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A geometric approach to the two-dimensional Tingley problem and geometric constants of Banach spaces

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1 Introduction

Throughout this note, the term "Banach space" always means a real Banach space. Let X and Y be Banach spaces. Then the classical Mazur-Ulam theorem states that if $T: X \to Y$ is a surjective isometry then T is affine. In 1972, Mankiewicz [12] extended this result by showing that if $U \subset X$ and $V \subset Y$ are open and connected and $T_0 : U \to V$ is a surjective isometry then there exists a surjective affine isometry $T: X \to Y$ such that $T_0 = T|_U$. From this, in particular, it turns out that every isometry from the unit ball of X onto that of Y can be extended to an isometric isomorphism between X and Y. Motivated by this observation, Tingley [17] proposed in 1987 the following problem. Let X and Y be Banach spaces, let S_X and S_Y denote the unit spheres of X and Y, and let $U \subset X$ and $V \subset Y$. Suppose that $T: U \to V$ is a surjective isometry.

$U = X$ and $V = Y$	T: affine	Mazur-Ulam
U, V: open and connected	T has an affine extension	Mankiewicz
$U = S_X$ and $V = S_Y$	T: [?]	**Tingley's problem**

To be precise, Tingley's problem is as follows.

Tingley's problem. *Let X and Y be Banach spaces. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then, does T_0 have a linear isometric extension $T: X \to Y$?*

Many papers, especially in the last decade, have been devoted to the problem, and is solved positively for some classical Banach spaces; see, for example, [2, 4, 9, 11, 18, 19]. The survey of Ding [5] is a good starting point to understanding the history of the problem.

Recently some mathematicians began to attack the problem on more general spaces, and developed various methods and notions.

(i) Somewhere-flat spaces (Cheng and Dong [3]).

(ii) Finite-dimensional polyhedral Banach spaces (Kadets and Martín [10]).

(iii) Sharp corner points (Ding and Li [6]).

(iv) The Tingley property (Tan and Liu [15]).

However, surprisingly, Tingley's problem remains open even if $X = Y$ and X is two-dimensional.

This note is a survey of recent work [16] which provides new geometric methods for the two-dimensional Tingley problem and some results on symmetric absolute normalized norms on $\mathbb{R}^2$.

2 New methods for Tingley's problem

A usual way to attack Tingley's problem is to show that the natural extension T of T_0 is linear, where T is given by

$$Tx = \begin{cases} \|x\|T_0\left(\dfrac{x}{\|x\|}\right) & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

In this section, we construct new methods for Tingley's problem on two-dimensional spaces. We first recall the following result of Tingley.

Lemma 2.1 (Tingley [17]). *Let X and Y be finite dimensional normed spaces. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then $T_0(-x) = -T_0x$ for all $x \in S_X$.*

It is known that if there exists a surjective isometry between the unit spheres of two finite dimensional normed spaces then the dimensions of the spaces coincide.

Lemma 2.2. *Let X be a two-dimensional normed space, and let Y be a normed space. If there exists a surjective isometry $T_0 : S_X \to S_Y$, then $\dim Y = 2$.*

The following two lemmas are key for our approach.

Lemma 2.3 ([16]). *Let X be a two-dimensional normed space. Suppose that $x, y \in S_X$, and that $x \pm y \neq 0$. Then there exists an element $z \in A(x, y)$ such that $\|z - x\| = \|z - y\| \leq \|x - y\|$. Furthermore, such an element is unique in $A(x, y)$.*

Lemma 2.4 ([16]). *Let X be a two-dimensional normed space, and let Y be a normed space. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. Then $T_0(A(x, y)) = A(T_0x, T_0y)$ whenever $x, y \in S_X$ and $x \pm y \neq 0$.*

We now present a new method for Tingley's problem on two-dimensional spaces.

Theorem 2.5 ([16]). *Let X be a two-dimensional normed space, and let Y be a normed space. Suppose that $T_0 : S_X \to S_Y$ is a surjective isometry. If there exists an isometric isomorphism $T : X \to Y$ such that $T_0x = Tx$ and $T_0y = Ty$ for some $x, y \in S_X$ with $x \pm y \neq 0$, then $T_0 = T|_{S_X}$.*

Suppose that T is a map from a set C into itself. Then an element $x \in C$ is said to be a fixed point of T if $Tx = x$. The set of all fixed points of T is denoted by $F(T)$. Applying the preceding theorem, we immediately have the following result.

Corollary 2.6 ([16]). *Let X be a two-dimensional normed space. Suppose that $T_0 : S_X \to S_X$ is a surjective isometry. If there exist $x, y \in S_X \cap F(T_0)$ such that $x \pm y \neq 0$, then $T_0 = I|_{S_X}$, where I is the identity map on X.*

3 Tingley's problem on symmetric absolute normalized norms on $\mathbb{R}^2$

In this section, we present some new sufficient conditions for Tingley's problem on symmetric absolute normalized norms on $\mathbb{R}^2$. We first note the following property.

Lemma 3.1 ([16]). *Let $\psi \in \Psi_2^S$. Suppose that T_0 is an isometry from the unit sphere of $(\mathbb{R}^2, \|\cdot\|_\psi)$ onto itself. Then $T_0(1,0) \neq \psi(1/2)^{-1}(1/2,1/2)$ if $\|\cdot\|_\psi$ is not $\pi/4$ rotation invariant.*

To present sufficient conditions for Tingley's problem, the following two geometric constants of a normed space X play important roles.

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\right\},$$
$$c'_{NJ}(X) = \inf\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\right\}.$$

These constants were introduced by Gao [7], and are naturally strongly related to the von Neumann-Jordan constant $C_{NJ}(X)$ given by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X,\ (x,y) \neq (0,0)\right\}.$$

In particular, the constant $C'_{NJ}(X)$ is called the modified von Neumann-Jordan constant, and has been studied in [1, 8, 14].

Define a partial order $\leq$ on Ψ_2 by declaring that $\varphi \leq \psi$ if $\varphi(t) \leq \psi(t)$ for all $t \in [0,1]$. Using Lemma 3.1 and the constants $C'_{NJ}(X)$ and $c'_{NJ}(X)$, we obtain the following sufficient conditions for Tingley's problem.

Theorem 3.2 ([16]). *Let $\psi \in \Psi_2^S$. Then Tingley's problem is affirmative if $X = Y = (\mathbb{R}^2, \|\cdot\|_\psi)$ and either of the following statements holds.*

(i) *$\psi \leq \psi_2$ and the function ψ_2/ψ on $[0,1/2]$ takes the maximum only at $t_0 \in (0,1/2]$.*

(ii) *$\psi \geq \psi_2$ and the function ψ_2/ψ on $[0,1/2]$ takes the minimum only at $t_0 \in (0,1/2]$.*

For incomparable cases, we have the following result.

Theorem 3.3 ([16]). *Let $\psi \in \Psi_2^S$. Then Tingley's problem is affirmative if $X = Y = (\mathbb{R}^2, \|\cdot\|_\psi)$ and either of the following statements holds.*

(i) *The function* ψ_2/ψ *on* $[0,1/2]$ *takes the minimum at* $1/2$ *and the maximum only at* $t_0 \in (0,1/2]$.

(ii) *The function* ψ_2/ψ *on* $[0,1/2]$ *takes the maximum at* $1/2$ *and the minimum only at* $t_0 \in (0,1/2]$.

These theorems provide many examples by easy arguments.

Example 3.4. Let $1 \le p < 2 < q \le \infty$, and let $2^{1/q-1/p} < \lambda < 1$. Then the function $\psi_2/\psi_{p,q,\lambda}$ is increasing on $[0,t_\lambda]$, and decreasing on $[t_\lambda, 1/2]$. Hence it takes the maximum only at t_λ. We remark that $\psi_{p,q,\lambda} \le \psi_2$ if and only if $2^{1/q-1/p} < \lambda \le 2^{1/2-1/p}$. If $2^{1/2-1/p} < \lambda < 1$, it turns out that $\psi_2/\psi_{p,q,\lambda}$ takes the minimum at $1/2$. Thus, in both cases, Tingley's problem is affirmative if $X = Y = (\mathbb{R}^2, \|\cdot\|_{\psi_{p,q,\lambda}})$.

Example 3.5. Let $0 < \omega < 1$ and $1 < q < \infty$. The two-dimensional Lorentz sequence space $d^{(2)}(\omega,q)$ is defined as the space $\mathbb{R}^2$ endowed with the norm

$$\|(x,y)\|_{\omega,q} = (\max\{|x|^q,|y|^q\} + \omega \min\{|x|^q,|y|^q\})^{1/q}.$$

Remark that $\|\cdot\|_{\omega,q}$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, and that the function $\psi_{\omega,q}$ associated with this norm is given by

$$\psi_{\omega,q}(t) = \begin{cases} ((1-t)^q + \omega t^q)^{1/q} & \text{if } 0 \le t \le 1/2, \\ (t^q + \omega(1-t)^q)^{1/q} & \text{if } 1/2 \le t \le 1. \end{cases}$$

We now consider the function $\psi_2/\psi_{\omega,q}$ on $[0,1/2]$. Then the first derivative is given by

$$\left(\frac{\psi_2}{\psi_{\omega,q}}\right)'(t) = \frac{((1-t)^q + \omega t^q)^{1/q-1}(t(1-t)^{q-1} - \omega t^{q-1}(1-t))}{\psi_2(t)\psi_{\omega,q}(t)^2}$$

for all $t \in (0,1/2)$. From this, one can easily check that the function $\psi_{\omega,q}$ satisfies the assumption of Theorems 3.2 or 3.3. Thus, we have an affirmative answer for Tingley's problem in the case of $X = Y = d^{(2)}(\omega,q)$.

Example 3.6. Let $0 < \omega < 1$ and $1 < q < \infty$. In [13], it was shown that $d^{(2)}(\omega,q)^*$ is isometrically isomorphic to the space $\mathbb{R}^2$ endowed with the norm $\|\cdot\|^*_{\omega,q}$ defined by

$$\|(x,y)\|^*_{\omega,q} = \begin{cases} (|x|^p + \omega^{1-p}|y|^p)^{1/p} & \text{if } |y| \le \omega|x|, \\ (1+\omega)^{1/p-1}(|x|+|y|) & \text{if } \omega|x| \le |y| \le \omega^{-1}|x|, \\ (\omega^{1-p}|x|^p + |y|^p)^{1/p} & \text{if } \omega^{-1}|x| \le |y|, \end{cases}$$

where $1/p + 1/q = 1$. The norm $\|\cdot\|^*_{\omega,q}$ is symmetric, absolute and normalize, and the corresponding function $\psi^*_{\omega,q}$ is given by

$$\psi^*_{\omega,q}(t) = \begin{cases} ((1-t)^p + \omega^{1-p}t^p)^{1/p} & \text{if } 0 \le t \le \omega/(1+\omega), \\ (1+\omega)^{1/p-1} & \text{if } \omega/(1+\omega) \le t \le 1/(1+\omega), \\ (t^p + \omega^{1-p}(1-t)^p)^{1/p} & \text{if } 1/(1+\omega) \le t \le 1. \end{cases}$$

We can conclude that Tingley's problem is affirmative if $X = Y = d^{(2)}(\omega,q)^*$ by an argument similar to that in the preceding example.

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