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Kardar-Parisi-Zhang equation and its approximation

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1 KPZ equation

The Kardar-Parisi-Zhang (KPZ) equation is a stochastic partial differential equation (SPDE), which describes the motion of a growing interface with a random fluctuation. Denoting the height of the interface at time t and position $x \in \mathbb{R}$ by $h = h(t, x)$, it has the form

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \dot{W}(t, x), \tag{1}$$

where $\dot{W}(t, x)$ is the space-time Gaussian white noise, whose covariance structure is given by

$$E[\dot{W}(t, x)\dot{W}(s, y)] = \delta(x - y)\delta(t - s). \tag{2}$$

We consider the equation (1) in one dimension. This equation is actually ill-posed because of inconsistency between the nonlinearity and the roughness of the noise. As is explained in Section 4 for linear stochastic equations under periodic boundary condition, the solution $h(t, x)$ is expected to be $(\frac{1}{2} - \varepsilon)$-Hölder continuous in the space variable x for every $\varepsilon > 0$, so that the nonlinear term $(\partial_x h)^2$ would diverge. In fact, instead of (1), the renormalized equation

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - \delta_x(x)\right) + \dot{W}(t, x), \tag{3}$$

has the meaning in the following sense: Its Cole-Hopf solution defined as the logarithm of the solution of the linear stochastic heat equation (SHE) with a multiplicative noise:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}(t, x), \tag{4}$$

i.e., $h(t, x) := \log Z(t, x)$ is a mathematically well-defined object and, by applying Itô's formula for this $h(t, x)$, we obtain (3) from (4) at least at a heuristic level. Note that, since $(dW(t, x))^2 = \delta_x(x)dt$ from (2), the term $-\frac{1}{2}\delta_x(x)$ appears in (3) as an Itô correction term.

2 Approximation of KPZ equation

To give a meaning to (3) more precisely, we need to consider approximation schemes for (3). First approximation is simple and introduced by

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - \xi^\varepsilon\right) + \dot{W}^\varepsilon(t,x), \tag{5}$$

where $\dot{W}^\varepsilon(t,x) = \dot{W} * \eta^\varepsilon(t,x)$ is a smeared noise defined by applying a usual convolution kernel η^ε which tends to δ_0 as $\varepsilon \downarrow 0$, and $\xi^\varepsilon = \eta_2^\varepsilon(0)$ with $\eta_2^\varepsilon = \eta^\varepsilon * \eta^\varepsilon$. Then, by Itô's formula, we easily see that the solution $h = h^\varepsilon$ of (5) is given by the Cole-Hopf transform $h^\varepsilon = \log Z^\varepsilon$ of the solution $Z = Z^\varepsilon$ of the following SHE with the smeared noise:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}^\varepsilon(t,x). \tag{6}$$

It is also easy to see that Z^ε converges to the solution Z of (4) as $\varepsilon \downarrow 0$. Thus, we can show that the solution h^ε of (5) converges to the Cole-Hopf solution of the KPZ equation. M. Hairer [2] has recently succeeded to give a meaning to (3), without bypassing the Cole-Hopf transform, and proved that the Cole-Hopf solution is the right solution of (3) under the periodic boundary condition.

In [1], we introduced a different type of approximation scheme for the KPZ equation:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - \xi^\varepsilon\right) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t,x). \tag{7}$$

This type of approximation is appropriate from the view point to identify the invariant measures, since the applications of a certain operator A (in our case, the convolution operator) to the noise term and the same operator A twice to the drift term usually do not change the structure of the invariant measures; note that the second derivative ∂_x^2 and the convolution operator commute. The Cole-Hopf transform applied to this equation leads to an SHE with a smeared noise having an extra complex nonlinear term involving a certain renormalization structure:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \frac{1}{2}Z\left\{\left(\frac{\partial_x Z}{Z}\right)^2 * \eta_2^\varepsilon - \left(\frac{\partial_x Z}{Z}\right)^2\right\} + Z\dot{W}^\varepsilon(t,x). \tag{8}$$

It is shown that, under the situation that the corresponding tilt process is stationary, this complex term (the middle term in the right hand side of (8)) can be replaced by a simple linear term divided by 24 in the limit, so that the limit equation is the linear SHE:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \frac{1}{24}Z + Z\dot{W}(t,x). \tag{9}$$

The constant $\frac{1}{24}$ is specific and frequently appears in KPZ related papers. The Wiener-Itô expansion and a similar method for establishing the so-called Boltzmann-Gibbs principle are effectively used to derive (9) from (8). As a result, it is shown that the distribution of a two-sided geometric Brownian motion with a height shift given by Lebesgue measure is invariant under the evolution determined by the SHE (4) on $\mathbb{R}$.

3 Multi-component KPZ equation

Multi-component KPZ equation is an extension of (1) and written for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d \in \mathbb{R}^d$ as

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \dot{W}^\alpha(t,x), \tag{10}$$

where $\Gamma^\alpha_{\beta\gamma}$ are constants which satisfy the condition $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \Gamma^\beta_{\gamma\alpha}$ for all α, β, γ and $\{\dot{W}^\alpha(t,x)\}_{\alpha=1}^d$ are independent space-time Gaussian white noises. We introduce its approximation:

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - \xi^\varepsilon \delta^{\beta\gamma}) * \eta_2^\varepsilon + \dot{W}^{\varepsilon,\alpha}(t,x), \tag{11}$$

and study its invariant measures.

4 Regularity of solutions of linear SPDEs

Finally, we explain the regularity of the solutions of linear SPDEs, and see that it is determined under the balance between the regularizing effect of the differential operators and the roughness of the noise. Let us consider the linear SPDE on a d-dimensional torus for simplicity:

$$\partial_t u = -(-\Delta)^\alpha u + \dot{W}(t,x), \quad x \in \mathbb{T}^d = [0,1)^d, \tag{12}$$

with $\alpha \in \mathbb{R}$ and give a proof of

$$u(t,\cdot) \in \bigcap_{s<\alpha-\frac{d}{2}} H^s(\mathbb{T}^d), \text{ a.s.} \tag{13}$$

Let $\{\phi_k, \lambda_k\}_{k=1}^\infty$ be the eigenfunctions and corresponding eigenvalues of $-\Delta$ on $\mathbb{T}^d$ such that $\{\phi_k\}_{k=1}^\infty$ is a complete orthonormal system of $L^2(\mathbb{T}^d)$; note that $\{\lambda_k\}$ behaves as $\lambda_k \sim ck^{2/d}$ with $c > 0$ as $k \to \infty$. Then,

$$E[\|u(t)\|^2_{H^s(\mathbb{T}^d)}] = E[\|(1-\Delta)^{s/2}u(t)\|^2_{L^2(\mathbb{T}^d)}]$$

$$= \sum_{k=1}^{\infty} (1+\lambda_k)^s E[u_k(t)^2],$$

where $u_k(t) := (u(t), \phi_k)_{L^2(\mathbb{T}^d)}$. It satisfies the stochastic differential equation (SDE):

$$du_k(t) = -\lambda_k^\alpha u_k(t)dt + dw_k(t),$$

with independent Brownian motions $w_k(t) := (W(t), \phi_k)$. This SDE can be easily solved:

$$u_k(t) = e^{-\lambda_k^\alpha t} u_k(0) + \int_0^t e^{-\lambda_k^\alpha (t-s)} dw_k(s),$$

and we have that

$$\begin{aligned} E[u_k(t)^2] &= e^{-2\lambda_k^\alpha t} u_k^2(0) + E\left[\left(\int_0^t e^{-\lambda_k^\alpha (t-s)} dw_k(s) \right)^2 \right] \\ &= e^{-2\lambda_k^\alpha t} u_k^2(0) + \int_0^t e^{-2\lambda_k^\alpha (t-s)} ds \\ &= e^{-2\lambda_k^\alpha t} u_k^2(0) + \frac{1}{2\lambda_k^\alpha} \left(1 - e^{-2\lambda_k^\alpha t}\right). \end{aligned}$$

Thus, if $u(0) \in H^s(\mathbb{T}^d)$,

$$E[\|u(t)\|^2_{H^s(\mathbb{T}^d)}] \sim \sum_{k=1}^{\infty} \frac{(1+\lambda_k)^s}{2\lambda_k^\alpha} < \infty \Longleftrightarrow s < \alpha - \tfrac{d}{2}.$$

This proves (13). In particular, in case $\alpha = 1$ and $d = 1$, we see that $u(t,\cdot) \in \bigcap_{s<1/2} H^s(\mathbb{T})$ (a.s.), and this suggests the ill-posedness of the KPZ equation (1).

References

[1] T. FUNAKI AND J. QUASTEL, *KPZ equation, its renormalization and invariant measures*, preprint.

[2] M. Hairer, *Solving the KPZ equation*, Ann. Math, **178** (2013), 559–664.

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