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A study on the linear complementarity representation of piecewise linear functions

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Abstract: This paper describes investigations on a representation of piecewise linear functions, called the linear complementarity representation (the LCR for short). Especially, the paper focuses on the P-representation and the ULT-representation, which are special types of the LCR, and explains their properties. Moreover, the paper discusses the minimization problem of the LCR, and explains investigations on this issue.

1 Introduction

Piecewise linear function plays an important role as an approximation function in many fields such as nonlinear circuit [3, 6], nonlinear controll [5], mathematical programming [1], and many other engineering applications, for the reason of possessing an ability of uniformly approximating a continuous function defined on a compact domain and a property of linearity on a neighborhood of almost every point in the domain. However, it also presents some problems in practical use: improving an approximation in accuracy causes the exponential increase of the number of parameters; it is not so easy to treat the expression of piecewise linear function based on its definition (e.g., [3]). Therefore, much research has been done in an effort to develop new efficient representation [6]. This paper report our present research findings on the linear complementarity representation.

In Section 2, we explain the definition of piecewise linear function. In Section 3, we introduce the linear complementarity representation and its special types, called the P-representation and the ULT-representation, and describe their properties as well as the construction method of a ULT-representation for a given piecewise linear function. In Sections 4 and 5, we formulate the minimization problem on the linear complementarity representation, and explain our research findings on this issue [8].

Throughout this paper, m and n indicate positive integers. For a positive integer l, the set of integers from 1 to l is denoted by $[l]$, i.e., $[l] = \{1, 2, \ldots, l\}$. The inner product of two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is denoted by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$. "Linear" should be read as "affine linear" in this paper. A convex set $R \subset \mathbb{R}^n$ is called a polyhedron if it can be represented as the intersection of finitely many closed half-spaces in $\mathbb{R}^n$. By definition, $\emptyset$ and $\mathbb{R}^n$ are polyhedra.

2 Piecewise linear function

Definition 2.1. [9] A finite family $\mathcal{R}$ of polyhedra in $\mathbb{R}^n$ is called a *polyhderal partition* of $\mathbb{R}^n$ if it satisfies the following conditions:
(i) $\bigcup \mathcal{R} = \mathbb{R}^n$;
(ii) $\operatorname{int} P \neq \emptyset$ for all $P \in \mathcal{R}$;
(iii) For each $P, Q \in \mathcal{R}$, $P \neq Q$ implies $\operatorname{int} P \cap \operatorname{int} Q = \emptyset$,
where "int" denotes the topological interior of a set.

Definition 2.2. [7, 9] A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *piecewise linear* if it is continuous on $\mathbb{R}^n$ and there exists a polyhedral partition $\mathcal{R}$ of $\mathbb{R}^n$ such that f is linear on each region $R \in \mathcal{R}$. A linear function $g : \mathbb{R}^n \to \mathbb{R}^m$ which coincides with f on some $R \in \mathcal{R}$ is said to be a linear component of f. We denote by **PWL** the family of all piecewise linear functions.

3 The linear complementarity representation

3.1 Definition

Definition 3.1. [6] A correspondence f from $\boldsymbol{x} \in \mathbb{R}^n$ to $\boldsymbol{y} \in \mathbb{R}^m$ is called a *linear complementarity correspondence*, an LCC for short, if there exist a nonnegative integer k, matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$, $C \in \mathbb{R}^{k\times n}$, and $D \in \mathbb{R}^{k\times k}$, and vectors $\boldsymbol{g} \in \mathbb{R}^m$ and $\boldsymbol{h} \in \mathbb{R}^k$ such that

$$\boldsymbol{y} = A\boldsymbol{x} + B\boldsymbol{u} + \boldsymbol{g}, \tag{1}$$

$$\boldsymbol{j} = C\boldsymbol{x} + D\boldsymbol{u} + \boldsymbol{h}, \tag{2}$$

$$\boldsymbol{u}, \boldsymbol{j} \geq \boldsymbol{0}, \ \langle \boldsymbol{u}, \boldsymbol{j} \rangle = 0. \tag{3}$$

The vectors $\boldsymbol{u}$ and $\boldsymbol{j}$ satisfying the equation (3) are called *complementarity vectors*, and the equations (1)–(3) are collectively called a *linear complementarity representation*. By convention, we abbreviate the representation (1)–(3) as $(A, B, \boldsymbol{g}; C, D, \boldsymbol{h})$.

Remark 3.1. Every linear function $A\boldsymbol{x}+\boldsymbol{g}$ has a representation $(A, O, \boldsymbol{g}; \boldsymbol{0}, 1, 0)$, where $A \in \mathbb{R}^{m\times n}$ and $\boldsymbol{g} \in \mathbb{R}^m$. By convention for the discussion of this paper, we assume that each linear function has a zero-dimensional representation $(A; \boldsymbol{g})$ instead of the above one-dimensional representation $(A, O, \boldsymbol{g}; \boldsymbol{0}, 1, 0)$. See Definition 4.1 in Subsection 4.2, for the definition of the dimension of a representation.

Remark 3.2. Calculation of a correspondence value $\boldsymbol{y}$ for each $\boldsymbol{x}$ is reduced to solving the linear complementarity problem (LCP for short) of the form $(D, \boldsymbol{q}(\boldsymbol{x}))$, where $\boldsymbol{q}(\boldsymbol{x}) = C\boldsymbol{x} + \boldsymbol{h}$. This problem is called the derived LCP from the triplet $(C, D, \boldsymbol{h})$. See Definition A.1 in Appendix A, for the definition of the LCP.

3.2 P-representation and ULT-representation

Definition 3.2. [2, 6] (a) *P-representation* is a linear complementarity representation whose coefficient D in (2) is a P-matrix (see Definition A.2.(i)). The family of LCCs having a P-representation is called *Class P*, and denoted by $\mathbf{P}$.
(b) *ULT-representation* is a linear complementarity representation whose coefficient D in (2) is a ULT-matrix (see Definition A.2.(ii)). The family of LCCs having a ULT-representation is called *Class ULT*, and denoted by $\mathbf{ULT}$.

Remark 3.3. By the definitions of P-matrix and ULT-matrix that $\mathbf{P} \supset \mathbf{ULT}$ holds. Though an LCC is, in general, a multi-valued function, Proposition A.1 in Appendix A states that every LCC in $\mathbf{P}$ becomes a single-valued function. As mentioned in Remark 3.1 that every linear function has a representation $(A, O, \boldsymbol{g}; \mathbf{0}, 1, 0)$. This is, in fact, a ULT-representation. Thus, every linear function belongs to both $\mathbf{P}$ and $\mathbf{ULT}$.

The next theorem, Theorem 3.1, indicates that $\mathbf{ULT}$ is closed under the operations of max and min compositions, and direct sum. In addition to this, the closedness is also true for the operations of composition and linear combination [9]. This theorem yields an operation formulae between two representations, and moreover, plays an important role in the construction of a ULT-representation for a given piecewise linear function. This theorem is also true for $\mathbf{P}$.

Theorem 3.1. [9] (i) If a function $f : \mathbb{R}^n \to \mathbb{R}$ has a ULT-representation, say $(A, B, g; C, D, h)$, and a function $f' : \mathbb{R}^n \to \mathbb{R}$ has a ULT-representation, say $(A', B', g'; C', D', h')$, then their max $f \vee f'$ and min $f \wedge f'$ have the ULT-representations:

$$f \vee f' : \left(A', \begin{pmatrix} O & B' & 1 \end{pmatrix}, g'; \begin{pmatrix} C \\ C' \\ A' - A \end{pmatrix}, \begin{pmatrix} D & O & \mathbf{0} \\ O & D' & \mathbf{0} \\ -B & B' & 1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{h}' \\ g' - g \end{pmatrix} \right),$$

$$f \wedge f' : \left(A, \begin{pmatrix} B & O & -1 \end{pmatrix}, g; \begin{pmatrix} C \\ C' \\ A' - A \end{pmatrix}, \begin{pmatrix} D & O & \mathbf{0} \\ O & D' & \mathbf{0} \\ -B & B' & 1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{h}' \\ g' - g \end{pmatrix} \right).$$

(ii) If a function $f : \mathbb{R}^n \to \mathbb{R}^m$ has a ULT-representation, say $(A, B, \boldsymbol{g}; C, D, \boldsymbol{h})$, and a function $f' : \mathbb{R}^n \to \mathbb{R}^{m'}$ has a ULT-representation, say $(A', B', \boldsymbol{g}'; C', D', \boldsymbol{h}')$, then their direct sum $f'' = f \oplus f' : \mathbb{R}^n \to \mathbb{R}^{m+m'}$ has the ULT-representation:

$$\left(\begin{pmatrix} A \\ A' \end{pmatrix}, \begin{pmatrix} B & O \\ O & B' \end{pmatrix}, \begin{pmatrix} g \\ g' \end{pmatrix}; \begin{pmatrix} C \\ C' \end{pmatrix}, \begin{pmatrix} D & O \\ O & D' \end{pmatrix}, \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{h}' \end{pmatrix} \right).$$

The next theorem states that P-representation and ULT-representation, individually, characterize all piecewise linear functions.

Theorem 3.2. [9] The family of all piecewise linear functions coinsides with Classes P and ULT, that is, $\mathbf{ULT} = \mathbf{P} = \mathbf{PWL}$.

3.3 Construction of a ULT-representation

By the following procedure, a ULT-representation of each scalar-valued piecewise linear function can be obtained by means of Theorem 3.1 and the below mentioned Theorem 3.3:
(i) Construct a max-min polynomial for a given piecewise linear function;
(ii) Transform each max and min operators to a ULT-representation by means of Theorem 3.1.(i), in recursively.

Theorem 3.3. [7] Every piecewise linear function $f : \mathbb{R}^n \to \mathbb{R}$ has a formula $f = \bigvee_{j \in J} \bigwedge_{i \in S_j} g_i$, where $\{g_i\}_{i \in I}$ is the finite family of all linear components of f, S_j is a subset of I, and J is a finite index set. This formular is called a max-min polynomial in the variables g_i.

Remark 3.4. Theorem 3.3 is also valid for the vector-valued piecewise linear function [7]. Therefore, by using also Theorem 3.1.(ii), we can construct a ULT-representation for every vector-valued piecewise linear function.

Example 3.1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ as defined in Figure 1.

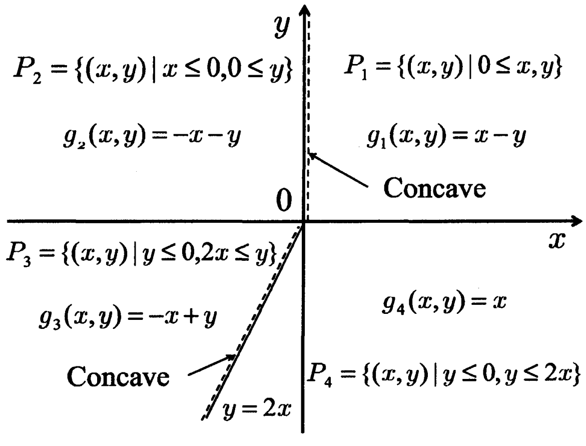

Figure 1: Two-variable piecewise linear function f

(i) The first step is the construction of a max-min polynomial of f. The procedure consists of the identifications of J and $\{S_j\}_{j \in J}$ in Theorem 3.3. The identification of J corresponds to the polyhedral partition of the domain. Partitioning of the domain is carried out at the place where f

becomes concave (see Figure 1). In this case, $P_1 \cup P_4$ corresponds to $j = 1$, and $P_2 \cup P_3$ corresponds to $j = 2$. Next, we determine S_j. S_j is the index set of linear component whose value is greater than or equal to f on a corresponding region. On $P_1 \cup P_4$, g_1 and g_4 are satisfied, and on $P_2 \cup P_3$, g_2 and g_3 are satisfied. Thus, we have $S_1 = \{1, 4\}$, $S_2 = \{2, 3\}$, and hence, we obtain the formula $f = (g_1 \wedge g_4) \vee (g_2 \wedge g_3)$.

(ii) The second step is the transformation to a ULT-representation. We begin with zero-dimensional ULT-representations of linear components of f, and transform each max and min operations to ULT-representations. Since g_1 and g_4 have ULT-representations $((1 \;\; -1); 0)$ and $((1 \;\; 0); 0)$, respectively (see Remark 3.1), a ULT-representation of $g_1 \wedge g_4$ is obtained as follows:

$$((1 \;\; -1), -1, 0; (0 \;\; 1), 1, 0).$$

Similarly, we have a ULT-representation of $g_2 \wedge g_3$ as:

$$((-1 \;\; -1), -1, 0; (0 \;\; 2), 1, 0).$$

Therefore, we obtain a ULT-representation $\mathcal{S} = (A, B, g; C, D, h)$ of $f = (g_1 \wedge g_4) \vee (g_2 \wedge g_3)$ as follows:

$$A = \begin{pmatrix} -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix}, \quad g = 0,$$
$$C = \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ -2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

4 Minimization of the linear complementarity representation

4.1 Problem institution

Consider again the ULT-representation $\mathcal{S}$ obtained in Example 3.1. Then, we can easily verify the existence of the relation $u_2(x, y) = 2u_1(x, y)$ $(\forall (x, y) \in \mathbb{R}^2)$ between u_1 and u_2 with a simple calculation. Thus, by eliminating the component u_2, the representation $\mathcal{S}$ results in the following representation $(A, B', g; C', D', h')$:

$$A = \begin{pmatrix} -1 & -1 \end{pmatrix}, \quad B' = \begin{pmatrix} -2 & 1 \end{pmatrix}, \quad g = 0,$$
$$C' = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \quad D' = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad h' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As in this case, the resulting representation, in generally, involves some redundancies. In practically, it is desirable that the obtained representation does not contain any redundancies. Then, how can we verify a given representation to be a minimum representation? In Subsection 4.2, we formulate the minimization of the representation in rigorously, and in Section 5, we will discuss it.

4.2 Problem formulation

Let m and n be arbitrary positive integers. For a positive integer k, we define the families of triplets $\mathbb{A}^k$ and $\mathbb{C}^k$ as follows:

$$\mathbb{A}^k = \{(A, B, \boldsymbol{g}) \mid A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{m\times k},\ \boldsymbol{g} \in \mathbb{R}^m\},$$
$$\mathbb{C}^k = \{(C, D, \boldsymbol{h}) \mid C \in \mathbb{R}^{k\times n},\ D \in \mathbb{R}^{k\times k},\ \boldsymbol{h} \in \mathbb{R}^k\}.$$

The family of all linear complementarity representations with k-dimensional complementarity vectors is denoted by $\mathbb{S}^k \triangleq \mathbb{A}^k \times \mathbb{C}^k$. By convention, we denote by $\mathbb{S}^0 = \{(A, \boldsymbol{g}) \mid A \in \mathbb{R}^{m\times n},\ \boldsymbol{g} \in \mathbb{R}^m\}$ the family of all representations of linear functions. Then, $\mathbb{S} \triangleq \bigcup_{k\geq 0} \mathbb{S}^k$ expresses the family of all linear complementarity representations. Similarly, when we denote by $\mathbb{S}^k_{\mathrm{ULT}}$ the family of all ULT-representations with the k-dimensional complementarity vectors, $\mathbb{S}_{\mathrm{ULT}} = \bigcup_{k\geq 0} \mathbb{S}^k_{\mathrm{ULT}}$ expresses the family of all ULT-representations. Note that $\mathbb{S}^0_{\mathrm{ULT}} = \mathbb{S}^0$. The "dimension" of a representation is defined as follows.

Definition 4.1. Let $\mathcal{S} \in \mathbb{S}$ be given, and let k be a nonnegative integer. We say $\mathcal{S}$ is *k-dimensional* if $\mathcal{S} \in \mathbb{S}^k$, denoted by $\dim(\mathcal{S})$.

Let f be an LCC. Then we denote by $\mathbb{S}(f)$ the family of all representations that characterize f. Similarly, we denote by $\mathbb{S}_{\mathrm{ULT}}(f)$ the family of all ULT-representations of f. The minimization problem is formulated in the following.

Definition 4.2. Let $\mathcal{S} \in \mathbb{S}(f)$. Then $\mathcal{S}$ is called a *minimum dimensional representation* (a minimum representation for short) of f if $\dim(\mathcal{S}) \leq \dim(\mathcal{T})$ for all $\mathcal{T} \in \mathbb{S}(f)$.

Problem 4.1. The *minimization problem* with respect to f consists of the following two requirements: For a given representation $\mathcal{S} \in \mathbb{S}(f)$,

(a) verify whether or not $\mathcal{S}$ is a minimum representation of f;

(b) find a minimum representation of f, when $\mathcal{S}$ is not minimum.

In the same manner, we can define the concept of minimum dimensional ULT-representation, and formulate the ULT-minimization problem: similarly, we can also formulate the P-minimization problem.

Definition 4.3. Let $\mathcal{S} \in \mathbb{S}_{\mathrm{ULT}}(f)$. Then $\mathcal{S}$ is called a *minimum dimensional ULT-representation* (a minimum ULT-representation for short) of f if $\dim(\mathcal{S}) \leq \dim(\mathcal{T})$ for all $\mathcal{T} \in \mathbb{S}_{\mathrm{ULT}}(f)$.

Problem 4.2. The *ULT-minimization problem* with respect to f consists of the following two requirements: For a given representation $\mathcal{S} \in \mathbb{S}_{\mathrm{ULT}}(f)$,

(a) verify whether or not $\mathcal{S}$ is a minimum ULT-representation of f;

(b) find a minimum ULT-representation of f, when $\mathcal{S}$ is not minimum.

5 Considerations on the minimum dimensionality

In this section, we will discuss how we verify the minimal dimensionality for a given representation. The key to our approach is that finding a minimum representation will be achieved by eliminating all redundancies of a given representation. In our present research, we have considered the ULT-representation only.

5.1 Redundancies of the complementatity vectors

We begin with the following three examples to discuss the reducibility. Each examples demonstrate different type of redundancies from one another.

Example 5.1. Let $\mathcal{S}_1 = (A_1, B_1, g_1; C_1, D_1, h_1)$ be the ULT-representation given by the following:

$$A_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}, \; g_1 = 0, \; C_1 = \begin{pmatrix} -3 & -6 \\ -4 & -8 \\ 4 & 8 \\ 6 & 12 \end{pmatrix}, \; D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \; \boldsymbol{h}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then, we can easily verify that there exist the following relations among the components of the complementarity vectors $u_i(\boldsymbol{x})$ $(i = 1, 2, 3, 4)$:

$$u_1(\boldsymbol{x}) = 3u_2(\boldsymbol{x}), \; u_3(\boldsymbol{x}) = 2u_4(\boldsymbol{x}), \; u_4(\boldsymbol{x}) = -2x_1 - 4x_2 + 2u_2(\boldsymbol{x}).$$

This would imply that the valiables $u_1(\boldsymbol{x})$, $u_3(\boldsymbol{x})$, and $u_4(\boldsymbol{x})$ are omitted from $\mathcal{S}_1$. Indeed, we can omit them from $\mathcal{S}_1$, and hence we find that $\mathcal{S}_1$ reduces to the following ULT-representation $\mathcal{S}_1' = (A_1', B_1', g_1'; C_1', D_1', h_1')$:

$$A_1' = \begin{pmatrix} -1 & -3 \end{pmatrix}, \; B_1' = 3, \; g_1' = 0, \; C_1' = \begin{pmatrix} 1 & 2 \end{pmatrix}, \; D_1' = 1, \; h_1' = 0.$$

Example 5.2. Let $\mathcal{S}_2 = (A_2, B_2, g_2; C_2, D_2, \boldsymbol{h}_2)$ be the ULT-representation given by the following:

$$A_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \; g_2 = 0, \; C_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \; D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \boldsymbol{h}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then, in this case, there is no relation between $u_1(\boldsymbol{x})$ and $u_2(\boldsymbol{x})$ as in Example 5.1. However, since the variable $u_2(\boldsymbol{x})$ is independently obtained from $u_1(\boldsymbol{x})$ and the variable $u_1(\boldsymbol{x})$ is vanished from the first formula of the original representation, we can omit $u_1(\boldsymbol{x})$ from $\mathcal{S}_2$. Thus $\mathcal{S}_2$ reduces to the following one-dimensional ULT-representation $\mathcal{S}_2' = (A_2', B_2', g_2'; C_2', D_2', \boldsymbol{h}_2')$:

$$A_2' = \begin{pmatrix} 1 & 1 \end{pmatrix}, \; B_2' = 1, \; g_2' = 0, \; C_2' = \begin{pmatrix} 2 & 1 \end{pmatrix}, \; D_2' = 1, \; h_2' = 0.$$

Example 5.3. Let $\mathcal{S}_3 = (\mathcal{A}_3, \mathcal{C}_3) \in \mathbb{S}^k_{\mathrm{ULT}}$ be given. Suppose there exist a positive integer $k' < k$, a triplet $\mathcal{C}'_3 \in \mathbb{C}^{k'}_{\mathrm{ULT}}$, and a matrix $E \in \mathbb{R}^{k \times k'}$ such that the solution $\boldsymbol{u}(\boldsymbol{x})$ to the derived LCP from $\mathcal{C}_3$ can be expressed as $\boldsymbol{u}(\boldsymbol{x}) = E\boldsymbol{u}'(\boldsymbol{x})$, where $\boldsymbol{u}'(\boldsymbol{x})$ is the solution to the derived LCP from $\mathcal{C}'_3$. Then $\mathcal{S}_3$ reduces to the ULT-representation $\mathcal{S}'_3 = (\mathcal{A}'_3, \mathcal{C}'_3) \in \mathbb{S}^{k'}_{\mathrm{ULT}}$, where $\mathcal{A}'_3 = (A_3, B_3E, \boldsymbol{g}_3)$ for $\mathcal{A}_3 = (A_3, B_3, \boldsymbol{g}_3)$.

As demonstrated above, there exist at least three types of redundancies of the complementarity vectors: (i) dependency among the components of the complementarity vectors (Ex.5.1), (ii) the erasability of the components of the complementarity vectors from the first formula caused by some columns of B being zero (Ex.5.2), (iii) representability of the original complementarity vectors by means of some lower-dimensional representation (Ex.5.3). Clearly, the minimum dimensionality requires the absence of redundancies of the complementarity vectors. We therefore conclude that the problem of finding a minimum dimensional representation results in the problem of eliminating redundant components of the complementarity vectors. We conjecture that the redundancies would be covered by the above mentioned three types. However, it has not been proven yet. This is a future work. So far, we have investigated the redundancies of (i) and (iii), and found that the redundancy of (i) is equivalent to the generalization of (iii), called the ULT-reducibility. In the next subsection, we will explain about this investigation.

5.2 ULT-reducibility

Firstly, we define the ULT-reducibility that is a generalization of the redundancy (iii).

Definition 5.1. Two representations $\mathcal{S}, \mathcal{T} \in \mathbb{S}$ are said to be *equivalent* to each other, denoted by $\mathcal{S} \cong \mathcal{T}$, if there exists an LCC f such that $\mathcal{S}, \mathcal{T} \in \mathbb{S}(f)$.

Definition 5.2. Let $\mathcal{C} \in \mathbb{C}^k_{\mathrm{ULT}}$. Then $\mathcal{C}$ is said to be *ULT-reducible* if there exist a nonnegative integer $k' < k$, and a triplet $\mathcal{C}' \in \mathbb{C}^{k'}_{\mathrm{ULT}}$ such that every representation that contains $\mathcal{C}$ is equivarlent to a ULT-representation that contains $\mathcal{C}'$ [i.e., for every $\mathcal{A} \in \mathbb{A}^k$, there exists $\mathcal{A}' \in \mathbb{A}^{k'}$ such that $(\mathcal{A}, \mathcal{C}) \cong (\mathcal{A}', \mathcal{C}')$]. If not, it is said to be *ULT-irreducible.*

$\mathcal{C}_3$ in Example 5.3 is ULT-reducible. Moreover, by Theorem 5.1 below, $\mathcal{C}_1$ in Example 5.1 is also ULT-reducible. On the other hand, $\mathcal{C}_2$ in Example 5.2 is ULT-irreducible.

Proposition 5.1 is an immediate concequence of Definition 4.3 and Definition 5.2. This implies that ULT-irreducibility of $\mathcal{C}$ is necessary for a given representation to be minimum dimensional. Example 5.2 is a counterexample for the sufficiency.

Proposition 5.1. *If $\mathcal{S} = (\mathcal{A}, \mathcal{C}) \in \mathbb{S}_{\mathrm{ULT}}(f)$ is a minimum dimensional representation of f, then $\mathcal{C}$ is ULT-irreducible.*

The following Theorem 5.1 shows that the redundancy of (i) and ULT-reducibility of $\mathcal{C}$ is equivalent. The condition (S) in Theorem 5.1 expresses a dependency among the components of the complementarity vectors.

Theorem 5.1. *Let k be a positive integer. Then $\mathcal{C} \in \mathbb{C}^k_{\mathrm{ULT}}$ is ULT-reducible if and only if the solution $\boldsymbol{u}(\boldsymbol{x})$ to the derived LCP from $\mathcal{C}$ satisfies the following condition:*
(S) For some $p = 1, 2, \ldots, k$, there exist $\{\lambda_i\}_{i<p} \subset \mathbb{R}$ and a linear function $l_p : \mathbb{R}^n \to \mathbb{R}$ such that

$$u_p(\boldsymbol{x}) = \sum_{i<p} \lambda_i u_i(\boldsymbol{x}) + l_p(\boldsymbol{x}) \quad (\forall \boldsymbol{x} \in \mathbb{R}^n).$$

6 Conclusion

In this paper, we introduced the linear complementarity representation of piecewise linear function and its special types, called the P-representation and the ULT-representation, and explained their fundamental properties as well as the construction method of a ULT-representation for a given piecewise linear function. Moreover, we formulated the minimization problem concerning to the linear complementarity representation, and mentioned our investigation obtained so far on this issue. It is a future work to clarify the relation between the minimum dimensionality and the redundancies discussed in Subsection 5.1.

A The linear complementarity problem

Let k be a positive integer, and let a matrix $D \in \mathbb{R}^{k\times k}$ and a vector $\boldsymbol{q} \in \mathbb{R}^k$ be given.

Definition A.1. [4] The *linear complementarity problem*, LCP for short, is to find a pair of vectors $\boldsymbol{u}, \boldsymbol{j} \in \mathbb{R}^k$ such that

$$\boldsymbol{j} = D\boldsymbol{u} + \boldsymbol{q}, \tag{4}$$

$$\boldsymbol{u}, \boldsymbol{j} \geq \boldsymbol{0}, \ \langle \boldsymbol{u}, \boldsymbol{j} \rangle = 0 \tag{5}$$

or to show that no such pair exists. We denote the above problem by the pair $(D, \boldsymbol{q})$. A pair $(\boldsymbol{u}, \boldsymbol{j})$ satisfying (5) is said to be complementary, and the one satisfying (4) and (5) is called a solution to the LCP $(D, \boldsymbol{q})$.

Definition A.2. (i) [4] *P-matrix* is a square matrix whose principal minors are all positive.
(ii) [6] *Unit lower triangular matrix*, ULT-matrix for short, is a lower triangular matrix whose diagonal elements are all one's.

A principal minor is the determinant of a principal sub-matrix of D. A principal sub-matrix is defined as $(d_{ij})_{i,j\in I}$, where $\emptyset \neq I \subset [k]$, for the original matrix $D = (d_{ij})_{i,j\in[k]}$. By definition, every ULT-matrix is a P-matrix.

In general, the LCP does not necessarily have a solution. Even if it has a solution, generally it is not necessarily unique. However, Proposition A.1 below claims that a P-matrix guarantees the uniqueness of solution.

Proposition A.1. [4] A matrix $D \in \mathbb{R}^{k\times k}$ is a P-matrix if and only if the LCP $(D, \boldsymbol{q})$ has a unique solution for every $\boldsymbol{q} \in \mathbb{R}^k$.

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