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COPULAS, QUASI-COPULAS AND MARKOV OPERATORS

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ABSTRACT. Copulas are important tools that are used to construct multivariate distributions and to investigate dependence structures between random variables. Needless to say, they have many applications in Applied Probability. In this paper, we view the copulas in the settings of Functional Analysis. We provide a short survey of the set of copulas with reference to norms on the set of copulas; the copula product of Darsow, Nguyen and Olsen (1992); their connection with Markov operators; the scalar product of Siburg and Stoimenov (2008); and their MacNeille completion. Many properties of copulas extend to quasi-copulas. Counter examples are given to show that the copula product of Darsow-Nguyen-Olsen does not extend to quasi-copulas, and that the connection between quasi-copulas and Markov operators does not extend to a suitable set of positive operators.

1. INTRODUCTION

Suppose we are dealing with two real-valued random variables X and Y, and we wish to compute the variance

$$\operatorname{Var}(Z) := \mathbb{E}[(Z - \mathbb{E}Z)^2]$$

of $Z = aX + bY$, assuming that

$$\mathbb{E}[X^2] < \infty \text{ and } \mathbb{E}[Y^2] < \infty.$$

Knowing only the marginal distribution functions

$$F(x) := \mathbb{P}[X \leq x] \text{ and } G(y) := \mathbb{P}[Y \leq y]$$

of X and Y, respectively, is not enough. We need the joint distribution (cumulative distribution function)

$$H(x, y) := \mathbb{P}[X \leq x, Y \leq y]$$

of the random vector (X, Y). A copula is a function which joins a multivariate distribution function to its marginal (one-dimensional) distribution functions.

In this paper, we give a survey of the copulas in the settings of Functional Analysis. In Section 2, we present the definition, examples, and differentiability properties of copulas. We recall the scalar product of copulas, introduced by Siburg and Stoimenov [12] which establishes geometrical properties of copulas in Section 3. The set of copulas also has an algebraic property, given by the so-called $*$-product (cf. Darsow et al. [3]). We recall the results concerning the $*$-product and its connection with the scalar product in Section 4. The set of copulas $\mathcal{C}$ is closed under convex combination. The linear span of $\mathcal{C}$ is a vector space, and furthermore, is a Banach space. We recall the norms for copulas (cf. Darsow and Olsen [2]) in Section 5. The copulas also have a one-to-one correspondence to **1**-preserving Markov operator (see Section 6). Finally, we consider the set of quasi-copulas, which is the MacNeille completion of the copulas in Section 7. Many properties of copulas extend to quasi-copulas. Counter examples are given to show that the $*$-product does not extend to quasi-copulas, and that the connection between quasi-copulas and Markov operators does not extend to a suitable set of positive operators.

2. Copulas

Throughout the text, we denote by I the interval $[0,1]$ and let $I^2 = I \times I$.

Definition 1. A **copula** C is a function $C : I^2 \to I$ with the following properties:
(a) $C(u,0) = C(0,v) = 0$ for all $u, v \in I$,
(b) $C(u,1) = u$ and $C(1,v) = v$ for all $u, v \in I$.
(c) $C(u_1,v_1) - C(u_1,v_2) - C(u_2,v_1) + C(u_2,v_2) \geq 0$ for every $u_1, u_1', v_1, v_2 \in I$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

Sklar (cf. Nelsen [7]) noted the following important connection between distribution functions and copulas.

Theorem 2 (Sklar's Theorem). *Let $H : [-\infty,\infty]^2 \to \mathbb{R}$ be a joint distribution function with margins F and G (that is, $F(x) = H(x,+\infty)$ and $G(y) = H(+\infty,y)$ for $x,y \in [-\infty,\infty]$). Then there exists a copula C such that*

$$H(x,y) = C(F(x), G(x))$$

for all $x,y \in [-\infty,\infty]$. If F and G are continuous, then this copula is unique. Otherwise, it is only unique on Ran(F) $\times$ Ran(G). Conversely, if $C : I^2 \to I$ is a copula and $F : [-\infty,\infty] \to I$ and $G : [-\infty,\infty] \to I$ are distribution functions, then $H(x,y) = C(F(x), G(x))$ defines a joint distribution function with margins F and G.

Example 3. (1) W, defined by

$$W(u,v) = \max\{u+v-1, 0\}$$

for all $u, v \in I$, is a copula.
(2) M, defined by

$$M(u,v) = \min\{u,v\}$$

for all $u, v \in I$, is a copula, called the **Fréchet-Hoeffding** copula.
(3) For any copula C, we have

$$W(u,v) \leq C(u,v) \leq M(u,v)$$

for all $u, v \in I$.

Copulas have the following differentiability properties: For any copula C,
(1) C is uniformly continuous, as for all $u_1, u_2, v_1, v_2 \in I$;

$$|C(u_2,v_2) - C(u_1,v_1)| \leq |u_2 - u_1| + |v_2 - v_1|.$$

(2) for any $v \in I$, the partial derivative $\partial C/\partial u$ exists for almost all $u \in I$;
(3) for any $u \in I$, the partial derivative $\partial C/\partial v$ exists for almost all $v \in I$;

(4) for such u, $v \in I$,

$$0 \leq \partial C(u,v)/\partial v \leq 1 \text{ and } 0 \leq \partial C(u,v)/\partial u \leq 1;$$

(5) $\frac{\partial^2 C(u,v)}{\partial u \partial v}$ and $\frac{\partial^2 C(u,v)}{\partial v \partial u}$ exist almost everywhere.

Furthermore, Darsow et al. [3] noted that any copula C may be approximated in the uniform norm by a copula B for which $\frac{\partial^2 B}{\partial x \partial y}$ and $\frac{\partial^2 B}{\partial y \partial x}$ are equal and bounded almost everywhere. A more precise formulation is given in the following theorem.

Theorem 4. *Let C be a copula. For every $\epsilon > 0$ there exists a copula B such that*
(a) $\|C - B\|_\infty := \sup\{|C(u,v) - B(u,v)| : (u,v) \in I^2\} < \epsilon$, and
(b) $\frac{\partial^2 B}{\partial x \partial y}$ and $\frac{\partial^2 B}{\partial y \partial x}$ are equal and bounded almost everywhere.

Another approximation result of Darsow et al. [3] is given in the following theorem.

Theorem 5. *Let $\mathcal{C}$ denote the set of all copulas. Then $\mathcal{C}$ is a compact convex subset of all continuous real-valued functions defined on I^2, under the topology of uniform convergence.*

3. Sobolev spaces and Copulas

An n-dimensional **multi-index** is an n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$, where each $\alpha_i \in \mathbb{N}$. For multi-indices α and β, define

$$\alpha + (-)\beta = (\alpha_1 + (-)\beta_1, \cdots, \alpha_n + (-)\beta_n) \quad \text{and} \quad |\alpha| = \sum_{i=1}^{n} \alpha_n.$$

If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index and $|\alpha| = k \in \mathbb{N}$, then α is said to be of **order** k. Define:

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

If Ω be an open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, and $1 \leq p \leq \infty$, then

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}$$

is called the **Sobolev space** of **order** k. We note that $\|\cdot\|_{W^{k,p}(\Omega)}$ defined by

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = \infty, \end{cases}$$

is a norm on $W^{k,p}(\Omega)$ and $W^{k,p}(\Omega)$ is a Banach space. We remark that $W^{k,p}(\Omega)$ is separable for $1 \leq p < \infty$.

For any $1 \leq p \leq \infty$, and for $k = 0$ or $k = 1$, we have that

$$\mathcal{C} \subset W^{k,p}(I^2).$$

For any $1 \leq p < \infty$, we have $\mathcal{C}$ is a complete metric space for the metric induced by $\|\cdot\|_{W^{1,p}(I)}$.

Let

$$W_0^{1,2}(I^2) := \{f \in W_0^{1,2}(I^2) : f \in C^0(I^2),\ f(0) = 0\}.$$

Then

$$\langle f, g\rangle = \int_{I^2} \nabla f \,.\, \nabla g \, d\lambda$$

defines scalar product on $W_0^{1,2}(I^2)$. Define on $\mathcal{C}$,

$$\|C\| = \Big(\int_{I\times I} |\nabla C|^2 \, d\lambda\Big)^{\frac{1}{2}}$$

and on $\mathcal{C} \times \mathcal{C}$

$$d(A, B) = \Big(\int_{I\times I} |\nabla A - \nabla B|^2 \, d\lambda\Big)^{\frac{1}{2}}$$

Siburg and Stoimenov [12] established geometric properties of copulas in terms of the metric d.

Theorem 6. *Let $A, B \in \mathcal{C}$. Then,*
(a) $\frac{1}{2} \leq \langle A, B\rangle \leq 1$;
(b) $\langle A, B\rangle = 1$ *iff* $\|A\| = 1 = \|B\|$ *and* $A = B$;
(c) $d(A, B) \leq 1$;
(d) $d(A, B) = 1$ *iff* $\|A\| = 1 = \|B\|$ *and* $\langle A, B\rangle = \frac{1}{2}$.

4. A Product of Copulas

Darsow et al. in [3] introduced an important product for copulas.

Definition 7. Let C_1 and C_2 be copulas. We define the *product* $C_1 * C_2$ of C_1 and C_2 by

$$(C_1 * C_2)(u, v) = \int_0^1 \partial_2 C_1(u, t) \cdot \partial_1 C_2(t, v) \, dt,$$

for $(u, v) \in I^2$ (where ∂_i denotes the partial derivative with respect to coordinate i).

We have the following results for the $*$-product on $\mathcal{C}$, i.e. the set of all copulas.

Theorem 8. *(a) Let C_1 and C_2 be copulas. Then $C_1 * C_2$ is a copula.*
(b) The $*$-product is not commutative.
(c) As a binary operation on $\mathcal{C}$, the $*$-product operation is right and left distributive over convex combinations.

Definition 9. A copula $C \in \mathcal{C}$ is **left invertible** if there exists $A \in \mathcal{C}$ such that

$$A * C = M \qquad (M(u, v) = \min\{u, v\})$$

The definition is similar for **right invertible**. Furthermore, $A \in \mathcal{C}$ is **invertible** if A is both right and left invertible.

Siburg and Stoimenov [12] related invertible copulas, with respect to the $*$-product, to the norm $\|\cdot\|$ introduced in Section 3.

Theorem 10. *Let $C \in \mathcal{C}$. Then,*
(a) $\frac{2}{3} \leq \|C\|^2 \leq 1$;
(b) $\|C\|^2 = \frac{2}{3}$ *iff* $C = P$ *(here* $P(u, v) = uv$*)*;
(c) $\|C\|^2 \in (\frac{5}{6}, 1]$ *if C is left or right invertible;*

(d) $\|C\|^2 = 1$ *iff* C *is invertible.*

Theorem 11. *Let* $C \in \mathcal{C}$. *Then the following are equivalent:*

(i) $\|C\| = 1$;
(ii) $\partial_1 C, \partial_2 C \in \{0, 1\}$ *a.e.;*
(iii) C *is invertible.*

5. Norms for Copulas

The norms described in this section were introduced by Darsow and Olsen [2]. By span($\mathcal{C}$) we mean the linear span of the copulas. Any element $A \in$ span($\mathcal{C}$) can be written in the form

$$A = sB - tC$$

where s and t are nonnegative and B and C are copulas.

5.1. **Minkowski norm.** Let $\mathcal{B} = \mathrm{co}(-\mathcal{C} \cup \mathcal{C})$ denotes the convex hull of the set $-\mathcal{C} \cup \mathcal{C}$. Note that span($\mathcal{C}$) $= \cup_{t \geq 0} t\mathcal{B}$. For any $A \in$ span($\mathcal{C}$) define

$$\begin{aligned} \|A\|_M &= \inf\{t > 0 \mid A \in t\mathcal{B}\} \\ &= \inf\{s + t \mid s, t \geq 0 \text{ and } A = sB - tC \text{ for some } B, C \in \mathcal{C}\}. \end{aligned}$$

We note that span($\mathcal{C}$) is a Banach algebra under the norm $\|\cdot\|_M$ (and the $*$-product given in Section 4).

5.2. **Sobolev norm.** Denote $W^{m,p}(\Omega)$ to be the Sobolev space

$$\{f \mid f \in L^p(\Omega) \text{ and } D^\alpha f \in L^p(\Omega), \text{ for all } \alpha \text{ where } |\alpha| \leq m\},$$

where $\Omega \subset \mathbb{R}^d$, a multi-index α is a d-tuple of nonnegative integers, $|\alpha|$ denotes the sum of the components of α, and D^α denotes a distributional partial derivative.

We note that

(a) $\mathcal{C} \subset W^{m,p}(I^2)$;
(b) $\mathcal{C}$ is a complete metric space under the Sobolev norms $\|\cdot\|_{1,p}$.
(c) On span($\mathcal{C}$) each of the Sobolev norms is dominated by, but not equivalent to, $\|\cdot\|_M$.
(d) On $\mathcal{C}$, the $*$-product (cf. Section 4) is jointly continuous with respect to $\|\cdot\|_{1,p}$.

5.3. **Jordan norm.** In Darsow et al. [3], it is noted that any copula C induces a unique doubly stochastic probability measure μ_C on I^2 via the assignment

$$\mu_C([x_1, x_2] \times [y_1, y_2]) = C(x_1, y_1) - C(x_2, y_1) - C(x_1, y_2) + C(x_2, y_2)$$

as the measure of a rectangle. Conversely, for any doubly stochastic measure μ, there is a unique copula C_μ defined via $C_\mu(x, y) = \mu([0, x] \times [0, y])$. Any $C \in$ span($\mathcal{C}$) induces a finite signed measure on I^2 via the definition of μ_C above. For any finite signed measure μ there exists measurable sets E^+ and E^- such that

$$E^+ \cap E^- = \emptyset, \text{ and } E^+ \cup E^- = I^2$$

and for all measurable sets F,

$$\mu(E^+ \cap F) \geq 0, \text{ and } \mu(E^- \cap F) \leq 0.$$

It follows that μ^+ and μ^- defined by

$$\mu^+(F) = \mu(E^+ \cap F), \quad \mu^-(F) = -\mu(E^- \cap F)$$

are measures. Furthermore, μ^+ and μ^- are unique. The decomposition $\mu = \mu^+ - \mu^-$ of a finite signed measure as the difference of two measures is called the Jordan decomposition. The set of finite signed measures is closed under real linear combination and that

$$\|\mu\|_J = \mu^+(I^2) + \mu^-(I^2)$$

defines a norm on the set under which the set is a Banach space. For $A \in \text{span}(\mathcal{C})$ we define the Jordan norm:

$$\|A\|_J = \|\mu_A\|_J.$$

The Jordan norm dominates, but is not equivalent to, the uniform norm and is dominated by, but is not equivalent to the Minkowski norm. We remark that $\text{span}(\mathcal{C})$ is not complete with respect to the Jordan norm.

6. 1-PRESERVING MARKOV OPERATORS

Consider an operator $T : L^1(I) \to L^1(I)$. Here, $L^1(I)$ denotes the space of all Lebesgue integrable functions on I. Denote $L^1_+(I)$ to be the set of all positive functions in $L^I(I)$.

Definition 12. We call T a **1-preserving Markov operator** if T is a Markov operator on $L^1(I)$; i.e., T is positive, and $\|Tf\|_1 = \|f\|_1$ for all $f \in L^1_+(I)$, with the additional property: $T\mathbf{1} = \mathbf{1}$, where $\mathbf{1}(x) = 1$ almost everywhere for all $x \in I$.

Darsow et al. [3] established that $\mathbf{1}$-preserving Markov operators are in one-to-one correspondence with copulas.

Theorem 13. *Let $C : I^2 \to I$ be a copula. Then the operator T_C defined on $L^1(I)$, by*

$$(1) \qquad (T_C f)(x) := \frac{d}{dx} \int_0^1 \partial_2 C(x,t) f(t)\ dt$$

for all $f \in L^1(I)$, is a $\mathbf{1}$-preserving Markov operator. Conversely, let $T : L^1(I) \to L^1(I)$ be a $\mathbf{1}$-preserving Markov operator. Then the function C_T defined on I^2 by

$$(2) \qquad C_T(x,y) := \int_0^x (T1_{[0,y]})(s)\ ds,$$

for $(x,y) \in I^2$, is a copula. Moreover, the maps $C \mapsto T_C$ and $T \mapsto C_T$ are inverses of each other.

Idempotent copulas (with respect to the $*$-product given in Section 4) are related to conditional expectations. We require a result noted by Douglas [4] to describe the relationship.

Theorem 14. *Let (Ω, Σ, μ) be a probability space and $1 \le p < \infty$. If $T : L^p(\mu) \to L^p(\mu)$ is a positive contractive projection with $T\mathbf{1} = \mathbf{1}$, then there exists a unique σ-algebra $\mathcal{F} \subseteq \Sigma$ such that*

$$Tf = \mathbb{E}(f|\mathcal{F}) \quad \text{for all} \quad f \in L^p(\mu).$$

Every conditional expectation has the property

$$\|\mathbb{E}(f|\mathcal{F})\| = \|f\| \quad \text{for all } f \geq 0,\ f \in L^1.$$

Thus, the result of Douglas shows that if $T : L^1 \to L^1$ is a positive contractive projection with $T\mathbf{1} = \mathbf{1}$, then T is also a $\mathbf{1}$-preserving Markov operator. Hence, there is a one-to-one correspondence between the class of Markov operators on $L^1(I)$ which hold the $\mathbf{1}$ function invariant and are projections, and the conditional expectations on $L^1(I)$; that is, a $\mathbf{1}$-preserving Markov operator is a conditional expectation if, and only if, it is a projection.

The following result concerning the relationship between idempotent copulas and conditional expectations, as mentioned above, can be found in Darsow et al. [3].

Theorem 15. *Let $C : I^2 \to I$ be a copula with the properties that $\frac{\partial^2 C}{\partial x \partial y}$ and $\frac{\partial^2 C}{\partial y \partial x}$ are bounded and equal almost everywhere. Let $T_C : L^1(I) \to L^1(I)$ denote the corresponding $\mathbf{1}$-preserving Markov operator. Then T_C is a conditional expectation if, and only if, C is idempotent with respect to the $*$-product (that is, $C * C = C$).*

7. Quasi-copulas

The concept of quasi-copula was introduced by Alsina et al. [1] in order to characterise operations on distribution functions that can or cannot be derived from operations on random variables [7, p. 1].

Definition 16. A function $S : I^2 \to I$ for which $S(t, 0) = S(0, t) = 0$ and $S(t, 1) = S(1, t) = t$ for all $t \in I$ is called a *quasi-copula* if

(1) S is non-decreasing in each component, and

(2) $|S(u_1, v_1) - S(u_2, v_2)| \leq |u_1 - u_2| + |v_1 - v_2|$ for all $u_1, u_2, v_1, v_2 \in I$.

Denote by $\mathcal{Q}$ the set of all quasi-copulas. By their respective definitions, we get that $\mathcal{C} \subseteq \mathcal{Q}$. Genest et al. [5] showed that there exists a quasi-copula which is not a copula, which implies that the inclusion $\mathcal{C} \subset \mathcal{Q}$ is strict.

Example 17. $Q : I^2 \to I$, defined by

$$Q(x, y) = xy + g(y)\sin(2\pi x),$$

where

$$g(y) = \begin{cases} 0, & 0 \leq y \leq 1/4 \\ (4y - 1)/24, & 1/4 \leq y \leq 1/2 \\ (1 - y)/12, & 1/2 \leq y \leq 1 \end{cases}$$

is a quasi-copula which is not a copula.

Denote the space of continuous functions on I^2 by $C(I^2)$. It is known that $(C(I^2), \leq)$ is a partially ordered set, where $\leq$ is defined by

$$B_1 \leq B_2 \Leftrightarrow B_1(u, v) \leq B_2(u, v) \text{ for all } u, v \in I.$$

We refer to $\leq$ as the *pointwise order*. Nelsen and Úbeda Flores [8] showed that $(\mathcal{Q}, \leq)$ is a complete lattice (i.e., for any $S \subseteq \mathcal{Q}$, the least upperbound $\bigvee S$ of S is in $\mathcal{Q}$ and the greatest lower bound $\bigwedge S$ of S is in $\mathcal{Q}$); and $(\mathcal{C}, \leq)$ is not a lattice. They also showed that $\mathcal{Q}$ is the Dedekind-MacNeille completion of $(\mathcal{C}, \leq)$. The following is a theorem by Nelsen [7, Theorem 8.5.] which gives a characterisation of quasi-copulas in terms of copulas:

Theorem 18. *Let $Q : I^2 \to I$. Then Q is a quasi-copula if and only if there exists $\emptyset \neq S \subset \mathcal{C}$ such that for all $(u,v) \in I^2$, $Q(u,v) = \sup\{C(u,v) : C \in S\}$.*

Open problem. There are several definitions of stochastic order given in Shaked and Shanthikumar [11]. Is there an appropriate ordering from these stochastic orderings for which $\mathcal{C}$ is a lattice?

7.1. **Extension.** Many properties of copulas extend to quasi-copulas. We give a few examples:

Theorem 19. *The set of all quasi-copulas $\mathcal{Q}$ is a convex compact subset of $C(I^2)$.*

Proof. The convexity follows by the fact that $\mathcal{Q}$ is closed under convex combination. Since I^2 is a compact set, it is sufficient to show that $\mathcal{Q}$ is equicontinuous and closed. Equicontinuity follows from Definition 16 (2). The quasi-copula properties are preserved by limits, hence for any uniform convergent sequence in $\mathcal{Q}$, the limit is again a quasi-copula, which implies that $\mathcal{Q}$ is closed. □

Theorem 20. *Let A be a quasi-copula and let $\epsilon > 0$ be given. Then there exists a quasi-copula B such that*

$$\|A - B\|_\infty < \epsilon,$$

where B has the property that $\frac{\partial^2 B}{\partial x \partial y}$ and $\frac{\partial^2 B}{\partial y \partial x}$ are equal and bounded almost everywhere.

Proof. The proof follows similarly from Hawke [6, Proposition 7.15], by replacing the definition of V_A associated with a copula A by

$$W_A([u_1, v_1] \times [u_2, v_2]) = A(u_2, v_2) - A(u_1, v_1) \text{ if } 0 \leq u_1 \leq u_2 \text{ and } 0 \leq v_1 \leq v_2$$

where A is a quasi-copula. □

The following quantity exists for any $A, B \in \mathcal{Q}$

$$\langle A, B \rangle := \int_{I^2} \nabla A \cdot \nabla B \, d\lambda.$$

Theorem 21. *$\langle \cdot, \cdot \rangle$ defines an inner product on $\mathcal{Q}$.*

Proof. By the properties of the partial derivatives, $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form with $\langle A, A \rangle \geq 0$ for all $A \in \mathcal{Q}$. If $\langle A, A \rangle = 0$, then $\nabla A = 0$ almost everywhere. The Lipschitz continuity of quasi-copulas asserts that quasi-copulas are absolutely continuous in each argument, so that it can be recovered from any of its partial derivatives by integration. Together with the fact that $A(0,0) = 0$, we conclude that $A = 0$. □

Consequently, the definition of the metric d in Section 3 and the geometric result of Siburg and Stoimenov extend from copulas to quasi-copulas. More precisely, the geometric result becomes:

Theorem 22. *Let $A, B \in \mathcal{Q}$. Then,*

(a) $\frac{1}{2} \leq \langle A, B \rangle \leq 1$;

(b) $d(A,B) \le 1$;
(c) $d(A,B) = 1$ *iff* $\|A\| = 1 = \|B\|$ *and* $\langle A,B\rangle = \frac{1}{2}$;
(d) $\langle A,B\rangle = 1$ *iff* $\|A\| = 1 = \|B\|$ *and* $A = B$.

7.2. **Counter examples and open problems.** There are properties of copulas that do not extend to quasi-copulas and we discuss some of them in this subsection.

Recall the copula C_T that can be defined for any **1**-preserving Markov operator T:

$$C_T = \int_0^x (T1_{[0,y]})(s)\,ds, \qquad \text{for all } (x,y) \in I^2.$$

Since C_T is a copula, it is also a quasi-copula. If we replace the copula C in the definition of T_C:

$$(T_C f)(x) := \frac{d}{dx}\int_0^1 \partial_2 C(x,t) f(t)\,dt$$

with a quasi-copula Q, the resulting operator T_Q is not necessarily a positive operator, as shown in the following example.

Example 23. Consider the proper quasi-copula Q in Example 17 by Genest et al. [5]. Let f be the function

$$f(t) = \begin{cases} 0, & 0 \le t \le 1/4 \\ 1, & 1/4 \le t \le 1/2 \\ 0, & 1/2 \le t \le 1. \end{cases}$$

Clearly, $0 \le f \in L_1(I)$. As

$$\begin{aligned} T_Q f(x) &= \frac{d}{dx}\int_0^1 \partial_2 Q(x,t) f(t)\,dt \\ &= \frac{d}{dx}\int_{\frac{1}{4}}^{\frac{1}{2}} \left[x + \frac{1}{6}\sin(2\pi x)\right] dt \\ &= \frac{1}{2}\frac{d}{dx}\left[x + \frac{1}{6}\sin(2\pi x)\right] = \frac{1}{2}\left[1 + \frac{\pi}{3}\cos(2\pi x)\right], \end{aligned}$$

$T_Q f$ is strictly negative when $\cos(2\pi x)$ is close to -1. Hence, T_Q is not a positive operator, thus T_Q cannot be a Markov operator.

Since $T_Q f$ is, in general, not necessarily positive, we have

$$\begin{aligned} \|T_Q f\|_1 &= \int_0^1 \left|\frac{d}{dx}\int_0^1 \partial_2 Q(x,t) f(t)\,dt\right| dx \\ &\ge \int_0^1 \frac{d}{dx}\int_0^1 \partial_2 Q(x,t) f(t)\,dt\,dx \\ &= \int_0^1 (\partial_2 Q(1,t) - \partial_2 Q(0,t)) f(t)\,dt = \int_0^1 f(t)\,dt = \|f\|_1. \end{aligned}$$

for all $f \in L^1(I)$ with $f \geq 0$ almost everywhere. However, T_Q is a $\mathbf{1}$-preserving map, as

$$\begin{aligned} T_Q\mathbf{1} &= \frac{d}{dx}\int_0^1 \partial_2 Q(x,t)\mathbf{1}(t)\,dt \\ &= \frac{d}{dx}\int_0^1 \partial_2 Q(x,t)\,dt = \frac{d}{dx}[Q(x,1) - Q(x,0)] = \frac{d}{dx}x = 1 \end{aligned}$$

for all $x \in I$.

Remark 24. The above arguments show that for any quasi-copula Q, the operator T_Q is not a $\mathbf{1}$-preserving Markov operator. This provides an alternative verification of the fact that $\mathcal{C}$ is strictly contained in $\mathcal{Q}$, as the set of all operators T_Q contains the class of $\mathbf{1}$-preserving Markov operators (which is in one-to-one correspondence with $\mathcal{C}$).

Open Problem. Which class of operators is isometrically isomorphic to $\mathcal{Q}$? Provided that such a class of operators can be found, is it possible to define an ordering on the class in such a way that the isometric isomorphism is also an order isomorphism if $\mathcal{Q}$ is endowed with an appropriate ordering from the stochastic orderings given in Shaked and Shanthikumar [11]?

Any quasi-copula is non-decreasing in each component (by definition), which implies that the partial derivative of a quasi-copula exist almost everywhere. Thus, the quantity

$$(P * Q)(u,v) = \int_0^1 \partial_2 P(u,t)\partial_1 Q(t,v)\,dt$$

exists for all $P, Q \in \mathcal{Q}$. However, the product $P * Q$ is not necessarily a quasi-copula, as shown in the next example.

Example 25. The $*$-product (as given in Section 4) is not closed in $\mathcal{Q}$.

Proof. Consider the function $F : I^2 \to \mathbb{R}$ defined for all $u, v \in I$ by:

$$F(u,v) = \begin{cases} \min\{u, v, \frac{1}{3}, u+v-\frac{2}{3}\}, & \frac{2}{3} \leq u+v \leq \frac{4}{3}; \\ \max\{u+v-1, 0\}, & \text{otherwise.} \end{cases}$$

We have

$$\partial_1 F(u,v) = \begin{cases} 0, & u+v \leq \frac{2}{3}; \\ 1, & \frac{2}{3} \leq u+v \leq 1 \text{ and } u \leq \frac{2}{3} \\ 0, & \frac{2}{3} \leq u+v \leq 1 \text{ and } u \geq \frac{2}{3} \\ 1, & 1 \leq u+v \leq \frac{4}{3} \text{ and } u \leq \frac{1}{3} \\ 0, & 1 \leq u+v \leq \frac{4}{3} \text{ and } u \geq \frac{1}{3} \\ 1, & u+v \geq \frac{4}{3}; \end{cases}$$

and

$$\partial_2 F(u,v) = \begin{cases} 0, & u+v \leq \frac{2}{3}; \\ 1, & \frac{2}{3} \leq u+v \leq 1 \text{ and } v \leq \frac{2}{3} \\ 0, & \frac{2}{3} \leq u+v \leq 1 \text{ and } v \geq \frac{2}{3} \\ 1, & 1 \leq u+v \leq \frac{4}{3} \text{ and } v \leq \frac{1}{3} \\ 0, & 1 \leq u+v \leq \frac{4}{3} \text{ and } v \geq \frac{1}{3} \\ 1, & u+v \geq \frac{4}{3}. \end{cases}$$

In particular, we have

$$(3)\qquad \partial_1 F\left(t,\frac{1}{2}\right)=\begin{cases}0, & t\in[0,\frac{1}{6});\\ 1, & t\in(\frac{1}{6},\frac{1}{2});\\ 0, & t\in(\frac{1}{2},\frac{5}{6});\\ 1, & t\in(\frac{5}{6},1];\end{cases}=\partial_2 F\left(\frac{1}{2},t\right);$$

and

$$(4)\qquad \partial_1 F\left(t,\frac{3}{4}\right)=\begin{cases}1, & t\in[0,\frac{1}{3});\\ 0, & t\in(\frac{1}{3},\frac{7}{12});\\ 1, & t\in(\frac{7}{12},1].\end{cases}$$

Thus, (3) and (4) give us

$$\begin{aligned}(F*F)\left(\frac{1}{2},\frac{1}{2}\right) &= \int_0^1 \partial_2 F\left(\frac{1}{2},t\right)\partial_1 F\left(t,\frac{1}{2}\right)\,dt\\ &= \int_{\frac{1}{6}}^{\frac{1}{2}} \partial_1 F\left(t,\frac{1}{2}\right)\,dt+\int_{\frac{5}{6}}^{1}\partial_1 F\left(t,\frac{1}{2}\right)\,dt=\frac{1}{2};\end{aligned}$$

and

$$\begin{aligned}(F*F)\left(\frac{1}{2},\frac{3}{4}\right) &= \int_0^1 \partial_2 F\left(\frac{1}{2},t\right)\partial_1 F\left(t,\frac{3}{4}\right)\,dt\\ &= \int_{\frac{1}{6}}^{\frac{1}{2}} \partial_1 F\left(t,\frac{3}{4}\right)\,dt+\int_{\frac{5}{6}}^{1}\partial_1 F\left(t,\frac{3}{4}\right)\,dt\\ &= \int_{\frac{1}{6}}^{\frac{1}{3}} dt+\int_{\frac{5}{6}}^{1} dt=\frac{1}{3}.\end{aligned}$$

We have $(F*F)(\frac{1}{2},\frac{1}{2}) > (F*F)(\frac{1}{2},\frac{3}{4})$, which shows that $F*F$ is not a quasi-copula. □

Open Problem. Can we define a product which is closed on $\mathcal{Q}$ and define a norm on span(Q) in such a way that span(Q) is a Banach algebra?

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