{-j}|\xi|) - \chi(2^{1-j}|\xi|)$ for $j \in \mathbb{Z}$. Then, for every $k \in \mathbb{Z}$ we have $\Phi(2^{-k}\xi) + \sum_{j=k+1}^{\infty} \varphi_j(\xi) = 1$. For a fixed $t \ge T$, choose k as the smallest positive integer such that $t \le 2^{2k}$; namely, $k \ge (\log_2 t)/2$. We then put

$$v^{(1)}(t) = \mathscr{F}^{-1}\left[\Phi(2^k\xi)\mathscr{F}[v(t)]\right],\ v^{(2)}(t) = \sum_{j=-k+1}^{k-1}\left[\varphi_j(\xi)\mathscr{F}[v(t)]\right],$$

$$v^{(3)}(t) = \sum_{j=k}^{\infty}\left[\varphi_j(\xi)\mathscr{F}[v(t)]\right].$$

Then we have $v(t) = v^{(1)}(t) + v^{(2)}(t) + v^{(3)}(t)$.

We first have

$$(3.26)\quad \left\|v^{(1)}(t)\right\|_\infty$$
$$\leq \|v(t)\|_8 \left\|2^{-2k}\left(\mathscr{F}^{-1}[\Phi]\right)\left(2^{-k}\cdot\right)\right\|_{8/7} = C2^{-k/4}\|v(t)\|_8 \leq Ct^{-1/2}\varepsilon.$$

We next observe that

$$\left\|v^{(2)}(t)\right\|_\infty \leq \sum_{j=-k+1}^{k-1} \|\mathscr{F}^{-1}[\varphi_j \mathscr{F}[v(t)]]\|_\infty$$
$$(3.27)\qquad \leq \sqrt{2k-1}\left(\sum_{j=-k+1}^{k-1} \|\mathscr{F}^{-1}[\varphi_j \mathscr{F}[v(t)]]\|_\infty^{\,2}\right)^{1/2}$$
$$\leq C\sqrt{\log t}\|v(t)\|_{\dot{B}^0_{\infty,2}} \leq C\varepsilon t^{-1/2}\sqrt{\log t}.$$

Finally, in order to estimate $\left\|v^{(3)}(t)\right\|_\infty$, we employ another representation

$$(3.28)\quad v(t) = \exp(-A)v(t-1)$$
$$+\int_{t-1}^{t} \exp(-(t-\tau)A)P\left[(w\cdot\nabla)v(\tau) + \left(v(\tau)\cdot\nabla\right)w + \left(v(\tau)\cdot\nabla\right)v(\tau)\right]d\tau.$$

Then the Sobolev embedding theorem and Corollary 3.10, (2) imply

$$(3.29)\quad \|\exp(-A)v(t-1)\|_{\dot{C}^{1/3}} \leq C\|\exp(-A)v(t-1)\|_{\dot{H}^{4/3}}$$
$$\leq C\|\nabla v(t-1)\|_2 \leq C\varepsilon(t-1)^{-1/2} \leq 2C\varepsilon t^{-1/2}$$

Next, for $\tau \in [t-1,t]$, we have

$$\|(w\cdot\nabla)v(\tau)\|_{8/5} \leq \|w\|_8\|\nabla v(\tau)\|_2 \leq C\|w\|_{\mathscr{X}(b)}\varepsilon t^{-1/2},$$
$$\left\|\left(v(\tau)\cdot\nabla\right)w\right\|_{8/5} \leq \|v(\tau)\|_8\|\nabla w\|_2 \leq C\|\nabla w\|_2\varepsilon t^{-3/8},$$

and

$$\left\|\left(v(\tau)\cdot\nabla\right)v(\tau)\right\|_{8/5} \leq C\varepsilon^2 t^{-7/8}.$$

Summing up these estimates we conclude that

$$\|P\left[(w\cdot\nabla)v(\tau) + \left(v(\tau)\cdot\nabla\right)w + \left(v(\tau)\cdot\nabla\right)v(\tau)\right]\|_{8/5}$$
$$\leq C\varepsilon(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2 + 1)t^{-3/8}.$$

Hence the Sobolev embedding theorem and Corollary 3.10, (2) imply
(3.30)

$$\|g(t)\|_{\dot{C}^{1/3}} \leq C\int_{t-1}^{t} (t-\tau)^{-2/3-5/4+1}$$
$$\|P\left[(w\cdot\nabla)v(\tau) + \left(v(\tau)\cdot\nabla\right)w + \left(v(\tau)\cdot\nabla\right)v(\tau)\right]\|_{8/5}d\tau$$
$$\leq C\varepsilon(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2 + 1)t^{-3/8}\int_{t-1}^{t} (t-\tau)^{-11/12}d\tau$$
$$\leq C\varepsilon(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2 + 1)t^{-3/8}$$

It follows from (3.28), (3.29) and (3.30) that

$$\|v(t)\|_{\dot{C}^{1/3}} \le C\varepsilon(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2 + 1)t^{-3/8}.$$

Since $\dot{C}^{1/3}$ coincides with $\dot{B}^{1/3}_{\infty,\infty}$, we have

$$\|\mathscr{F}^{-1}[\varphi_j \mathscr{F}[v(t)]]\|_\infty \le C2^{-j/3}\varepsilon(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2 + 1)t^{-3/8}.$$

Summing up we obtain

$$\begin{aligned} \left\|v^{(3)}(t)\right\|_\infty &\le \sum_{j=k}^{\infty} \|\mathscr{F}^{-1}[\varphi_j \mathscr{F}[v(t)]]\|_\infty \\ &\le C2^{-k/3}\varepsilon(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2 + 1)t^{-3/8} \\ &\le C\varepsilon(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2 + 1)t^{-3/8-1/6} \\ &\le C\varepsilon(\|w\|_{\mathscr{X}(b)} + \|\nabla w\|_2 + 1)t^{-1/2}. \end{aligned} \tag{3.31}$$

Summing up (3.26), (3.31) and (3.27) we conclude (1.20).

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E-mail address: masao.yamazaki@waseda.jp