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SOME RESULTS ON SET-VALUED STOCHASTIC INTEGRALS WITH RESPECT TO POISSON JUMP IN AN M-TYPE 2 BANACH SPACE

JINPING ZHANG*, ITARU MITOMA, AND YOSHIKI OKAZAKI

1. INTRODUCTION

Probability theory is an important tool of modeling randomness in a practical problem. But besides randomness, in the real world, there exists other kind of uncertainties such as imprecision or vagueness. Set-valued functions are employed to model the imprecision in applied field such as in Economics, control theory (see for example [1]). Integrals of set-valued functions have been received much attention with widespread applications, see for example [2, 7, 9, 10] etc. Recently, stochastic integrals for set-valued stochastic processes with respect to the Brownian motion and martingales have been received much attention, e.g. see [12, 13, 18, 23, 32, 37]. Correspondingly, the set-valued stochastic differential equations are studied, e.g. see [23, 25, 33, 34, 35, 36]. Michta (2011) [22] extended the integrator to a larger class: semimartingales. But the integrably boundedness of the corresponding set-valued stochastic integrals are not obtained since the semimartingales may not be of finite variation. In such cases, the set-valued stochastic integrals may not be well defined as Ogura pointed out [25].

The Poisson stochastic processes are special. They play important roles both in the random mathematics (c.f. [11, 8, 17]) and in applied fields, for example, in the financial mathematics [17]. If the characteristic measure $\nu$ of a stationary Poisson process $p$ is finite, then both of the Poisson random measure $N(dsdz)$ (where $z \in Z$, the state space of $p$) and the compensated Poisson random measure $\hat{N}(dsdz)$ are of finite variation a.s. We will give some results (without giving proof since the page limitation) on the set-valued stochastic integrals with respect to the Poisson random measure $N(dsdz)$, $\hat{N}(dsdz)$. For the detail proof, the reader can refer to [31, 38]. For example, the stochastic integrals for set-valued $\mathcal{F}$-predictable (see Definition 3.2) processes with respect to $N(dsdz)$ and $\hat{N}(dsdz)$ are $L^2$-integrably bounded. For Brownian or Martingale integrator with continuous part, the integrable boundedness are not obtained until now. Furthermore, if the $\sigma$-algebra $\mathcal{F}$ is separable, then the integral $\{I_t(F)\}$ of convex set-valued stochastic process will not become a set-valued martingale, which is very different from single valued case. We would like to pointed out that there is a gap in the proof of Theorem 3.7 in [31] about the set-valued martingale property of set-valued stochastic integral with respect to the compensated Poisson measure, which is corrected and proven in [38].

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Key words and phrases. M-type 2 Banach space, set-valued stochastic processes, Set-valued stochastic integral, Poisson random measure, Compensated Poisson random measure.

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This paper is organized as follows: In Section 2 we give the notations and the preliminaries in the set-valued theory. Section 3 is on the definitions and results of stochastic integrals for set-valued $\mathcal{F}$-predictable processes with respect to $N(ds dz)$ and $\tilde{N}(ds dz)$.

2. Preliminaries

Let $\Omega, \mathcal{F}, P$ be a complete probability space, $\{\mathcal{F}_t\}_{t \geq 0}$ a filtration satisfying the usual conditions, that is: $\mathcal{F}_0$ includes all $P$-null sets in $\mathcal{F}$, the filtration is non-decreasing and right continuous. Let $\mathcal{B}(E)$ be the Borel field of a topological space $E$, $(X, \| \cdot \|)$ a separable Banach space equipped with the norm $\| \cdot \|$ and $\mathbf{K}(X)$ (resp. $\mathbf{K}_b(X), \mathbf{K}_c(X)$) the family of all nonempty closed (resp. closed bounded, closed convex) subsets of $X$. Let $1 \leq p < +\infty$ and $L^p(\Omega, \mathcal{F}, P; X)$ (denoted briefly by $L^p(\Omega; X)$) be the Banach space of equivalence classes of $X$-valued $\mathcal{F}$-measurable functions $f : \Omega \rightarrow X$ such that the norm

$$\| f \|_p = \left\{ \int_{\Omega} \| f(\omega) \|^p dP \right\}^{1/p}$$

is finite. An $X$-valued function $f$ is called $L^p$-integrable if $f \in L^p(\Omega; X)$.

A set-valued function $F : \Omega \rightarrow \mathbf{K}(X)$ is said to be measurable if for any open set $O \subset X$, the inverse $F^{-1}(O) := \{ \omega \in \Omega : F(\omega) \cap O \neq \emptyset \}$ belongs to $\mathcal{F}$. Such a function $F$ is called a set-valued random variable. Let $\mathcal{M}(\Omega, \mathcal{F}, P; \mathbf{K}(X))$ be the family of all set-valued random variables, which is briefly denoted by $\mathcal{M}(\Omega; \mathbf{K}(X))$.

For any open subset $O \subset X$, set

$$Z_O := \{ E \in \mathbf{K}(X) : E \cap O \neq \emptyset \},$$

$$C := \{ Z_O : O \subset X, \text{ O is open} \},$$

and let $\sigma(C)$ be the $\sigma$-algebra generated by $C$.

A set-valued function $F : \Omega \rightarrow \mathbf{K}(X)$ is measurable if and only if $F$ is $\mathcal{F}/\sigma(C)$-measurable.

For $A, B \in 2^X$ (the power set of $X$), $H(A, B) \geq 0$ is defined by

$$H(A, B) := \max \{ \sup_{x \in A} \inf_{y \in B} \| x - y \|, \sup_{y \in B} \inf_{x \in A} \| x - y \| \}.$$

$H(A, B)$ for $A, B \in \mathbf{K}_b(X)$ is called the Hausdorff metric. It is well-known that $\mathbf{K}_b(X)$ equipped with the $H$-metric denoted by $(\mathbf{K}_b(X), H)$ is a complete metric space.

The following results are also well-known. (see e.g. [9], [19], [24]).

**Proposition 2.1.** (i) For $A, B, C, D \in \mathbf{K}(X)$, we have

$$H(A + B, C + D) \leq H(A, C) + H(B, D),$$

$$H(A \oplus B, C \oplus D) = H(A + B, C + D),$$

where $A \oplus B := \{ a + b : a \in A, \ b \in B \}$.

(ii) For $A, B \in \mathbf{K}(X)$, $\mu \in \mathbb{R}$, we have

$$H(\mu A, \mu B) = |\mu| H(A, B).$$
For $F \in \mathcal{M}(\Omega, K(X))$, the family of all $L^p$-integrable selections is defined by

$$S^p_F(\mathcal{F}) := \{ f \in L^p(\Omega, \mathcal{F}, P; X) : f(\omega) \in F(\omega) \ a.s. \}.$$ 

In the following, $S^p_F(\mathcal{F})$ is denoted briefly by $S^p_F$. If $S^p_F$ is nonempty, $F$ is said to be $L^p$-integrable. $F$ is called $L^p$-integrably bounded if there exits a function $h \in L^p(\Omega, \mathcal{F}, P; \mathbb{R})$ such that $\|x\| \leq h(\omega)$ for any $x$ and $\omega$ with $x \in F(\omega)$. It is equivalent to that $\|F\|_K \in L^p(\Omega; \mathbb{R})$, where $\|F(\omega)\|_K := \sup_{a \in F(\omega)} \|a\|$. The family of all measurable $K(X)$-valued $L^p$-integrably bounded functions is denoted by $L^p(\Omega, \mathcal{F}, P; K(X))$. Write it for brevity as $L^p(\Omega; K(X))$.

The integral (or expectation) of a set-valued random variable $F$ was defined by Aumann in 1965 (\cite{Aumann}):

$$E[F] := \{ E[f] : f \in S^p_F \}.$$ 

**Proposition 2.2.** (\cite{Aumann}) Let $F \in \mathcal{M}(\Omega, X)$, $1 \leq p < +\infty$. Then $F$ is $L^p$-integrably bounded if and only if $S^p_F$ is nonempty and bounded in $L^p(\Omega; X)$.

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers and $\mathcal{B}_+ := \mathcal{B}(\mathbb{R}_+)$. $\mathbb{N}$ denotes the set of natural numbers. An $X$-valued stochastic process $f = \{f_t : t \geq 0\}$ (or denoted by $f = \{f(t) : t \geq 0\}$) is defined as a function $f : \mathbb{R}_+ \times \Omega \rightarrow X$ with the $\mathcal{F}$-measurable section $f_t$, for $t \geq 0$. We say $f$ is measurable if $f$ is $\mathcal{B}_+ \otimes \mathcal{F}$-measurable. The process $f = \{f_t : t \geq 0\}$ is called $\mathcal{F}_t$-adapted if $f_t$ is $\mathcal{F}_t$-measurable for every $t \geq 0$. $f = \{f_t : t \geq 0\}$ is called predictable if it is $\mathcal{P}$-measurable, where $\mathcal{P}$ is the $\sigma$-algebra generated by all left continuous and $\mathcal{F}_t$-adapted stochastic processes.

In a fashion similar to the $X$-valued stochastic process, a set-valued stochastic process $F = \{F_t : t \geq 0\}$ is defined as a set-valued function $F : \mathbb{R}_+ \times \Omega \rightarrow K(X)$ with $\mathcal{F}$-measurable section $F_t$ for $t \geq 0$. It is called measurable if it is $\mathcal{B}_+ \otimes \mathcal{F}$-measurable, and $\mathcal{F}_t$-adapted if for any fixed $t$, $F_t(\cdot)$ is $\mathcal{F}_t$-measurable. $F = \{F_t : t \geq 0\}$ is called predictable if it is $\mathcal{P}$-measurable.

**Definition 2.3.** (see \cite{Aumann}) An integrable bounded convex set-valued $\mathcal{F}_t$-adapted stochastic process $\{F_t, \mathcal{F}_t : t \geq 0\}$ is called a set-valued $\mathcal{F}_t$-martingale if for any $0 \leq s \leq t$ it holds that $E[F_t|\mathcal{F}_s] = F_s$ in the sense of $S^1_{E[F_t|\mathcal{F}_s]}(F_s) = S^1_{F_s}(F_s)$.

It is called a set-valued submartingale (supermartingale) if for any $0 \leq s \leq t$, $E[F_t|\mathcal{F}_s] \supset F_s$ (resp. $E[F_t|\mathcal{F}_s] \subset F_s$) in the sense of $S^1_{E[F_t|\mathcal{F}_s]}(F_s) \supset S^1_{F_s}(F_s)$ (resp. $S^1_{E[F_t|\mathcal{F}_s]}(F_s) \subset S^1_{F_s}(F_s)$).

### 3. Stochastic Integrals with respect to Poisson Point Processes

#### 3.1. Single Valued Stochastic Integrals w.r.t. Poisson Point Processes

Let $X$ be a separable Banach space and $Z$ be another separable Banach space with $\sigma$-algebra $B(Z)$. A point function $p$ on $Z$ means a mapping $p : D_p \rightarrow Z$, where the domain $D_p$ is a countable subset of $[0, T]$. $p$ defines a counting measure $N_p(dt \mu)$ on $[0, T] \times Z$ (with the product $\sigma$-algebra $B([0, T]) \otimes B(Z)$) by

$$N_p((0, t], U) := \# \{ \tau \in D_p : \tau \leq t, p(\tau) \in U \},$$

$$t \in (0, T], \ U \in B(Z). \quad (3.1)$$

For $0 \leq s < t \leq T$,

$$N_p((s, t], U) := N_p((0, t], U) - N_p((0, s], U). \quad (3.2)$$
In the following, we also write $N_p((0, t], U)$ as $N_p(t, U)$.

A point process is obtained by randomizing the notion of point functions. If there is a continuous $\mathcal{F}_t$-adapted increasing process $\tilde{N}_p$ such that for $U \in \mathcal{B}(Z)$ and $t \in [0, T]$, $\tilde{N}_p(t, U) := N_p(t, U) - \hat{N}_p(t, U)$ is an $\mathcal{F}_t$-martingale, then the random measure $\{\tilde{N}_p(t, U)\}$ is called the compensator of the point process $p$ (or $\{N_p(t, U)\}$ and the process $\{\hat{N}_p(t, U)\}$ is called the compensated point process.

A point process $p$ is called the Poisson Point Process if $N_p(dt, dz)$ is a Poisson random measure on $[0, T] \times Z$. A Poisson point process is stationary if and only if its intensity measure $\nu_p(dt, dz) = E[N_p(dt, dz)]$ is of the form

$$\nu_p(dt, dz) = d\nu(dz)$$

for some measure $\nu(dz)$ on $(Z, \mathcal{B}(Z))$. $\nu(dz)$ is called the characteristic measure of $p$.

Let $\nu$ be a $\sigma$-finite measure on $(Z, \mathcal{B}(Z))$, (i.e. there exists $U_i \in \mathcal{B}(Z), i \in \mathbb{N}$, pairwise disjoint such that $\nu(U_i) < \infty$ for all $i \in \mathbb{N}$ and $Z = \bigcup_{i=1}^{\infty} U_i$, $p = (p_t)$ be the $\mathcal{F}_t$-adapted stationary Poisson point process on $Z$ with the characteristic measure $\nu$ such that the compensator $\hat{N}_p(t, U) = E[N_p(t, U)] = \nu(U)$ (non-random).

The above definitions and notations of Poisson point processes come from [11] and [30]. For convenience, we will omit the subscript $p$ in the above notations.

**PROPOSITION 3.1.** ([31]) Assume $\nu(Z)$ is finite. Then for any $U \in \mathcal{B}(Z)$, both $\{N(t, U), t \in [0, T]\}$ and $\{\tilde{N}(t, U), t \in [0, T]\}$ are stochastic processes with finite variation a.s.

For convenience, from now on, we suppose $\nu$ is a finite measure in the measurable space $(Z, \mathcal{B}(Z))$.

**DEFINITION 3.2.** An $X$-valued mapping $f(t, z, \omega)$ defined on $[0, T] \times Z \times \Omega$ is called $\mathcal{S}$-predictable if the mapping $(t, z, \omega) \rightarrow f(t, z, \omega)$ is $\mathcal{S}/\mathcal{B}(X)$-measurable, where $\mathcal{S}$ is the smallest $\sigma$-algebra on $[0, T] \times Z \times \Omega$ with respect to which all mappings $g : [0, T] \times Z \times \Omega \rightarrow X$ satisfying (i) and (ii) below are measurable:

(i) for each $t \in [0, T]$, the mapping $(z, \omega) \rightarrow g(t, z, \omega)$ is $\mathcal{B}(Z) \otimes \mathcal{F}_t$-measurable;

(ii) for each $(z, \omega) \in Z \times \Omega$, the mapping $t \rightarrow g(t, z, \omega)$ is left continuous.

**REMARK 3.3.** (see e.g. [30]) $\mathcal{S} = \mathcal{P} \otimes \mathcal{B}(Z)$, where $\mathcal{P}$ denotes the $\sigma$-field on $[0, T] \times \Omega$ generated by all left continuous and $\mathcal{F}_t$-adapted processes.

Set

$$\mathcal{L} = \left\{ f(t, z, \omega) : f \text{ is } \mathcal{S}\text{-predictable and } \right\}$$

$$E \left[ \int_0^T \int_Z \|f(t, z, \omega)\|^2 \nu(dz)dt \right] < \infty$$

equipped with the norm

$$\|f\|_{\mathcal{S}} := \left( E \left[ \int_0^T \int_Z \|f(t, z, \omega)\|^2 \nu(dz)dt \right] \right)^{1/2}.$$

Let $S$ be the subspace of those $f \in \mathcal{L}$ for which there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ such that
SOME RESULTS

\[ f(t, z, \omega) = f(0, z, \omega) \chi_{\{0\}}(t) + \sum_{i=1}^{n} \chi_{(t_{i-1}, t_i]}(t)f(t_{i-1}, z, \omega). \]

Let \( f \) be in \( S \) and

(3.4) \[ f(t, z, \omega) = f(0, z, \omega) \chi_{\{0\}}(t) + \sum_{i=1}^{n} \chi_{(t_{i-1}, t_i]}(t)f(t_{i-1}, z, \omega), \]

where \( 0 = t_0 < t_1 < \cdots < t_n = T \) is a partition of \([0, T]\). Define

\[ J_T(f) = \int_{0}^{T+} \int_{Z} f(s-, z, \omega) N(dtdz) \]

(3.5) \[ := \sum_{i=1}^{n} \int_{Z} f(t_{i-1}, z, \omega) N((t_{i-1}, t_i], dz), \]

and

\[ I_T(f) = \int_{0}^{T+} \int_{Z} f(s-, z, \omega) \tilde{N}(dtdz) \]

(3.6) \[ := \sum_{i=1}^{n} \int_{Z} f(t_{i-1}, z, \omega) \tilde{N}((t_{i-1}, t_i], dz), \]

where \( \int_{Z} f(t_{i-1}, z, \omega) N((t_{i-1}, t_i], dz) \) and \( \int_{Z} f(t_{i-1}, z, \omega) \tilde{N}((t_{i-1}, t_i]dz) \) are the Bochner integrals.

The notation \( \int_{0}^{T+} \) means \( \int_{(0,T]} \).

For any integer \( 0 \leq k \leq n \), let

\[ M_k = \sum_{i=1}^{k} \int_{Z} f(t_{i-1}, z, \omega) \tilde{N}((t_{i-1}, t_i], dz) \]

then \( M_k \) is \( \mathcal{F}_{t_k} \)-measurable, \( E[M_k] = 0 \), \( E[I_T(f)] = E[M_n] = 0 \) and

(3.7) \[ E[M_k | \mathcal{F}_{t_k-1}] = E[(M_{k-1} + \int_{Z} f(t_{k-1}, z, \omega) \tilde{N}((t_{k-1}, t_k], dz) | \mathcal{F}_{t_k-1}] \]

\[ = M_{k-1} + E[\int_{Z} f(t_{k-1}, z, \omega) \tilde{N}((t_{k-1}, t_k], dz) | \mathcal{F}_{t_k-1}] \]

\[ = M_{k-1} + \int_{Z} f(t_{k-1}, z, \omega) E[\tilde{N}((t_{k-1}, t_k], dz)] = M_{k-1}. \]

For any \( t \in (0, T] \), define

\[ J_t(f) = \int_{0}^{t+} \int_{Z} f(s-, z, \omega) N(dzds) \]

(3.8) \[ := \sum_{i=1}^{n} \int_{Z} f(t_{i-1}, z, \omega) N((t_{i-1} \land t, t_i \land t], dz), \]

and

\[ I_t(f) = \int_{0}^{t+} \int_{Z} f(s-, z, \omega) \tilde{N}(dzds) \]

(3.9) \[ := \sum_{i=1}^{n} \int_{Z} f(t_{i-1}, z, \omega) \tilde{N}((t_{i-1} \land t, t_i \land t], dz). \]
LEMMA 3.4. ([31]) For any $f \in \mathcal{S}$, both $\{I_t(f)\}$ and $\{J_t(f)\}$ are $\mathcal{F}_t$-adapted integrable processes. Moreover, $\{I_t(f)\}$ is an $X$-valued right continuous martingale. And for any $t \in (0, T]$,

$$E[\int_0^{t+} \int_Z f(s-, z, \omega) \tilde{N}(dsdz)] = 0,$$

$$E[\int_0^{t+} \int_Z f(s-, z, \omega) N(dsdz)] = \int_0^{t+} \int_Z E[f(s-, z, \omega)] ds \nu(dz).$$

In order to extend the integrand from the step function which belongs to $\mathcal{S}$ to a more general case (belongs to $\mathcal{L}$), it is necessary to add some assumption in the Banach space $X$. Now we assume $X$ is of $M$-type 2 below.

DEFINITION 3.5. ([5]) A Banach space $(X, \| \cdot \|)$ is called $M$-type 2 if and only if there exists a constant $C_X > 0$ such that for any $X$-valued martingale $\{M_k\}$, it holds that

$$\sup_k E[\|M_k\|^2] \leq C_X \sum_k E[\|M_k - M_{k-1}\|^2].$$

THEOREM 3.6. ([31]) Let $X$ be of $M$-type 2 and $(Z, \mathcal{B}(Z))$ a separable Banach space with finite measure $\nu$. Let $p$ be a stationary Poisson process with the characteristic measure $\nu$ and let $f$ be in $\mathcal{S}$. Then there exists a constant $C$ such that

$$E[\sup_{0 < s \leq t} \left\| \int_0^{s+} \int_Z f(\tau-, z, \omega) \tilde{N}(d\tau dz) \right\|^2] \leq C \int_0^{t} \int_Z E[\|f(s, z, \omega)\|^2] ds \nu(dz),$$

and

$$E[\sup_{0 < s \leq t} \left\| \int_0^{s+} \int_Z f(\tau-, z, \omega) N(d\tau dz) \right\|^2] \leq C \int_0^{t} \int_Z E[\|f(s, z, \omega)\|^2] ds \nu(dz),$$

where $C$ depends on the constant $C_X$ in Definition 3.5.

LEMMA 3.7. ([31]) $\mathcal{S}$ is dense in $\mathcal{L}$ with respect to the norm $\| \cdot \|_{\mathcal{L}}$.

By Lemma 3.7, for any $f \in \mathcal{L}$, there exist a sequence $\{f^n : n \in \mathbb{N}\}$ in $\mathcal{S}$ such that $\{f^n\}$ converges to $f$ with respect to $\| \cdot \|_{\mathcal{L}}$ and the sequence

$$\left\{ \int_0^{t+} \int_Z f^n(s-, z, \omega) \tilde{N}(dsdz), n \in \mathbb{N} \right\}$$

converges to a limit in $L^2$-sense. We denote the limit by

$$I_t(f) = \int_0^{t+} \int_Z f(s-, z, \omega) \tilde{N}(dsdz),$$

which is called the stochastic integral of $f$ with respect to the compensated Poisson random measure $\tilde{N}(dsdz)$. Similarly, we can define the stochastic integral of $f$ with respect to the Poisson random measure $N(dsdz)$, denoted by

$$J_t(f) = \int_0^{t+} \int_Z f(s-, z, \omega) N(dsdz).$$
Similarly, for any $0 < s < t < T$,  
\[
\int_{s}^{t} \int_{Z} f(\tau-, z, \omega) \tilde{N}_{p}(d\tau dz)
\]
and  
\[
\int_{s}^{t} \int_{Z} f(\tau-, z, \omega) N(d\tau dz)
\]
can be well defined.

**Remark 3.8.** When the measure $\nu$ is finite, for any $U \in \mathcal{B}(Z)$, the processes \{\(N(t, U)\)\} and \{\(\tilde{N}(t, U)\)\} are both of finite variation a.s. Then the stochastic integrals coincide with the Lebesgue-Stieltjes integrals.

**Corollary 3.9.** ([31]) Let $X$ be of $M$-type 2 and $(Z, \mathcal{B}(Z))$ a separable Banach space with finite measure $\nu$. Let $p$ be a stationary Poisson process with the characteristic measure $\nu$ and let $f$ be in $\mathcal{L}$. Then there exists a constant $C$ such that  
\[
E\left[ \sup_{0<s\leq t} \left\| \int_{0}^{s+} \int_{Z} f(\tau-, z, \omega) \tilde{N}(d\tau dz) \right\|^2 \right] \leq C \int_{0}^{t} \int_{Z} E\left[ \left\| f(s, z, \omega) \right\|^2 \right] ds \nu(dz),
\]
and  
\[
E\left[ \sup_{0<s\leq t} \left\| \int_{0}^{s+} \int_{Z} f(\tau-, z, \omega) N(d\tau dz) \right\|^2 \right] \leq C \int_{0}^{t} \int_{Z} E\left[ \left\| f(s, z, \omega) \right\|^2 \right] ds \nu(dz),
\]
where $C$ depends on the constant $C_X$ in Definition 3.5.

**Corollary 3.10.** ([31]) For any $f \in \mathcal{L}$, both \{\(I_t(f)\)\} and \{\(J_t(f)\)\} are $\mathcal{F}_t$-adapted square-integrable processes. Moreover, \{\(I_t(f)\)\} is an $X$-valued right continuous martingale. And for any $t \in (0, T]$,
\[
E\left[ \int_{0}^{t+} \int_{Z} f(s-, z, \omega) \tilde{N}(dsdz) \right] = 0,
\]
\[
E\left[ \int_{0}^{t+} \int_{Z} f(s-, z, \omega) N(dsdz) \right] = \int_{0}^{t} \int_{Z} E[f(s, z, \omega)] ds \nu(dz),
\]

### 3.2. Set-Valued Stochastic Integrals w.r.t. Poisson Point Processes

A set-valued stochastic process $F = \{F_t\} : [0, T] \times Z \times \Omega \rightarrow K(X)$ is called $\mathcal{S}$-predictable if $F(z, t, \omega)$ is $\mathcal{S}/\sigma(C)$-measurable.

Set  
\[
\mathcal{M} = \left\{ F(t, z, \omega) : F \text{ is } \mathcal{S}-\text{predictable and } \int_{0}^{T} \int_{Z} \|F(t, z, \omega)\|^2_K dt \nu(dz) < \infty \right\}
\]
Given a set-valued stochastic process \( \{ F(t, z, \omega) \} \), the \( X \)-valued stochastic process \( \{ f(t, z, \omega) \} \) is called an \( \mathcal{S} \)-selection if \( f(t, z, \omega) \in F(t, z, \omega) \) for all \((t, z, \omega)\) and \( f \in \mathcal{S} \). By Proposition ??, for \( F \in \mathcal{A} \), the \( \mathcal{S} \)-selection exists and satisfies \( E \left[ \int_0^T \int_Z ||f(t, z, \omega)||^2 dt \nu(dz) \right] < \infty \) since
\[
E \left[ \int_0^T \int_Z ||f(t, z, \omega)||^2 dt \nu(dz) \right] \leq E \left[ \int_0^T \int_Z ||F(t, z, \omega)||^2 dt \nu(dz) \right] < \infty,
\]
which means \( f \in \mathcal{L} \). The family of all \( f \) which belongs to \( \mathcal{L} \) and satisfies \( f(t, z, \omega) \in F(t, z, \omega) \) for a.e. \((t, z, \omega)\) is denoted by \( S(F) \), that is
\[
S(F) = \{ f \in \mathcal{L} : f(t, z, \omega) \in F(t, z, \omega) \text{ for a.e. } (t, z, \omega) \}.
\]

Set
\[
\tilde{\Gamma}_t := \left\{ \int_0^{t+} \int_Z f(s-, z, \omega) \tilde{N}(dsdz) : (f(t))_{t \in [0, T]} \in S(F) \right\},
\]
\[
\Gamma_t := \left\{ \int_0^t \int_Z f(s-, z, \omega) N(dsdz) : (f(t))_{t \in [0, T]} \in S(F) \right\}.
\]

**Remark 3.11.** It is easy to see for any \( t \in [0, T] \), \( \tilde{\Gamma}_t \) and \( \Gamma_t \) are the subsets of \( L^2(\Omega, \mathcal{F}_t, P; X) \). Furthermore, if \( \{ F_t, \mathcal{F}_t : t \in [0, T] \} \) is convex, then so are \( \tilde{\Gamma}_t \) and \( \Gamma_t \).

Let \( \text{def}\tilde{\Gamma}_t \) (resp. \( \text{def}\Gamma_t \)) denote the decomposable set of \( \tilde{\Gamma}_t \) (resp. \( \Gamma_t \)) with respect to \( \mathcal{F}_t \), \( \overline{\text{def}\tilde{\Gamma}_t} \) (resp. \( \overline{\text{def}\Gamma_t} \)) the decomposable closed hull of \( \tilde{\Gamma}_t \) (resp. \( \Gamma_t \)) with respect to \( \mathcal{F}_t \), where the closure is taken in \( L^1(\Omega, X) \). That is to say, for any \( g \in \text{def}\tilde{\Gamma}_t \) (resp. \( \overline{\text{def}\tilde{\Gamma}_t} \)) and any given \( \epsilon > 0 \), there exists a finite \( \mathcal{F}_t \)-measurable partition \( \{ A_1, \ldots, A_m \} \) of \( \Omega \) and \( (f^1(t))_{t \in [0, T]}, \ldots, (f^m(t))_{t \in [0, T]} \in S(F) \) such that
\[
\| g - \sum_{k=1}^m \chi_{A_k} \int_0^{t+} \int_Z f^k(s-, z, \omega) \tilde{N}(dsdz) \|_{L^1} < \epsilon.
\]

(resp. \( \| g - \sum_{k=1}^m \chi_{A_k} \int_0^t \int_Z f^k(s-, z, \omega) N(dsdz) \|_{L^1} < \epsilon \))

Similar to Theorem 4.1 in [32], we have

**Theorem 3.12.** Let \( \{ F_t, \mathcal{F}_t : t \in [0, T] \} \in \mathcal{A} \), then for any \( t \in [0, T] \), \( \overline{\text{def}\Gamma_t} \subset L^1(\Omega, \mathcal{F}_t, P; X) \). Moreover, there exists a set-valued random variable \( J_t(F) \in \mathcal{M}(\Omega, \mathcal{F}_t, P; K(X)) \) such that \( S_{\text{def}\Gamma_t}(F_t) = \text{def}\Gamma_t \). Similarly, there exists a set-valued random variable \( I_t(F) \in \mathcal{M}(\Omega, \mathcal{F}_t, P; K(X)) \) such that \( S_{\text{def}\Gamma_t}(F_t) = \overline{\text{def}\Gamma_t} \).

**Definition 3.13.** The set-valued stochastic processes \( \{ J_t(F) \}_{t \in [0, T]} \) and \( \{ I_t(F) \}_{t \in [0, T]} \) defined as above are called the stochastic integrals of \( \{ F_t, \mathcal{F}_t : t \in [0, T] \} \in \mathcal{A} \) with respect to the Poisson random measure \( N(dsdz) \) and the compensated random measure \( \tilde{N}(dsdz) \) respectively. For each \( t \), we denote \( I_t(F) = \int_0^t \int_Z F(s-, z, \omega) \tilde{N}(dsdz) \), \( J_t(F) = \int_0^{t+} \int_Z F(s-, z, \omega) N(dsdz) \). Similarly, for \( 0 < s < t \), we can also define the set-valued random variable \( I_s(F) = \int_s^t \int_Z F(\tau-, z, \omega) \tilde{N}(d\tau dz) \), \( J_s(F) = \int_s^t \int_Z F(\tau-, z, \omega) N(d\tau dz) \).

For brevity, the integral \( \int_0^t \int_Z h(s-, z, \omega) \tilde{N}(dsdz) \) (\( \int_0^{t+} \int_Z h(s-, z, \omega) \tilde{N}(dsdz) \)) also will be denoted by \( \int_0^t \int_Z h_s \tilde{N}(dsdz) \) (\( \int_0^{t+} \int_Z h_s \tilde{N}(dsdz) \) resp.), where \( h \) is an \( X \)-valued or \( K(X) \)-valued integrand.
PROPOSITION 3.14. ([38]) Assume set-valued stochastic processes \( \{F_t, \mathcal{F}_t : t \in [0, T]\} \) and \( \{G_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M} \). Then
\[
J_t(F + G) = \text{cl} \{J_t(F) + J_t(G)\} \text{ a.s and } I_t(F + G) = \text{cl} \{I_t(F) + I_t(G)\} \text{ a.s.,}
\]
where the \( \text{cl} \) stands for the closure in \( X \).

THEOREM 3.15. ([31]) Assume a set-valued stochastic process \( \{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M} \). Then \( \{J_t(F)\} \) and \( \{I_t(F)\} \) are integrably bounded.

THEOREM 3.16. ([31, 38]) Let a convex set-valued stochastic process \( \{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M} \), then the stochastic integral \( \{I_t(F), \mathcal{F}_t : t \in [0, T]\} \) is a set-valued submartingale but not a set-valued martingale.

REMARK 3.17. With the assumption of \( \mathcal{F} \) being separable with respect to the probability measure \( P \), Theorem 3.7 in [31] pointed out that the integral \( \{I_t(F)\} \) is a set-valued martingale. But unfortunately, now we found there is a gap in the proof. In fact, \( \{I_t(F)\} \) is not a set-valued martingale except for special case (the singletons). The counterexample and rigorous proof are given in [38].

THEOREM 3.18. ([38]) Assume a set-valued stochastic process \( \{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M} \). Then both \( \{J_t(F)\} \) and \( \{I_t(F)\} \) are \( L^2 \)-integrably bounded.

THEOREM 3.19. ([31]) (Castaing representation of set-valued stochastic integral)
Assume \( \mathcal{F} \) is separable with respect to the probability measure \( P \). Then for a set-valued stochastic process \( \{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M} \), there exists a sequence \( \{(f_t^i)_{t \in [0, T]} : i = 1, 2, \ldots \} \subset S(F) \) such that for each \( t \in [0, T], z \in \mathcal{Z}, F(t, z, \omega) = d\{(f_t^i(z, \omega)) : i = 1, 2, \ldots \} \text{ a.s.}, \)
\[
I_t(F)(\omega) = \text{cl}\left\{ \int_0^{t+} \int_{\mathcal{Z}} f_s^i(z, \omega)\tilde{N}(dsdz)(\omega) : i = 1, 2, \ldots \right\} \text{ a.s.}
\]
and
\[
J_t(F)(\omega) = \text{cl}\left\{ \int_0^{t+} \int_{\mathcal{Z}} f_s^i(z, \omega)N(dsdz)(\omega) : i = 1, 2, \ldots \right\} \text{ a.s.}
\]

THEOREM 3.20. ([38]) Assume \( \mathcal{F} \) is separable with respect to \( P \). Let \( \{F_t\}_{t \in [0, T]} \) and \( \{G_t\}_{t \in [0, T]} \) be set-valued stochastic processes in \( \mathcal{M} \). Then for all \( t \), it follows that
\[
E[H(\int_0^{t+} \int_{\mathcal{Z}} F(s, - z, \omega)N(dsdz), \int_0^{t+} \int_{\mathcal{Z}} G(s, - z, \omega)N(dsdz))] \leq E[\int_0^{t+} \int_{\mathcal{Z}} H(F(s, z, \omega), G(s, z, \omega))N(dsdz)]
\]
\[
= E[\int_0^t \int_{\mathcal{Z}} H(F(s, z, \omega), G(s, z, \omega))dsdz]
\]
and
\[
E[H^2(\int_0^{t+} \int_{\mathcal{Z}} F(s, - z, \omega)N(dsdz), \int_0^{t+} \int_{\mathcal{Z}} G(s, - z, \omega)N(dsdz))] \leq CE[\int_0^{t+} \int_{\mathcal{Z}} H^2(F(s, z, \omega), G(s, z, \omega))N(dsdz)]
\]
\[
= CE[\int_0^t \int_{\mathcal{Z}} H^2(F(s, z, \omega), G(s, z, \omega))dsdz]
\]
where \( C \) is the constant appearing in Corollary 3.9.

REFERENCES

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