数理解析研究所講究録
第 1906 巻 2014 年 155-161

Valuations on distributive lattices

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ABSTRACT. This article summarizes the problems of interest and the results presented by the authors at the workshop. The full article with their complete proofs will be made available under the same title by the authors at the appropriate juncture.

1. DOMAINS AND TOPOLOGIES

Let L be a partially ordered set (poset) equipped with partial order $\leq$. A subset E of L is called *directed* if it is nonempty and every pair of elements from E has an upper bound in E, and L is called a *directed complete poset* (or, *dcpo* for short) if $\sup E$ exists for any directed subset E. A subset E is called a *lower* set if $x \in E$ and $y \leq x$ imply $y \in E$, and a lower set E is called an *ideal* if it is also directed. For example, $\langle x\rangle := \{z : z \leq x\}$ is an ideal, called *principal* ideal. An element x of L is said to be "way below" y, denoted by $x \ll y$, if we can find $x \leq w$ for some $w \in E$ whenever a directed set $E \subseteq L$ satisfies $y \leq \sup E$. An element x is called *isolated from below* if $x \ll x$. A dcpo L is called a *domain* if (i) $\langle\!\langle x\rangle\!\rangle := \{z : z \ll x\}$ is an ideal and (ii) it satisfies $x = \sup\langle\!\langle x\rangle\!\rangle$ for any $x \in L$. Every domain possesses the following property, called the *strong interpolation property*: If $x \ll z$ and $z \neq x$ then there exists some $y \neq x$ interpolating $x \ll y \ll z$.

A poset is said to be a *semilattice* if $x \wedge y := \inf\{x, y\}$, called *meet*, exists for every pair $\{x, y\}$. Similarly we can define a *sup-semilattice* if $x \vee y := \sup\{x, y\}$, called *join*, exists for any $\{x, y\}$. A poset is called a *lattice* if the meet and the join exist for every pair, and it is said to be a *complete lattice* if the supremum and the infimum exist for every subset. A domain is called a *continuous lattice* if it is a complete lattice. Throughout this paper we frequently refer to [6] for their extensive treatise of continuous lattices and domains. The notable exception is our choice of notation $\langle\cdot\rangle$ and $\langle\!\langle\cdot\rangle\!\rangle$ for generators (which are $\downarrow\cdot$ and $\twoheaddownarrow\cdot$ in [6]). A poset with converse (or "dual") order relation $\leq^*$ is called *dual*, denoted by L^*. A subset is called *filtered* in L if it is directed in L^*. An *upper* set of L is dually defined as a lower set of L^*, and a filtered upper set is simply called a *filter*. In an analogous manner to $\langle x\rangle$ and $\langle\!\langle x\rangle\!\rangle$, we write $\langle x\rangle^* := \{z : x \leq z\}$ and $\langle\!\langle x\rangle\!\rangle^* := \{z : x \ll z\}$ (which are respectively expressed as $\uparrow x$ and $\twoheaduparrow x$ in [6]). For a subset A we can write $\langle A\rangle := \{z : z \leq x \text{ for some } x \in A\}$ and $\langle\!\langle A\rangle\!\rangle := \{z : z \ll x \text{ for some } x \in A\}$. Again, analogously we can introduce $\langle A\rangle^*$ and $\langle\!\langle A\rangle\!\rangle^*$.

We will call a domain L *continuous sup-semilattice* if it is a sup-semilattice. If a sup-semilattice L is a dcpo then it suffices to check $x = \sup\langle\!\langle x\rangle\!\rangle$ in order to see whether it is a domain, or equivalently, to find some $z \ll x$ with $z \not\leq y$ whenever $x \not\leq y$. A continuous sup-semilattice L is unital, containing the top element $\hat{1} := \sup L$. If it also has the bottom element $\inf L$ then it becomes a continuous lattice. Regardless of whether there exists a bottom element or not, we can always form a continuous lattice, denoted by $\check{L} := L \cup \{\hat{0}\}$, by adjoining a bottom element $\hat{0}$. The extended interval $(-\infty, +\infty]$ is a continuous sup-semilattice without the bottom element, and $x < y$ is equivalent to $x \ll y$. The top element $+\infty$ is not isolated from below.

Suppose that L is a continuous sup-semilattice. Then we can introduce a *Scott open* set U if (i) it is an upper set and (ii) $U \cap E \neq \varnothing$ holds whenever a directed subset E satisfies $\sup E \in U$. By Scott(L) we denote the family of Scott open subsets in L. The poset Scott(L) ordered by inclusion is a continuous lattice, in which $U \ll V$ if $U \subseteq \langle A\rangle^*$ holds for some finite subset A of V. By $\mathcal{S} :=$ Scott$(L) \setminus \{\varnothing\}$ we denote the collection of nonempty Scott open subsets in L. Then $\mathcal{S}$ is a continuous sup-semilattice ordered by inclusion. We view Scott(L) as the continuous lattice by adjoining the bottom element $\varnothing$ to $\mathcal{S}$, and denote it by $\check{\mathcal{S}}$. If $\hat{1}$ is isolated then $\mathcal{S}$ itself becomes a continuous lattice with the bottom element $\{\hat{1}\}$. For $x \in L$ we can define a filter $\mathcal{S}_x := \{U \in \mathcal{S} : x \in U\}$. Given a directed subset $\mathcal{E}$ of $\mathcal{S}$ satisfying $\sup \mathcal{E} \in \mathcal{S}_x$, we can find some $U \in \mathcal{E}$ which contains x so that $U \in \mathcal{S}_x$; thus, $\mathcal{S}_x$ is Scott-open. In fact, the collection $\mathcal{S}_x$, $x \in L$, becomes an open subbase for the Scott topology on the domain $\mathcal{S}$.

The Scott topology is not a Hausdorff space; thus, a refinement can be made by introducing a closed upper set $\langle x\rangle^*$, called the *Lawson topology.* A subbase of the topology can be formed by all the Scott open subsets U and all the lower subsets of the form $L \setminus \langle x\rangle^*$. The Lawson topology of the domain $(-\infty, +\infty]$, for example, is homeomorphic to the metric space $(0, 1]$. In general, equipped with the Lawson topology, a continuous sup-semilattice L is locally compact Hausdorff (LCH), and $\check{L}$ is the one-point compactification of L. (cf. Theorem III-1.9 of [6]).

A map from L to another domain is called *Scott-continuous* if it is continuous under their respective Scott topologies. The Scott-continuity implies monotonicity, and it has the following equivalent condition: f is Scott-continuous if and only if $f(x) = \sup f(\langle\!\langle x\rangle\!\rangle)$ for every $x \in L$ (Proposition II-2.1 of [6]). Equipped with the Lawson topology on the respective domains, a continuous map is called *Lawson-continuous.* In continuous lattices, a net $\{x_\alpha\}$ is said to be *liminf-converge* to x if $x = \liminf_{\alpha'} x_{\alpha'}$ for every subnet $\{x_{\alpha'}\}$, which agrees with convergence in the Lawson topology (Theorem III-3.17 of [6]).

A real-valued function φ on a domain L is Scott-continuous if and only if it is monotone and lower semicontinuous (l.s.c.) with respect to the Lawson topology. A Scott-continuous nonnegative function φ will be called *s-monotone* in this paper. Furthermore, we assume in the rest of paper that L is a continuous sup-semilattice which is also a semilattice (hence a lattice), and that the top element $\hat{1}$ is isolated from below. Thus, $\mathcal{S}$ is a continuous lattice with the bottom element $\{\hat{1}\}$, and it is viewed as a compact Hausdorff space equipped with the Lawson topology.

Example 1.1. The collection $\mathcal{K}$ of compact sets on an LCH space R can be viewed as a dcpo with reverse inclusion, and a lattice with the top element $\varnothing$. It has the bottom element, namely the entire space R, only when it is compact. We can show that $E \ll F$ in $\mathcal{K}$ if $F \subseteq \mathrm{int}(E)$ (cf. Proposition I-1.24.2 of [6]). The lattice $\mathcal{K}$ is a continuous sup-semilattice, and the top element $\varnothing$ is isolated from below. A set function φ over $\mathcal{K}$ is said to be *continuous on the right* if for any $E \in \mathcal{K}$ and $\varepsilon > 0$ there is an open neighborhood G of E such that $|\varphi(E) - \varphi(F)| < \varepsilon$ for all $E \subseteq F \subseteq G$. If φ is nonnegative, decreasing [i.e., monotone in the lattice $\mathcal{K}$] and continuous on the right, then φ is s-monotone (cf. Theorem 4.6.18 in Berg et al [1]).

2. Completely s-monotone functions

The collection $\mathrm{OFilt}(L)$ of Scott open filters becomes a base for the Scott topology on L. The semilattice $\mathrm{OFilt}(L)$ ordered by inclusion is denoted by $\mathcal{F}$, and it has the bottom element $\{\hat{1}\}$. A continuous semilattice is called *multiplicative* if $a \wedge b \ll x \wedge y$ holds whenever $a \ll x$ and $b \ll y$. In this section and again in Section 4 we will assume that L is multiplicative. Then $\mathcal{F}$ becomes a complete lattice (see [8]).

Since $\mathcal{F}$ is a Scott open base for L, we can express a principal filter $\langle U \rangle_{\mathcal{S}}^* := \{W \in \mathcal{S} : U \subseteq W\}$ on $\mathcal{S}$ by the intersection of $\langle V \rangle_{\mathcal{S}}^*$, $V \in \mathcal{F}$, satisfying $V \subseteq U$ if $U \in \mathcal{S}$. Thus, we can find that $\mathcal{S}_x$, $x \in L$, and $\mathcal{S} \setminus \langle V \rangle_{\mathcal{S}}^*$, $V \in \mathcal{F}$, form an open subbase for the Lawson topology on $\mathcal{S}$.

Lemma 2.1. *$\mathcal{F}$ is a compact subset of $\mathcal{S}$.*

Let F be a semilattice, and let ϕ be a nonnegative function on F. Then we can introduce a *difference operator* ∇_z by $\nabla_z \phi(x) = \phi(x) - \phi(x \wedge z)$, and the *successive difference operator* $\nabla_{z_1,\ldots,z_n}$ recursively by $\nabla_{z_1,\ldots,z_n}\phi = \nabla_{z_n}(\nabla_{z_1,\ldots,z_{n-1}}\phi)$ for $n = 2, 3, \ldots$. The operator $\nabla_{z_1,\ldots,z_n}$ does not depend on an order of z_i's, nor a repetition of elements, and therefore, it is denoted by ∇_A for a finite subset $A = \{z_1, \ldots, z_n\}$. The function ϕ is called *completely monotone* if $\nabla_A \phi \geq 0$ holds for every nonempty finite subset A.

Proposition 2.2. *Suppose that m is a finite measure on F, and that $\langle x \rangle$ is measurable for each $x \in F$. Then $\phi(x) = m(\langle x \rangle)$ is completely monotone.*

The converse of Proposition 2.2 is also true if F is finite. We say "x covers z" in F if $z < x$ and there is no other element between z and x.

We set $r(x) = \phi(x)$ for the bottom element $x = \bigwedge F$, and $r(x) = \nabla_A \phi(x)$ with the collection A of all the elements covered by x if x is not the bottom element of F. Thus, we can construct a signed measure $m(E) = \sum_{x \in E} r(x)$ for each $E \subseteq F$. It is a straightforward exercise to show $\phi(x) = m(\langle x \rangle)$ for every $x \in F$. In fact, such a measure m corresponds uniquely to ϕ, and r is known as the *Möbius inverse* of ϕ. Particularly, r is nonnegative (i.e., m is a measure on F) if ϕ is completely monotone.

Let φ be an s-monotone function on L. Then φ is called *completely s-monotone* if it is also completely monotone.

Proposition 2.3. *A completely s-monotone φ has a unique representation μ on $\mathcal{F}$.*

3. Completely s-alternating functions

Recall that $\check{L} = L \cup \{\hat{0}\}$ is the one-point compactification of L, and $\check{\mathcal{S}} = \mathrm{Scott}(L)$ is the continuous lattice, having the respective bottom element $\hat{0}$ and $\varnothing$. Here we introduce the subspace $\check{\mathcal{L}} := \{\check{L} \setminus \langle z \rangle : z \in \check{L}\}$ of the compact Hausdorff space $\check{\mathcal{S}}$. An open subbase of $\check{\mathcal{L}}$ consists of $\check{\mathcal{L}}_x$, $x \in L$, and $\check{\mathcal{L}} \setminus \langle U \rangle^*_{\check{\mathcal{S}}}$, $U \in \mathcal{S}$, where $\langle U \rangle^*_{\check{\mathcal{S}}} := \{W \in \check{\mathcal{S}} : U \subseteq W\}$ is a closed upper set of $\check{\mathcal{S}}$.

Proposition 3.1. *The bijective map $\xi(x) = \check{L} \setminus \langle x \rangle$ is homeomorphic from $\check{L}$ to $\check{\mathcal{L}}$.*

We define the operator ∇_A on the "dual" poset of a sup-semilattice, and call it the *dual* successive difference operator, denoted by Δ_A. It can be constructed with the dual difference operator $\Delta_{z_1}\phi(x) = \phi(x) - \phi(x \vee z_1)$, and recursively by $\Delta_{z_1,\dots,z_n}\phi = \Delta_{z_n}(\Delta_{z_1,\dots,z_{n-1}}\phi)$ for $n = 2, 3, \dots$. A function ϕ is said to be *completely alternating* if $\Delta_A \phi \le 0$ holds for any nonempty finite subset A.

Let φ be a monotone nonnegative function on L. Then we can naturally extend it to $\check{\varphi}$ on $\check{L}$ by setting $\check{\varphi}(\hat{0}) = 0$, and correspond it to the *dual* monotone function $\varphi^*(x) = \check{\varphi}(\hat{1}) - \check{\varphi}(x)$ on the dual poset $\check{L}^*$. The completely alternating property of φ is dually characterized by the complete monotonicity of φ^*.

An s-monotone function φ is called *completely s-alternating* if it is completely alternating.

Let $\mathcal{L} := \check{\mathcal{L}} \setminus \{\varnothing\}$. By Proposition 3.1 we can observe that $\check{\mathcal{L}}$ is compact on $\check{\mathcal{S}}$, and therefore, that $\mathcal{L}$ is closed in $\mathcal{S}$. Thus, Corollary ?? and Lemma ?? together imply the existence of representation μ on $\mathcal{L}$ for completely s-alternating φ. We can view μ as a measure on $\check{\mathcal{L}}$ by setting $\mu(\{\varnothing\}) = 0$. By using the homeomorphism ξ of Proposition 3.1 we can construct a Radon measure R on $\check{L}$ by setting $R(B) = \mu(\xi(B))$ for any Borel-measurable subset B of $\check{L}$. Clearly it satisfies

$$R(\check{L} \setminus \langle x \rangle^*) = \check{\varphi}(x) \text{ for all } x \in \check{L} \text{ and } R(\{\hat{1}\}) = 0. \tag{1}$$

It should be noted that $R(\check{L}) = \varphi(\hat{1})$.

Conversely, suppose that R represents $\check{\varphi}$ in the sense of (1). Then $\varphi^*(x) = R(\langle x\rangle^*)$ is completely monotone on the dual $\check{L}^*$ by Proposition 2.2. Since the increasing net $\check{L} \setminus \langle z\rangle^*$, $z \in \langle\!\langle x\rangle\!\rangle$, converges to $\check{L} \setminus \langle x\rangle^*$, by MCT we have $R(\check{L} \setminus \langle x\rangle^*) = \sup_{z\in\langle\!\langle x\rangle\!\rangle} R(\check{L} \setminus \langle z\rangle^*)$. Hence, we found φ completely s-alternating.

Proposition 3.2. *The following statements are equivalent: (i) φ is completely s-alternating; (ii) there exists a unique Radon measure R on $\check{L}$ satisfying (1).*

Let R be the representation of φ in Proposition 3.2, and let $\check{L} \setminus \langle\!\langle z\rangle\!\rangle^*$, $z \in L^*$, be a decreasing net which converges to $\{\hat{0}\}$. Then we obtain $R(\{\hat{0}\}) = \inf_{z\in L^*} R(\check{L} \setminus \langle\!\langle z\rangle\!\rangle^*)$, which is equal to $\inf_{z\in L^*} R(\check{L} \setminus \langle z\rangle^*) = \inf_{z\in L} \varphi(z)$. Thus, R is nondegenerate at $\hat{0}$ if φ satisfies $\inf_{z\in L} \varphi(z) = 0$, and we simply call φ *nondegenerate* in such a case.

Corollary 3.3. *If φ is completely s-monotone and nondegenerate then there exists a unique Radon measure R on L such that $\varphi^*(x) = R(\langle x\rangle^*)$ for all $x \in L$.*

4. Main results

The lattice L is distributive if for any nonempty finite subset A of L we have

$$x \wedge \bigvee A = \bigvee (x \wedge A), \tag{2}$$

where $\bigvee A$ denotes the least upper bound of A and $x \vee A := \{x \vee z : z \in A\}$. The distributivity (2) is dually characterized by $x \vee \bigwedge A = \bigwedge (x \vee A)$. It allows us to construct a finite distributive sublattice of L by first generating a sup-subsemilattice (or a subsemilattice, instead) then extending it to a subsemilattice (a sup-subsemilattice, respectively).

Proposition 4.1. *The following statements are equivalent:*

(i) *φ is completely monotone and completely alternating;*
(ii) *φ is monotone, and satisfies*

$$\varphi(x) + \varphi(y) = \varphi(x \wedge y) + \varphi(x \vee y) \tag{3}$$

for any $x, y \in L$.

A monotone function φ is called *valuation* (or *module*) if (3) holds. In what follows we assume that L is multiplicative and distributive, and that φ is completely s-monotone. By Proposition 4.1 the valuation φ has a representation of φ on $\mathcal{L}$ and another on $\mathcal{F}$. In fact, we will see in Corollary 4.4 that they are the same, uniquely representing φ on the intersection $\mathcal{L} \cap \mathcal{F}$.

The notion of valuation can be generalized. For $k = 1, 2, \ldots$, φ is called *k-valuation* if

$$\nabla_B \varphi \left(\bigwedge_{\{x,y\} \subseteq B} x \vee y \right) = 0 \tag{4}$$

holds for every $(k+1)$-element antichain B [i.e., a subset B consisting of $(k+1)$ pairwise incomparable elements]. It should be noted that a 1-valuation is simply a valuation, and that k-valuation φ is not necessarily k'-valuation if $k > k'$.

Proposition 4.2. *A k-valuation φ is also k'-valuation if $k < k'$.*

Consider the complete lattice $\check{L}$ for which the distributivity of L is inherited. An element x of $\check{L}$ is called *prime* if $\check{L} \setminus \langle x \rangle$ is a filter in L. An element z of $\check{L}$ is called *irreducible* if $z = x \wedge y$ implies $z = x$ or $z = y$. The lattice $\check{L}$ is distributive, and the notions of being irreducible and prime are identical except for the top element $\hat{1}$ which is irreducible but not prime.

By $\check{P}$ we denote the collection of prime elements in $\check{L}$. Observe that $\hat{0} \in \check{P}$, and that

$$\mathcal{P} := \mathcal{L} \cap \mathcal{F} = \left\{ \check{L} \setminus \langle z \rangle : z \in \check{P} \right\}$$

is a compact subset of $\mathcal{S}$.

We define a continuous map π_k from $(V_1, \ldots, V_k) \in \mathcal{P}^k$ to $\mathcal{F}$ by setting $\pi_k(V_1, \ldots, V_k) = \bigcap_{i=1}^k V_i$. Since the product space $\mathcal{P}^k$ is compact, so is the image

$$\pi\left(\mathcal{P}^k\right) = \left\{ \check{L} \setminus \langle A \rangle : A \subseteq \check{P}, 1 \leq |A| \leq k \right\},$$

which we denote by $\mathcal{P}_k$.

For each $w \in \check{L}$ we can consider the complete sublattice $\langle w \rangle^*$ of $\check{L}$, and the corresponding surjective map λ_w from $U \in \mathcal{F}$ to $\lambda_w(U) \in \mathcal{F}_w := \mathcal{F} \cap \langle w \rangle^*$ by setting $\lambda_w(U) := U \cap \langle w \rangle^*$. Here $\lambda_{\hat{0}}$ is the identity map from $\check{\mathcal{F}}$ to itself. It is easily verified that $\mathcal{F}_w$ is compact, and that λ_w is continuous.

By $\mathcal{P}_{w,k}$ we denote the compact subset $\lambda_w(\mathcal{P}_k)$ of $\mathcal{F}_w$; in fact, we can express it as

$$\mathcal{P}_{w,k} = \{ \langle w \rangle^* \setminus \langle A \rangle : A \subseteq \check{P}_w,\ 0 \leq |A| \leq k \},$$

where $\check{P}_w := \check{P} \cap \langle w \rangle^*$ is the collection of prime elements in $\langle w \rangle^*$.

Proposition 4.3. *Let $\mathbb{F}$ be the poset of finite sublattices of L. Then there exists a net $\{\mu_F\}_{F \in \mathbb{F}}$ of measures on $\mathcal{F}$ so that each μ_F represents φ on F, and satisfies for $w \in F$,*

$$\mu_F \left(\mathcal{F} \setminus \lambda_w^{-1}(\check{\mathcal{P}}_{w,k}) \right) \leq \sum \nabla_B \varphi \left(\bigwedge_{\{x,y\} \subseteq B} x \vee y \right) \tag{5}$$

where the summation is over antichains B of $F \cap \langle w \rangle^*$ with $|B| \geq k+1$.

Suppose that μ is the limit of a converging subnet $\{\mu_{F'}\}$ of Proposition 4.3. Then by Proposition 2.3 μ must be unique, for which we can establish the following properties of valuations.

Corollary 4.4. *A k-valuation φ has a unique representation on $\mathcal{P}_k$.*

In the next corollary we also assume that there is a countable sequence $\{w_i\}$ of L such that $L = \bigcup_{i=1}^{\infty} \langle w_i \rangle^*$, that is, that L possesses *σ-compactness property.* Then we can construct the F_σ set $\mathcal{P}_{w,\infty} := \bigcup_{k=1}^{\infty} \lambda_w^{-1}(\mathcal{P}_{w,k})$, and the $F_{\sigma\delta}$ set $\mathcal{P}_\infty = \bigcap_{i=1}^{\infty} \mathcal{P}_{w_i,\infty}$.

We call φ a locally finite valuation if for any $w \in L$ and $\delta > 0$ we can find some integer n so that for every finite sublattice F of $\langle w \rangle^*$,

$$\sum \nabla_B \varphi \left(\bigwedge_{\{x,y\} \subseteq B} x \vee y \right) \leq \delta$$

where the summation is over antichains B of F with $|B| \geq n$.

Corollary 4.5. *If φ is completely s-monotone and locally finite valuation then it has a unique representation on $\mathcal{P}_\infty$.*

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