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Generalized Split Feasibility Problems and Nonlinear Analysis

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Abstract. In this article, motivated by the idea of the split feasibility problem and results for solving the problem, we consider generalized split feasibility problems. Then, using nonlinear analysis, we establish weak and strong convergence theorems which are related to the problems. As applications, we get well-known and new weak and strong convergence theorems which are connected with fixed point problem, split feasibility problem and equilibrium problem.

1 Introduction

Let H be a real Hilbert space and let C be a non-empty, closed and convex subset of H. A mapping $U : C \to H$ is called *inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\langle x-y, Ux-Uy\rangle \geq \alpha\|Ux-Uy\|^2, \quad \forall x,y \in C.$$

Such a mapping U is called *α-inverse strongly monotone.* Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be non-empty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility peoblem* [6] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [5] considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [5] is to find a point $z \in H_1$ such that

$$z \in \left(\cap_{i=1}^{m} A_i^{-1}0\right) \cap (\cap_{j=1}^{n} T_j^{-1}(B_j^{-1}0)),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I-P_Q)A$ in the split feasibility peoblem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [2], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is non-empty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I-P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility and generalized split feasibility problems including the split common null point problem; see, for instance, [5, 7, 17, 33]. In the study, the authors used established results for solving the problems. In particular, established convergence theorems have been used for finding solutions of the problems. We also know many existence and convergence theorems for inverse strongly monotone mappings in Hilbert spaces; see, for instance, [9, 12, 16, 18, 23, 24, 28].

In this article, motivated by the idea of the split feasibility problem and results for solving the problem, we consider generalized split feasibility problems. Then, using nonlinear analysis, we obtain weak and strong convergence theorems which are related to the problems. We first obtain some fundamental properties for inverse strongly monotone mappings and resolvents of maximal monotone operators in Hilbert spaces. For example, we extend the result of (1.1) from metric projections to nonexpansive mappings. Then using these properties, we establish two weak convergence theorems and two strong convergence theorems for finding solutions of the generalized split feasibility peoblems. The results are generalizations of weak and strong convergence theorems which have already been obtained. As applications, we get well-known and new weak and strong convergence theorems which are connected with fixed point problem, the split feasibility problem and an equilibrium problem.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. From [27] we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$ we have that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

We also know that for $x, y, u, v \in H$

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

A Hilbert space satisfies Opial's condition, that is,

$$\liminf_{n\to\infty} \|x_n - u\| < \liminf_{n\to\infty} \|x_n - v\|$$

if $x_n \rightharpoonup u$ and $u \neq v$; see [19]. Let C be a non-empty, closed and convex subset of H and let $T\colon C \to H$ be a mapping. We denote by $F(T)$ be the set of fixed points of T. A mapping $T : C \to H$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ is called *firmly nonexpansive* if $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in C$. If a mapping T is firmly nonexpansive, then it is nonexpansive. If $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex; see [27]. For a non-empty, closed and convex subset C of H, the nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such a mapping P_C is also called the *metric projection* of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see, for instance, [25]. Let B be a set-valued mapping of H into 2^H. The effective domain of B

is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A set-valued mapping B is said to be *monotone* on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone mapping B on H is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} \colon H \to D(B)$, which is called the *resolvent* of B for $r > 0$. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$. The following lemma is crucial in order to prove the main theorems.

Lemma 2.1 ([24]). *Let H be a Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

From Lemma 2.1, we have that

$$\|J_s x - J_t x\| \leq (|s-t|/s)\,\|x - J_s x\| \tag{2.3}$$

for all $s, t > 0$ and $x \in H$; see also [10, 25]. The following lemmas are also used to prove the main theorems.

Lemma 2.2 ([22]). *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$ and let $\{v_n\}$ and $\{w_n\}$ be sequences in H such that $\limsup_{n\to\infty} \|v_n\| \leq c$, $\limsup_{n\to\infty} \|w_n\| \leq c$ and $\limsup_{n\to\infty} \|\alpha_n v_n + (1-\alpha_n)w_n\| = c$ for some c. Then $\lim_{n\to\infty} \|v_n - w_n\| = 0$.*

Lemma 2.3 ([29]). *Let H be a Hilbert space and let E be a non-empty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - x\| \leq \|x_n - x\|$ for all $n \in \mathbb{N}$ and $x \in E$, then $\{P_E x_n\}$ converges strongly to some $z \in E$, where P_E is the metric projection on H onto E.*

Using Opial's theorem [19], we have the following lemma; see, for instance, [27].

Lemma 2.4. *Let H be a Hilbert space and let $\{x_n\}$ be a sequence in H such that there exists a non-empty subset $E \subset H$ satisfying (i) and (ii):*

(i) For every $x^* \in E$, $\lim_{n\to\infty} \|x_n - x^*\|$ exists:
(ii) if a subsequence $\{x_{n_j}\} \subset \{x_n\}$ converges weakly to x^*, then $x^* \in E$.

Then there exists $x_0 \in E$ such that $x_n \rightharpoonup x_0$.

We also know the following lemmas:

Lemma 2.5 ([3], [32]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6 ([14]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the floowing hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \ldots$ and $\tau(n) \to \infty$;
(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3 Weak Convergence Theorems

Let H be a Hilbert space and let S be a firmly nonexpansive mapping of H into itself with $F(S) \neq \emptyset$. Then we have that

$$\langle x - Sx, Sx - y \rangle \geq 0 \tag{3.1}$$

for all $x \in H$ and $y \in F(S)$. In fact, we have that for all $x \in H$ and $y \in F(S)$

$$\begin{aligned}\langle x - Sx, Sx - y\rangle &= \langle x - y + y - Sx, Sx - y\rangle \\ &= \langle x - y, Sx - y\rangle + \langle y - Sx, Sx - y\rangle \\ &\geq \|Sx - y\|^2 - \|Sx - y\|^2 \\ &= 0.\end{aligned}$$

The following lemmas which were proved by Takahashi, Xu and Yao [30] are crucial for proving our main theorems.

Lemma 3.1 ([30]). *Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then a mapping $A^*(I - T)A : H_1 \to H_1$ is $\frac{1}{2\|AA^*\|}$-inverse strongly monotone.*

Lemma 3.2 ([30]). *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rA^*(I - T)A)z$;
(ii) $0 \in A^*(I - T)Az + Bz$;
(iii) $z \in B^{-1}0 \cap A^{-1}F(T)$.

Now we can prove a weak convergence theorem which is related to the split feasibility problem and generalizes Reich's theorem [20] in a Hilbert space.

Theorem 3.3 ([30]). *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. For any $x_1 = x \in H_1$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy the following:*

$$\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty, \quad 0 < a \leq \lambda_n \leq \frac{1}{\|AA^*\|} \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty.$$

Then $x_n \rightharpoonup z_0 \in B^{-1}0 \cap A^{-1}F(T)$, *where* $z_0 = \lim_{n\to\infty} P_{B^{-1}0\cap A^{-1}F(T)} x_n$.

Let H be a Hilbert space and let C be a non-empty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [13] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2, \quad \forall x, y \in C. \tag{3.2}$$

We prove a weak convergence theorem which is governed generalized hybrid mappings.

Theorem 3.4 ([30]). *Let* H_1 *and* H_2 *be real Hilbert spaces and let* C *be a non-empty, closed and convex subset of* H_1. *Let* $B : H_1 \to 2^{H_1}$ *be a maximal monotone mapping such that the domain of* B *is included in* C *and let* $J_\lambda = (I + \lambda B)^{-1}$ *be the resolvent of* B *for* $\lambda > 0$. *Let* $S : C \to C$ *be a generalized hybrid mapping and let* $T : H_2 \to H_2$ *be a nonexpansive mapping. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Suppose that* $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. *For any* $x_1 = x \in C$, *define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)S(J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n), \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\}$ *and* $\{\lambda_n\}$ *satisfy the following:*

$$0 < c \leq \beta_n \leq d < 1 \text{ and } 0 < a \leq \lambda_n \leq b < \frac{1}{\|AA^*\|}.$$

Then the sequence $\{x_n\}$ *converges weakly to a point* $z_0 \in F(S) \cap B^{-1}0 \cap A^{-1}F(T)$, *where* $z_0 = \lim_{n\to\infty} P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}(x_n)$.

4 Strong Convergence Theorems

In this section, we first prove a strong convergence theorem which generalizes Wittmann's strong convergence theorem [31] in a Hilbert space.

Theorem 4.1 ([1]). *Let* H_1 *and* H_2 *be Hilbert spaces. Let* $B : H_1 \to 2^{H_1}$ *be a maximal monotone mapping and let* $J_\lambda = (I + \lambda B)^{-1}$ *be the resolvent of* B *for* $\lambda > 0$. *Let* $T : H_2 \to H_2$ *be a nonexpansive mapping. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Suppose that* $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence in* H_1 *such that* $u_n \to u$. *Let* $x_1 = x \in H_1$ *and let* $\{x_n\} \subset H_1$ *be a sequence generated by*

$$x_{n+1} = \alpha_n u_n + (1-\alpha_n)J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n$$

for all $n \in \mathbb{N}$, *where* $\{\lambda_n\} \subset (0,\infty)$ *and* $\{\alpha_n\} \subset (0,1)$ *satisfy*

$$0 < a \leq \lambda_n \leq \frac{1}{\|A^*A\|}, \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then $\{x_n\}$ *converges strongly to* $z_0 \in B^{-1}0 \cap A^{-1}F(T)$, *where* $z_0 = P_{B^{-1}0\cap A^{-1}F(T)}u$.

Next, we prove another strong convergence theorem.

Theorem 4.2 ([1]). *Let* H_1 *and* H_2 *be Hilbert spaces. Let* $B : H_1 \to 2^{H_1}$ *be a maximal monotone mapping and let* $J_\lambda = (I+\lambda B)^{-1}$ *be the resolvent of* B *for* $\lambda > 0$. *Let* $T : H_2 \to H_2$ *be a nonexpansive mapping. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Suppose that* $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence in* H_1 *such that* $u_n \to u$. *Let* $x_1 = x \in H_1$ *and let* $\{x_n\} \subset H_1$ *be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n)J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n)$$

for all $n \in \mathbb{N}$, *where* $\{\lambda_n\} \subset (0,\infty)$, $\{\beta_n\} \subset (0,1)$ *and* $\{\alpha_n\} \subset (0,1)$ *satisfy*

$$0 < a \le \lambda_n \le \frac{1}{\|A^*A\|}, \quad 0 < c \le \beta_n \le d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ *converges strongly to* $z_0 \in B^{-1}0 \cap A^{-1}F(T)$, *where* $z_0 = P_{B^{-1}0\cap A^{-1}F(T)}u$.

5 Applications

Let H be a Hilbert space and let f be a proper, lower semicontinuous and convex function of H into $(-\infty,\infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \le f(y), \ \ \forall y \in H\}$$

for all $x \in H$. By Rockafellar [21], it is shown that ∂f is maximal monotone. Let C be a non-empty, closed and convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Then $i_C : H \to (-\infty,\infty]$ is a proper, lower semicontinuous and convex function on H and hence ∂i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda > 0$ as follows:

$$J_\lambda x = (I + \lambda \partial i_C)^{-1}x, \quad \forall x \in H, \ \lambda > 0.$$

On the other hand, for any $u \in C$, we also define the normal cone $N_C(u)$ of C at u as follows:

$$N_C(u) = \{z \in H : \langle z, y - u\rangle \le 0, \ \ \forall y \in C\}.$$

Then we have that for any $x \in C$

$$\begin{aligned}\partial i_C(x) &= \{z \in H : i_C(x) + \langle z, y - x\rangle \le i_C(y), \quad \forall y \in H\} \\ &= \{z \in H : \langle z, y - x\rangle \le 0, \quad \forall y \in C\} \\ &= N_C(x).\end{aligned}$$

Thus we have that

$$\begin{aligned}u = J_\lambda x &\Leftrightarrow (I + \lambda \partial i_C)^{-1} x = u \Leftrightarrow x \in u + \lambda \partial i_C(u) \\ &\Leftrightarrow x \in u + \lambda N_C(u) \Leftrightarrow x - u \in \lambda N_C(u) \\ &\Leftrightarrow \langle x - u, y - u\rangle \le 0, \quad \forall y \in C \\ &\Leftrightarrow P_C(x) = u.\end{aligned}$$

Putting $B = \partial i_C$ in Theorems 3.3 and 3.4, we have $J_{\lambda_n} = P_C$ for any $n \in \mathbb{N}$. Using this fact, we have the following theorem of Reich [20] from Theorem 3.3; see also Takahashi [26].

Theorem 5.1 ([20]). *Let C be a non-empty, closed and convex subset of a Hilbert space H and let $U : C \to C$ be a nonexpansive mapping such that $F(U) \neq \emptyset$. For any $x_1 = x \in C$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) U x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset (0,1)$ satisfies

$$\sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty.$$

Then $x_n \rightharpoonup z_0 \in F(U)$, where $z_0 = \lim_{n\to\infty} P_{F(U)} x_n$.

Proof. Set $H_1 = H_2 = H$, $B = \partial i_C$, $T = UP_C$ and $A = I$ in Theorem 3.3. Then we have that $F(T) = F(U)$ and $J_\lambda = P_C$ for all $\lambda > 0$. Furthermore, putting $\lambda_n = 1$ for all $n \in \mathbb{N}$, we have that for $x_1 = x \in C$ and $n \in \mathbb{N}$

$$\begin{aligned}x_{n+1} &= \beta_n x_n + (1 - \beta_n) U x_n \\ &= \beta_n x_n + (1 - \beta_n) P_C U P_C x_n \\ &= \beta_n x_n + (1 - \beta_n) P_C T x_n \\ &= \beta_n x_n + (1 - \beta_n) P_C (x_n - (I - T) x_n) \\ &= \beta_n x_n + (1 - \beta_n) P_C (I - 1 \cdot I^*(I - T) I) x_n.\end{aligned}$$

Thus we have the desired result from Theorem 3.3. □

Using Theorem 4.1, we also have Wittmann's strong convergence theorem [31].

Theorem 5.2 ([31]). *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let U be a nonexpansive mapping of C into itself such that $F(U) \neq \emptyset$. Let $u \in C$, $x_1 = x \in C$ and let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) U x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $F(U)$, where $z_0 = P_{F(U)}u$.

Let C be a non-empty, closed and convex subset of a real Hilbert space H, let $f : C \times C \to \mathbb{R}$ be a bifunction. Then we consider the following equilibrium problem: Find $z \in C$ such that

$$f(z, y) \geq 0, \quad \forall y \in C. \tag{5.1}$$

The set of such $z \in C$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{z \in C : f(z, y) \geq 0, \ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$.

We know the following lemmas; see, for instance, [4] and [8].

Lemma 5.3 ([4]). *Let C be a nonempty closed convex subset of H, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 5.4 ([8]). *For $r > 0$ and $x \in H$, define the resolvent $T_r : H \to C$ of f for $r > 0$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}, \quad \forall x \in H.$$

Then, the following hold:

(i) T_r is single-valued;
(ii) T_r is firmly nonexpansive, i.e., for all $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(f)$;
(iv) $EP(f)$ is closed and convex.

Takahashi, Takahashi and Toyoda [24] showed the following.

Theorem 5.5 ([24]). *Let C be a no-nempty, closed and convex subset of a Hibert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Define A_f as follows:*

$$A_f(x) = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then $EP(f) = A_f^{-1}(0)$ and A_f is maximal monotone which the domain of A_f is included in C. Furthermore,

$$T_r(x) = (I + rA_f)^{-1}(x), \quad \forall r > 0.$$

We obtain the following theorem from Theorem 3.3.

Theorem 5.6. *Let H_1 and H_2 be Hilbert spaces. Let C be a non-empty, closed and convex subset of H_1. Let $f : C\times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 5.5. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap A^{-1}F(T) \neq \emptyset$. For $x_1 = x \in H_1$, define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n), \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the following:

$$\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty, \quad 0 < a \leq \lambda_n \leq \frac{1}{\|AA^*\|} \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty.$$

Then $x_n \rightharpoonup z_0 \in EP(f) \cap A^{-1}F(T)$, where $z_0 = \lim_{n\to\infty} P_{EP(f)\cap A^{-1}F(T)}x_n$.

Proof. Define A_f for the bifunction f and set $B = A_f$ in Theorem 3.3. Thus we have the desired result from Theorem 3.3. □

As in the proof of Theorem 5.6, we obtain the following result from Theorem 3.4.

Theorem 5.7. *Let H_1 and H_2 be Hilbert spaces. Let C be a non-empty, closed and convex subset of a real Hilbert space H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 5.5. Let $S : C \to C$ be a generalized hybrid mapping and let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $F(S) \cap EP(f) \cap A^{-1}F(T) \neq \emptyset$. For $x_1 = x \in C$, define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)ST_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy

$$0 < c \leq \beta_n \leq d < 1 \quad \text{and} \quad 0 < a \leq \lambda_n \leq b < \frac{1}{\|AA^*\|}.$$

Then the sequence $\{x_n\}$ converges weakly to a point $z_0 \in F(S) \cap EP(f) \cap A^{-1}F(T)$, where $z_0 = \lim_{n\to\infty} P_{F(S)\cap EP(f)\cap A^{-1}F(T)}x_n$.

Using Theorem 4.1, we obtain the following strong convergence theorem.

Theorem 5.8. *Let H_1 and H_2 be Hilbert spaces. Let C be a non-empty, closed and convex subset of H_1. Let $f : C\times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 5.5. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap A^{-1}F(T) \neq \emptyset$. Let $u \in H_1$, $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0,\infty)$ and $\{\alpha_n\} \subset (0,1)$ satisfy

$$0 < a \leq \lambda_n \leq \frac{1}{\|A^*A\|}, \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $EP(f)\cap A^{-1}F(T)$, where $z_0 = P_{EP(f)\cap A^{-1}F(T)}u$.

Proof. Define A_f for the bifunction f and set $B = A_f$ and set $u_n = u$ for all $n \in \mathbb{N}$ in Theorem 4.1. Thus we have the desired result from Theorem 4.1. $\square$

As in the proof of Theorem 5.8, we obtain the following result from Theorem 4.2.

Theorem 5.9. *Let H_1 and H_2 be Hilbert spaces. Let C be a non-empty, closed and convex subset of a real Hilbert space H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 5.5. Let $S : C \to C$ be a generalized hybrid mapping and let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap A^{-1}F(T) \neq \emptyset$. Let $u \in H_1$, $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)T_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq \frac{1}{\|A^*A\|}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $EP(f) \cap A^{-1}F(T)$, where $z_0 = P_{EP(f)\cap A^{-1}F(T)}u$.

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