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アダマール空間における写像族の共通不動点近似
Approximation of a common fixed point of mappings in a Hadamard space

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1 Introduction

In this short note, we consider the approximation of common fixed point, that is, the problem to find a sequence converging to a point $z \in X$ such that $z = T_i z$ for every $i \in I$, where $\{T_i\}$ is a given family of mappings defined on a metric space X.

We will focus on the following iterative method proved by Takahashi, Takeuchi, and Kubota [10], which is called the shrinking projection method.

Theorem 1 (Takahashi-Takeuchi-Kubota [10]). *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) = \{z \in C : z = Tz\}$ is nonempty. Let $\{\alpha_n\}$ be a sequence in $[0, a]$, where $0 < a < 1$. For a point $x \in H$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$
\begin{aligned}
&y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\
&C_{n+1} = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\} \cap C_n, \\
&x_{n+1} = P_{C_{n+1}} x
\end{aligned}
$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{F(T)} x \in C$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H.

We remark that the original result of the theorem above deals with a family of nonexpansive mappings. This method has been generalized to the setting of Banach spaces; see also Kimura, Nakajo, and Takahashi [7], Kimura and Takahashi [8].

Recently, the author proved the following result for a nonexpansive mapping defined on a Hadamard space [6]:

Theorem 2 (Kimura [6]). *Let X be a Hadamard space and suppose that a subset $\{z \in X : d(v,z) \leq d(u,z)\}$ is convex for every $u, v \in X$. Let $T : X \to X$ be a nonexpansive mapping such that the set $F(T)$ of fixed points is nonempty. Let $\{\epsilon_n\}$ be a sequence of nonnegative numbers and $\epsilon_0 = \limsup_{n\to\infty} \epsilon_n$. For given points $x_0 \in X$, $x_1 \in X$ and $C_1 = X$, generate a sequence $\{x_n\}$ as follows:*

$$C_{n+1} = \{z \in X : d(Tx_n, z) \leq d(x_n, z)\} \cap C_n,$$
$$x_{n+1} \in C_{n+1} \text{ such that } d(x_0, x_{n+1})^2 \leq d(x_0, C_{n+1})^2 + \epsilon_{n+1}^2,$$

for each $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} d(x_n, Tx_n) \leq 2\epsilon_0.$$

Moreover, if $\epsilon_0 = 0$, then $\{x_n\}$ converges to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection of X onto $F(T)$.

This result shows that the iterative sequence generated as above approximates a fixed point in a certain sense even though we do not require for error terms to converge to 0.

In this paper, we study an iterative scheme generated by a modified version of the shrinking projection method for a finite family of nonexpansive mappings defined on a Hadamard space. We consider an error for obtaining the value of a metric projection and show that the sequence still has a nice property for approximating a common fixed point of the mappings. See also the related results [3, 4, 5, 6].

2 Preliminaries

Let X be a metric space with a metric d. For $x, y \in X$, a mapping $c : [0, l] \to X$ with $l \geq 0$ is called a geodesic with endpoints x, y if $c(0) = x$, $c(l) = y$, and $d(c(t), c(s)) = |t - s|$ for $t, s \in [0, l]$. If a geodesic with endpoints x, y exists for every $x, y \in X$, then we call X a geodesic metric space. In what follows, we assume that a geodesic is uniquely detarmined for every $x, y \in X$. The image of a geodesic with endpoints x, y is called a geodesic segment joining x and y, and we denote it by $[x, y]$. A subset C of X is said to be convex if $[x, y]$ is included in C for any $x, y \in C$.

A geodesic triangle with vertices $x, y, z \in X$ is a union of geodesic segments $[x, y]$, $[y, z]$, and $[z, x]$. We denote it by $\triangle(x, y, z)$. A comparison triangle $\triangle(\overline{x}, \overline{y}, \overline{z})$ in $\mathbb{R}^2$ for $\triangle(x, y, z)$ is a triangle in the 2-dimensional Euclidean space $\mathbb{R}^2$ whose length of each edge is equal to that of the corresponding edge. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = |\overline{x} - \overline{p}|_{\mathbb{R}^2}$. If for any $p, q \in \triangle(x, y, z)$ and their comparison points $\overline{p}, \overline{q} \in \triangle(\overline{x}, \overline{y}, \overline{z})$, the inequality

$$d(p, q) \leq |\overline{p} - \overline{q}|_{\mathbb{R}^2}$$

holds for all triangles in X, we call X a CAT(0) space. A complete CAT(0) space is called a Hadamard space.

For $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that $d(x,z) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$. We denote it by $tx \oplus (1-t)y$. By using the CAT(0) inequality, we have that

$$d(z, tx \oplus (1-t)y)^2 \leq td(z,x)^2 + (1-t)d(z,y)^2 - t(1-t)d(x,y)^2$$

for every $x, y, z \in X$ and $t \in [0,1]$.

A mapping $T : X \to X$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x,y)$ holds for every $x, y \in X$. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{z \in X : Tz = z\}$. We know that $F(T)$ is closed and convex if T is nonexpansive.

Let C be a nonempty closed convex subset of a Hadamard space X. Then for each $x \in X$, there exists a unique point $y_x \in C$ such that $d(x, y_x) = \inf_{y \in C} d(x,y)$. The mapping $x \mapsto y_x$ is called a metric projection onto C and is denoted by P_C. We know that P_C is nonexpansive.

For more details of these notions, see [1].

We need the following lemma for the main theorem, which can be easily deduced from the result in [2].

Lemma 1 (Kimura [2]). *Let X be a Hadamard space, $\{C_n\}$ a sequence of nonempty closed convex subsets in X, and $C_0 = \bigcap_{n=1}^{\infty} C_n$. If C_0 is nonempty, then $\{P_{C_n} x\}$ converges to $P_{C_0} x \in X$ for every $x \in X$, where P_K is the metric projection onto a nonempty closed convex subset K of X.*

3 An approximate sequence to a common fixed point

In this section, we show the main result of this work, which shows convergence of an iterative sequence generated by a modified version of the shrinking projection method. We consider a calculation error for each step, and give an upper bound of the limit of the distance between a point in the sequence and its image of the mappings.

Theorem 3. *Let X be a Hadamard space and suppose that a subset $\{z \in X : d(v,z) \leq d(u,z)\}$ is convex for every $u, v \in X$. Let $\{T_j : j = 0, 1, \ldots, k-1\}$ be a family of nonexpansive mappings such that $F = \bigcap_{j=0}^{k-1} F(T_j)$ is nonempty. Let $\{\epsilon_n\}$ be a sequence in $[0, \infty[$ and let $\epsilon_0 = \limsup_{n\to\infty} \epsilon_n$. For a given point $u \in X$ generate a sequence $\{x_n\}$ as follows: $x_1 = u$, $C_1 = X$, and*

$$\begin{aligned} C_{n+1} &= \{z \in X : d(T_{(n \bmod k)} x_n, z) \leq d(x_n, z)\} \cap C_n, \\ x_{n+1} &\in C_{n+1} \text{ such that } d(u, x_{n+1})^2 \leq d(u, C_{n+1})^2 + \epsilon_{n+1}^2, \end{aligned}$$

for each $n \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} d(x_n, T_j x_n) \leq 4\epsilon_0$$

for every $j \in \{0, 1, \ldots, k-1\}$. *Moreover, if* $\epsilon_0 = 0$, *then* $\{x_n\}$ *converges to* $P_F x_0$, *where* P_F *is the metric projection of* X *onto* F.

For the proof, we employ the technique used in [6]. For the sake of completeness, we give the whole proof.

Proof. We first prove that $\{x_n\}$ is well defined by showing that each C_n is nonempty. Since $d(T_j x, z) \leq d(x, z)$ for every $x \in X$, $z \in F$, and $j = 0, 1, \ldots, k-1$, we have that $F \subset C_n$ for all $n \in \mathbb{N}$. By assumption F is nonempty and so is C_n. Thus the sequence $\{x_n\}$ is well defined. By the continuity of the mertic d, we have that C_n is closed. Since C_n is convex by the assumption of the space, we can define the metric projection P_{C_n} of X onto C_n. Let $p_n = P_{C_n} u$ for all $n \in \mathbb{N}$. Then, by Lemma 1, $\{p_n\}$ converges to $p_0 = P_{C_0} u$, where $C_0 = \bigcap_{n=1}^{\infty} C_n$. Since $x_n \in C_n$ and $d(u, C_n) = d(u, p_n)$, we have that

$$d(u, x_n)^2 \leq d(u, p_n)^2 + \epsilon_n^2$$

for every $n \in \mathbb{N}$. We also have that

$$\begin{aligned} d(p_n, u)^2 &\leq d(\alpha p_n \oplus (1-\alpha) x_n, u)^2 \\ &\leq \alpha d(p_n, u)^2 + (1-\alpha) d(x_n, u)^2 - \alpha(1-\alpha) d(p_n, x_n)^2 \end{aligned}$$

for $\alpha \in]0, 1[$, and hence

$$\alpha d(p_n, x_n)^2 \leq d(x_n, u)^2 - d(p_n, u)^2 \leq \epsilon_n^2.$$

Tending $\alpha \to 1$, we have that $d(p_n, x_n)^2 \leq \epsilon_n^2$, that is,

$$d(p_n, x_n) \leq \epsilon_n$$

for every $n \in \mathbb{N}$. Since $p_n \in C_n$, we also get that

$$d(T_{(n \bmod k)} x_n, p_n) \leq d(x_n, p_n) \leq \epsilon_n$$

for every $n \in \mathbb{N}$.

Let $j \in \{0, 1, \ldots, k-1\}$ and $n \in \mathbb{N}$. Then there exists $i \in \{0, 1, \ldots, k-1\}$ such that

$$(n + i) \bmod k = j.$$

Then we have that

$$\begin{aligned} d(x_n, T_j x_n) &\leq d(x_n, p_{n+i}) + d(p_{n+i}, T_j x_{n+i}) + d(T_j x_{n+i}, T_j x_n) \\ &\leq d(x_n, p_{n+i}) + d(p_{n+i}, T_j x_{n+i}) + d(x_{n+i}, x_n) \\ &\leq d(x_n, p_{n+i}) + d(p_{n+i}, T_{((n+i) \bmod k)} x_{n+i}) + d(x_{n+i}, x_n) \\ &\leq d(x_n, p_{n+i}) + d(p_{n+i}, x_{n+i}) + d(x_{n+i}, x_n) \\ &\leq d(x_n, p_{n+i}) + \epsilon_{n+i} + d(x_{n+i}, x_n). \end{aligned}$$

On the other hand, since

$$d(x_n, p_{n+i}) \leq d(x_n, p_n) + d(p_n, p_{n+i}) \leq \epsilon_n + d(p_n, p_{n+i})$$

and

$$\begin{aligned} d(x_{n+i}, x_n) &\leq d(x_{n+i}, p_{n+i}) + d(p_{n+i}, p_n) + d(p_n, x_n) \\ &\leq \epsilon_{n+i} + \epsilon_n + d(p_{n+i}, p_n), \end{aligned}$$

it follows that

$$d(x_n, T_j x_n) \leq 2(\epsilon_n + \epsilon_{n+i} + d(p_{n+i}, p_n))$$

for every $n \in \mathbb{N}$. Since i can be taken only a finite number of values, we have that

$$\limsup_{n\to\infty} d(x_n, T_j x_n) \leq 4\epsilon_0.$$

For the latter part of the theorem, suppose that $\epsilon_0 = 0$. Then

$$\limsup_{n\to\infty} d(x_n, p_n) \leq \limsup_{n\to\infty} \epsilon_n = 0.$$

It implies that $\lim_{n\to\infty} d(x_n, p_n) = 0$ and thus $\{x_n\}$ converges to $p_0 = P_{C_0}u$. We also have that $\{T_j x_n\}$ converges to p_0 for each $j \in \{0, 1, \ldots, k-1\}$ since

$$0 \leq \liminf_{n\to\infty} d(x_n, T_j x_n) \leq \limsup_{n\to\infty} d(x_n, T_j x_n) \leq 4\epsilon_0 = 0.$$

Since a nonexpansive mapping T_j is continuous, we have that

$$T_j p_0 = T_j \left(\lim_{n\to\infty} x_n \right) = \lim_{n\to\infty} T_j x_n = p_0,$$

that is, $p_0 \in F = \bigcap_{j=0}^{k-1} F(T_j)$. Since $F \subset C_0$, we get that

$$p_0 = P_{C_0}u = P_F u,$$

which completes the proof. □

We remark that there are a number of examples of the spaces satisfying the assumptions in the main theorem, such as nonempty closed convex subsets of Hilbert spaces, those of real Hilbert balls, and others. See, for example, [1, 9].

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