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A SOLUTION TO THE ASM-DPP-TSSCPP BIJECTION PROBLEM IN THE PERMUTATION CASE

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ABSTRACT. We give bijections between permutations and two types of plane partitions, descending (DPP) and totally symmetric self-complementary (TSSCPP). These bijections map the inversion number of the permutation to nice statistics on these DPPs and TSSCPPs. We also discuss the possible extension of this approach to finding bijections between alternating sign matrices and all DPPs and TSSCPPs.

1. INTRODUCTION

Alternating sign matrices (ASMs) and their equinumerous friends, descending plane partitions (DPPs) and totally symmetric self-complementary plane partitions (TSSCPPs), have been bothering combinatorialists for decades by the lack of an explicit bijection between any two of the three sets of objects. (See [6] [7] [1] [12] [5] for these enumerations and bijective conjectures and [3] for the story behind these papers.) In this summary paper, we outline the bijections from [8] and [9] between permutation matrices (which are a subclass of ASMs) and subclasses of DPPs and TSSCPPs. The DPP-permutation bijection proceeds in such a way that the inversion number of the permutation matrix equals the number of parts of the DPP. For the TSSCPP-permutation bijection, we identify the subclass of TSSCPPs corresponding to permutations and give a bijection which yields a direct interpretation for the inversion number on these permutation TSSCPPs.

In Section 2, we define ASMs, DPPs, and TSSCPPs and give bijections within their respective families. We recall the standard bijection from ASMs to monotone triangles. We outline a known bijection from TSSCPPs to non-intersecting lattice paths and then transform these to new objects we call *boolean triangles.*

In Section 3, we identify the permutation subclass of TSSCPPs in terms of the boolean triangles of Section 2. We use this characterization to present a direct bijection between this subclass of TSSCPPs and permutation matrices. Then, in Section 4, we give a bijection between permutation matrices and descending plane partitions with no special parts.

It is not obvious how to extend this bijection to all ASMs, DPPs, and TSSCPPs. In Section 5, we say a few words on the outlook of the general bijection problem.

2. THE OBJECTS AND THEIR ALTER EGOS

In this section, we first define ASMs and recall the standard bijection to monotone triangles. We then define DPPs. Finally, we define TSSCPPs and give bijections with non-intersecting lattice paths and new objects we call boolean triangles. In the following two sections, we give bijections from permutation ASMs to subclasses of TSSCPPs and DPPs via these intermediary objects.

Definition 1. An *alternating sign matrix* (ASM) is a square matrix with entries 0, 1, or -1 whose rows and columns each sum to 1 and such that the nonzero entries in each row and column alternate in sign.

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$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

FIGURE 1. The seven 3×3 ASMs.

See Figure 1 for the seven 3×3 ASMs. It is clear that the alternating sign matrices with no -1 entries are the permutation matrices.

Alternating sign matrices are known to be in bijection with monotone triangles, which are certain strict Gelfand-Tsetlin patterns. See Figure 2 for the seven monotone triangles of order 3 and Figure 7 for the indexing of a general monotone triangle.

Definition 2. A *monotone triangle* of order n is a triangular arrays of integers with i integers in row i for all $1 \leq i \leq n$, bottom row $1\ 2\ 3\ \cdots\ n$, and integer entries $a_{i,j}$ for $1 \leq i \leq n$, $n-i \leq j \leq n-1$ such that $a_{i,j-1} \leq a_{i-1,j} \leq a_{i,j}$ and $a_{i,j} < a_{i,j+1}$.

$$\begin{array}{ccccc}&&1&&\\&1&&2&\\1&&2&&3\end{array}\quad\begin{array}{ccccc}&&1&&\\&1&&3&\\1&&2&&3\end{array}\quad\begin{array}{ccccc}&&2&&\\&1&&2&\\1&&2&&3\end{array}\quad\begin{array}{ccccc}&&2&&\\&1&&3&\\1&&2&&3\end{array}\quad\begin{array}{ccccc}&&2&&\\&2&&3&\\1&&2&&3\end{array}\quad\begin{array}{ccccc}&&3&&\\&1&&3&\\1&&2&&3\end{array}\quad\begin{array}{ccccc}&&3&&\\&2&&3&\\1&&2&&3\end{array}$$

FIGURE 2. The seven monotone triangles of order 3, listed in order corresponding to Figure 1.

It is well-known that monotone triangles of order n are in bijection with $n \times n$ alternating sign matrices via the following map [3]. For each row of the ASM note which columns have a partial sum (from the top) of 1 in that row. Record the numbers of the columns in which this occurs in increasing order. This process yields a monotone triangle of order n. Note that entries $a_{i,j}$ in the monotone triangle satisfying the strict diagonal inequalities $a_{i,j-1} < a_{i-1,j} < a_{i,j}$ are in bijection with the -1 entries of the corresponding ASM. Also, recall that the *inversion number* of an ASM A is defined as $I(A) = \sum A_{ij}A_{k\ell}$ where the sum is over all i, j, k, ℓ such that $i > k$ and $j < \ell$. This definition extends the usual notion of inversion in a permutation matrix.

We now define descending plane partitions.

Definition 3. A *descending plane partition* (DPP) is an array of positive integers $\{d_{i,j}\}$ with $i \leq j$ (that is, with the ith row indented by $i-1$ units) with weak decrease across rows, strict decrease down columns, and in which the number of parts in each row is strictly less than the largest part in that row and is greater than or equal to the largest part in the next row.

$$\begin{array}{cccccc}d_{1,1} & d_{1,2} & d_{1,3} & \cdots\cdots\cdots & & d_{1,\lambda_1}\\ & d_{2,2} & d_{2,3} & \cdots\cdots & d_{2,\lambda_2} & \\ & & \ddots & & \iddots & \\ & & & d_{\ell,\ell}\ \cdots\ d_{\ell,\lambda_\ell} & & \end{array}$$

FIGURE 3. The general form of a descending plane partition.

Definition 4. A descending plane partition is *of order n* if its largest part is at most n.

Definition 5. A *special part* of a descending plane partition is a part $d_{i,j}$ such that $d_{i,j} \leq j - i$.

See Figure 3 for the general form of a DPP and Figure 4 for the seven DPPs of order 3. The only DPP in Figure 4 with a special part is 3 1. The 1 is a special part since $1 = d_{1,2} \leq 2 - 1$.

We now turn our attention to totally symmetric self-complementary plane partitions.

$\emptyset \qquad 2 \qquad 3 \qquad 3\ 1 \qquad 3\ 2 \qquad 3\ 3 \qquad \begin{array}{cc}3&3\\ &2\end{array}$

FIGURE 4. The seven descending plane partitions of order 3.

Definition 6. A *plane partition* is a two dimensional array of positive integers which weakly decreases across rows from left to right and down columns.

We can visualize a plane partition as a stack of unit cubes pushed up against the corner of a room. If we identify the corner of the room with the origin and the room with the positive orthant, then denote each unit cube by its coordinates in $\mathbb{N}^3$, we obtain the following equivalent definition. A plane partition π is a finite set of positive integer lattice points (i, j, k) such that if $(i, j, k) \in \pi$ and $1 \leq i' \leq i$, $1 \leq j' \leq j$, and $1 \leq k' \leq k$ then $(i', j', k') \in \pi$. A plane partition is *totally symmetric* if whenever $(i, j, k) \in \pi$ then all six permutations of (i, j, k) are also in π.

Definition 7. A *totally symmetric self-complementary plane partition* (TSSCPP) inside a $2n \times 2n \times 2n$ box is a totally symmetric plane partition which is equal to its complement, that is, the collection of empty cubes in the box is of the same shape as the collection of cubes in the plane partition itself.

$$\begin{array}{cccccc}6&6&6&3&3&3\\6&6&6&3&3&3\\6&6&6&3&3&3\\3&3&3&&&\\3&3&3&&&\\3&3&3&&&\end{array}\qquad\begin{array}{cccccc}6&6&6&4&3&3\\6&6&5&3&3&2\\6&6&6&3&3&3\\4&3&3&1&&\\3&3&3&&&\\3&3&2&&&\end{array}\qquad\begin{array}{cccccc}6&6&6&4&3&3\\6&6&5&4&3&3\\6&6&4&3&2&2\\4&4&3&2&&\\3&3&2&&&\\3&3&2&&&\end{array}\qquad\begin{array}{cccccc}6&6&6&5&5&3\\6&5&5&4&3&1\\6&5&4&3&2&1\\5&4&3&2&1&\\5&3&2&1&1&\\3&1&1&&&\end{array}$$

$$\begin{array}{cccccc}6&6&6&5&5&3\\6&5&5&3&3&1\\6&5&5&3&3&1\\5&3&3&1&1&\\5&3&3&1&1&\\3&1&1&&&\end{array}\qquad\begin{array}{cccccc}6&6&6&5&4&3\\6&6&5&3&3&2\\6&5&5&3&3&1\\5&3&3&1&1&\\4&3&3&1&&\\3&2&1&&&\end{array}\qquad\begin{array}{cccccc}6&6&6&5&4&3\\6&6&5&4&3&2\\6&5&4&3&2&1\\5&4&3&2&1&\\4&3&2&1&&\\3&2&1&&&\end{array}$$

FIGURE 5. TSSCPPs inside a $6 \times 6 \times 6$ box

See Figure 5 for the seven TSSCPPs of order 3.

In [4], Di Francesco gives a bijection from TSSCPPs of order n to a collection of nonintersecting lattice paths. The bijection proceeds by taking a fundamental domain of the TSSCPP, and instead of reading the number of boxes in each stack, one looks at the paths going alongside those boxes. This yields a collection of nonintersecting paths with two types of steps. With a slight further deformation, he obtains that the following objects are in bijection with TSSCPPs. See Figure 6.

Proposition 8 (Di Francesco). *Totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box are in bijection with nonintersecting lattice paths (NILP) starting at $(i, -i)$, $i = 1, 2, \ldots, n-1$, and ending at positive integer points on the x-axis of the form $(r_i, 0)$, $i = 1, 2, \ldots, n-1$, making only vertical steps $(0, 1)$ or diagonal steps $(1, 1)$.*

In [4], Di Francesco uses the Lindström-Gessel-Viennot formula for counting nonintersecting lattice paths via a determinant evaluation to give an expression for the generating function of TSSCPPs with a weight of τ per vertical step. We will show that when restricted to permutation

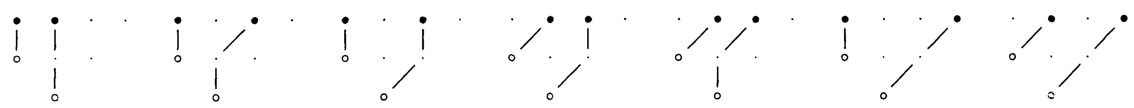


FIGURE 6. The seven TSSCPP NILP of order 3.

TSSCPPs, this weight corresponds to the inversion number of the permutation. Note that the distribution of the number of vertical steps in all TSSCPP NILPs does not correspond to the inversion number distribution on ASMs.

With another slight deformation, we obtain a tableaux version of these NILPs. See Figures 7 and 8.

Definition 9. A *boolean triangle* of order n is a triangular integer array $\{b_{i,j}\}$ for $1 \leq i \leq n-1$, $n-i \leq j \leq n-1$ with entries in $\{0,1\}$ such that the diagonal partial sums satisfy

$$1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \geq \sum_{i=j}^{i'} b_{i,n-j}. \tag{1}$$

$$\begin{array}{ccccccccc}
 & & & & b_{1,n-1} & & & & \\
 & & & b_{2,n-2} & & b_{2,n-1} & & & \\
 & & b_{3,n-3} & & b_{3,n-2} & & b_{3,n-1} & & \\
 & & & & \vdots & & & & \\
b_{n-1,1} & & & b_{n-1,2} & \cdots & b_{n-1,n-2} & & & b_{n-1,n-1}
\end{array}$$

FIGURE 7. The indexing on a generic monotone or boolean triangle

$$\begin{array}{ccccccc}
\begin{array}{cc} \multicolumn{2}{c}{1} \\ 1 & 1 \end{array} &
\begin{array}{cc} \multicolumn{2}{c}{1} \\ 1 & 0 \end{array} &
\begin{array}{cc} \multicolumn{2}{c}{0} \\ 1 & 1 \end{array} &
\begin{array}{cc} \multicolumn{2}{c}{0} \\ 0 & 1 \end{array} &
\begin{array}{cc} \multicolumn{2}{c}{1} \\ 0 & 0 \end{array} &
\begin{array}{cc} \multicolumn{2}{c}{0} \\ 1 & 0 \end{array} &
\begin{array}{cc} \multicolumn{2}{c}{0} \\ 0 & 0 \end{array}
\end{array}$$

FIGURE 8. The seven TSSCPP boolean triangles of order 3, listed in order corresponding to Figure 6 (and Figure 2 via the bijection of Theorem 12).

Proposition 10. *Boolean triangles of order n are in bijection with TSSCPPs inside a $2n \times 2n \times 2n$ box.*

Proof. The bijection proceeds by replacing each vertical step of the NILP with a 1 and each diagonal step with a 0 and vertically reflecting the array. The inequality on the partial sums is equivalent to the condition that the lattice paths are nonintersecting. □

3. THE TSSCPP-PERMUTATION BIJECTION

In this section, we give a bijection between $n \times n$ permutation matrices and a subclass of totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box, preserving the inversion number statistic and two boundary statistics. First, we identify the permutation subclass of TSSCPPs.

Definition 11. Let *permutation TSSCPPs* of order n be all TSSCPPs of order n whose corresponding boolean triangles have weakly decreasing rows. (In the NILP picture, each row has some number of vertical steps followed by some number of diagonal steps.)

It is easy to see that there are $n!$ permutation TSSCPPs. The condition on the boolean triangle that the rows be weakly decreasing means that all the 1's must be left-justified, thus the defining partial sum inequality (1) is never violated. To construct a permutation TSSCPP, freely choose any number of left-justified 1's in each row of the boolean triangle and the rest zeros; there are $i+1$ choices for row i, and the choices are all independent.

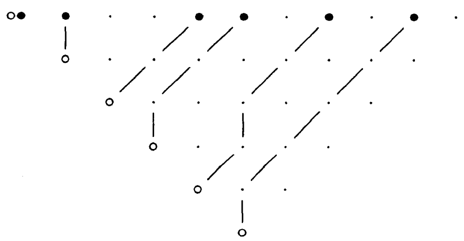
We are now ready to state and prove our main theorem.

Theorem 12. *There is a natural, statistic-preserving bijection between $n \times n$ permutation matrices with inversion number p whose 1 in the last row is in column k and whose 1 in the last column is in row ℓ and permutation TSSCPPs of order n with p zeros in the boolean triangle, exactly $n-k$ of which are contained in the last row, and for which the lowest 1 in diagonal $n-1$ is in row $\ell - 1$.*

Proof. We first describe the bijection map. An example of this bijection is shown in Figure 10.

Begin with a permutation TSSCPP of order n. Consider its associated boolean triangle $b = \{b_{i,j}\}_{1 \le i \le n-1, n-i \le j \le n-1}$. Define $a = \{a_{i,j}\}_{1 \le i \le n, n-i \le j \le n-1}$ as follows: $a_{n,j} = j+1$ and for $i < n$, $a_{i,j} = a_{i+1,j}$ if $b_{i,j} = 0$ and $a_{i,j} = a_{i+1,j-1}$ if $b_{i,j} = 1$. We claim a is a monotone triangle. Clearly $a_{i,j-1} \le a_{i-1,j} \le a_{i,j}$. Also, $a_{i,j} < a_{i,j+1}$, since if $a_{i,j} = a_{i,j+1}$, then $a_{i,j} = a_{i+1,j}$ and $a_{i,j+1} = a_{i+1,j+1}$ so that we would need $b_{i,j} = 0$ and $b_{i,j+1} = 1$. This contradicts the fact that the rows of permutation boolean triangles must weakly decrease. Furthermore, a is a monotone triangle with no -1's in the corresponding ASM, since each entry is defined to be equal to one of its diagonal neighbors in the row below. This process is clearly invertible.

TSSCPP



$\Leftrightarrow$

Boolean triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 0 & & 0 & & 0 & & 0
\end{array}$$

$\Leftrightarrow$

Monotone triangle

$$\begin{array}{ccccccccccc}
 & & & & & 4 & & & & & \\
 & & & & 4 & & 6 & & & & \\
 & & & 3 & & 4 & & 6 & & & \\
 & & 3 & & 4 & & 5 & & 6 & & \\
 & 1 & & 3 & & 4 & & 5 & & 6 & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6
\end{array}$$

$\Leftrightarrow$

Permutation matrix

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

FIGURE 9. An example of the bijection. The matrix on the lower right is the permutation 463512 which has 11 inversions. These inversions correspond to the 11 diagonal steps of the TSSCPP on the upper left.

We now show that this map takes a permutation TSSCPP boolean triangle with p zeros to a permutation matrix with p inversions. Recall that the inversion number of any ASM A (with the matrix entry in row i and column j denoted A_{ij}) is defined as $I(A) = \sum A_{ij} A_{k\ell}$ where the sum is over all i, j, k, ℓ such that $i > k$ and $j < \ell$. This definition extends the usual notion of inversion in a permutation matrix. In [10] we found that $I(A)$ satisfies $I(A) = E(A) + N(A)$, where $N(A)$ is the number of -1's in A and $E(A)$ is the number of entries in the monotone triangle equal to their southeast diagonal neighbor (entries $a_{i,j}$ satisfying $a_{i,j} = a_{i+1,j}$). Since in our case, $N(A) = 0$

and $E(A)$ equals the number of zeros in the corresponding TSSCPP boolean triangle, we have that $I(A)$ equals the number of zeros in b.

We can see that the zeros of b correspond to permutation inversions directly by noting that to convert from the monotone triangle representation of a permutation to a usual permutation σ such that $i \to \sigma(i)$, we set $\sigma(i)$ equal to the unique new entry in row i of the monotone triangle. Thus for each entry of the monotone triangle $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$, there will be an inversion in the permutation between $a_{i,j}$ and $\sigma(i+1)$. This is because $a_{i,j} = \sigma(k)$ for some $k \leq i$ and $\sigma(k) = a_{i,j} > \sigma(i)$. These entries $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$ correspond exactly to zeros in row i of the boolean triangle b. Thus if a permutation TSSCPP has p zeros in its boolean triangle, its corresponding permutation will have p inversions.

Also, observe that if the number of zeros in the last row of the boolean triangle is k, then the 1 in the bottom row of the permutation matrix will be in column $n-k$. So the missing number in the penultimate monotone triangle row shows where the last row of the boolean triangle transitions from ones to zeros. So by the bijection between monotone triangles and ASMs, the 1 in the last row of A is in column $n-k$.

Finally, if the lowest 1 in diagonal $n-1$ of the boolean triangle is in row $\ell - 1$, this means that the entries $\{a_{i,n-1}\}$ for $\ell \leq i \leq n$ are all equal to n. So the 1 in the last column of the permutation matrix is in row ℓ. □

See Figure 10 for an example of this bijection.

4. The DPP-permutation bijection

In this section we give a bijection between descending plane partitions of order n with no special parts, p parts, and k parts equal to n and $n \times n$ permutation matrices with p inversions and a 1 in row $n-k$ of the last column. We will need the following lemma.

Lemma 13. *There is a natural part-preserving bijection between descending plane partitions of order n with no special parts and partitions with largest part at most n and with at most $i-1$ parts equal to i for all $i \leq n$.*

The bijection map is very simple, so we will state the map, but refer to [8] for the proof that it is a well-defined bijection. To map from the DPPs to the partitions, take all the parts of the DPP and put them in one row in decreasing order. To map from the partitions to the DPPs, insert the parts of the partition into the shape of a DPP in decreasing order, putting as many parts in a row as possible and then moving on to the first position in the next row whenever the next part to be inserted would be forced to be special if added to the current row. So this bijection is simply a rearrangement of parts.

We use this lemma in the following bijection between permutations and descending plane partitions with no special parts.

Theorem 14. *There is a simple bijection between descending plane partitions of order n with a total of p parts, k parts equal to n, and no special parts and $n \times n$ permutation matrices with inversion number p whose 1 in the last column is in row $n-k$.*

Proof. We first describe the bijection map. An example of this bijection is shown in Figure 10.

Begin with a DPP δ of order n with no special parts. From Lemma 13 we know that the parts of δ form a partition with largest part at most n and at most $i-1$ parts equal to i for all $i \leq n$. Use these parts to make a monotone triangle of order n in the following way. The bottom row of a monotone triangle is always $1\ 2\ 3 \cdots n$. Let c_i denote the number of parts of δ equal to i. Beginning with $i = n$, make a continuous path (border strip) of i's in the triangle starting at the i in the bottom row and at each step going northeast if possible or else northwest. The path continues until there are a total of c_i northwest steps in the path. In this way, the path stays as far to the east as possible and has exactly c_i entries equal to their southeast diagonal neighbor. Decrement i by one and repeat until reaching $i = 1$. Since there are at most $i - 1$ parts equal to i, this process is well-defined. The resulting array is a monotone triangle of order n such that there are no entries satisfying $a_{i,j-1} < a_{i-1,j} < a_{i,j+1}$ (i.e. either $a_{i,j} = a_{i-1,j}$ or $a_{i-1,j} = a_{i,j-1}$). Thus the monotone triangle corresponds to an $n \times n$ permutation matrix A, since permutation matrices are alternating sign matrices with no -1 entries.

DPP		Monotone triangle		Permutation matrix
$\begin{array}{ccccc} 6 & 6 & 6 & 6 & 5 \\ & 5 & 4 & 4 & 4 \\ & & 3 & 3 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccccccccccc} &&&&&4&&&&& \\ &&&&\mathbf{4}&&\mathbf{6}&&&& \\ &&&3&&4&&\mathbf{6}&&& \\ &&\mathbf{3}&&\mathbf{4}&&\mathbf{5}&&\mathbf{6}&& \\ &1&&\mathbf{3}&&\mathbf{4}&&\mathbf{5}&&\mathbf{6}& \\ 1&&2&&3&&4&&5&&6 \end{array}$	$\Leftrightarrow$	$\begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix}$

FIGURE 10. An example of the bijection. The bold entries in the monotone triangle are the entries equal to their southeast diagonal neighbor. These are exactly the parts of the DPP. Note that the matrix on the right represents the permutation 463512 which has 11 inversions. These inversions correspond to the 11 parts of the DPP on the left.

The inverse map first takes a permutation matrix A to its monotone triangle. We claim that the parts of the corresponding DPP δ are exactly the entries of the monotone triangle which are equal to their southeast diagonal neighbor, that is, entries $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$. Because of the shape of the monotone triangle, there are at most $i - 1$ such entries equal to i. Thus these entries form a partition with largest entry at most n and at most $i - 1$ parts equal to i for all $i \leq n$. Using Lemma 13 the parts of this partition can always be formed into a unique DPP.

This is a bijection because the monotone triangle entries $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$ are exactly the entries coming from the northwest steps in the border strips which are exactly the entries of δ.

We now show that this map takes a DPP with p parts to a permutation matrix with p inversions. We again use the fact, noted in [10], that the inversion number of an ASM, $I(A)$, satisfies $I(A) = E(A) + N(A)$, where $N(A)$ is the number of -1's in A and $E(A)$ is the number of entries in the monotone triangle equal to their southeast diagonal neighbor (i.e. entries $a_{i,j}$ satisfying $a_{i,j} = a_{i+1,j}$). Since in our case, $N(A) = 0$ and $E(A)$ equals the number of parts of the corresponding DPP, we have that $I(A)$ equals the number of parts of δ.

We can see that the parts of δ correspond to permutation inversions directly by noting that to convert from the monotone triangle representation of a permutation to a usual permutation σ such that $i \to \sigma(i)$, one simply sets $\sigma(i)$ equal to the unique new entry in row i of the monotone triangle. Thus for each entry of the monotone triangle $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$, there will be an inversion in the permutation between $a_{i,j}$ and $\sigma(i+1)$. This is because $a_{i,j} = \sigma(k)$ for some $k \leq i$ and $\sigma(k) = a_{i,j} > \sigma(i)$. These entries $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$ are exactly the parts of the DPP. Thus if a DPP has p parts, its corresponding permutation will have p inversions.

Also, observe that if the number of parts equal to n in δ is k, then k determines the position of the 1 in the last column of the permutation matrix. This is because the path of n's in the monotone triangle must consist of exactly k northwest steps (no northeast steps). So by the bijection between monotone triangles and ASMs, the 1 in the last column of A is in row $n - k$. So finally, we have a bijection between descending plane partitions of order n with a total of p parts, k parts equal to n, and no special parts and $n \times n$ permutation matrices with inversion number p whose 1 in the last column is in row $n-k$. □

See Figure 10 for an example of this bijection.

5. Toward a bijection between all ASMs, DPPs, and TSSCPPs

In this Section, we discuss some of the challenges to finding the ASM-DPP-TSSCPP bijection in full generality.

As discussed in the proof of Theorems 12 and 14, the inversion number of an ASM satisfies $I(A) = E(A) + N(A)$ where $N(A)$ is the number of -1's in A and $E(A)$ is the number of entries in the monotone triangle equal to their southeast diagonal neighbor [10]. Thus there should be a one-to-one correspondence between these diagonal equalities of the monotone triangle and the non-special parts of the DPP. Difficulties arise quickly, though, since even for $n = 4$ there are examples of DPPs, such as $\begin{array}{ccc} 4 & 4 & 3 \\ & 3 & 1 \end{array}$, whose non-special parts cannot correspond exactly to diagonal equalities of the same number in the monotone triangle. In the above example, 1 is a special part and 4 4 3 3 are non-special parts. If the parts 4 4 3 3 are each entries in the monotone triangle equal to their southeast neighbors, there is no way to fill in the rest of the entries of the monotone triangle so that there is a -1 in the ASM (to correspond to the special part of 1 in the DPP). Evidently, the addition of special parts to the DPP makes the relationship between non-special parts and monotone triangle diagonal equalities more complex.

Another complicating factor is that, though there is at most one way to write any collection of numbers as a DPP with no special parts (as shown in Lemma 13), the same is not true of DPPs with special parts. For example, the parts 5 5 5 3 1 can form either the DPP $\begin{array}{cccc} 5 & 5 & 5 & 1 \\ & 3 & & \end{array}$ or $\begin{array}{ccc} 5 & 5 & 5 \\ & 3 & 1 \end{array}$. Therefore, the position of the parts in the DPP matters when special parts are present.

While DPPs have the property that the number of parts equals the inversion number of the ASM (this is now proved, though not bijectively [2]), TSSCPPs do not have such a statistic as of yet. We showed that the number of diagonal steps in a permutation-NILP gives the inversion number of the permutation matrix, but this is not true for general TSSCPPs and ASMs. Furthermore, while the number of special parts of a DPP corresponds to the number of -1's in the ASM, there is no such statistic on TSSCPPs. It would seem reasonable to conjecture that the -1 of the ASM should correspond to all instances of a vertical step followed by a diagonal step as you go from left to right along a row of the NILP (or a 0 followed by a 1 as you go across a row of the boolean triangle). This holds up to $n = 4$, and it seems to hold for arbitrary n in the special cases of one -1 and the maximal number of -1's ($\lfloor \frac{n^2}{4} \rfloor$). But for the number of -1's between 1 and $\lfloor \frac{n^2}{4} \rfloor$, these statistics diverge.

Di Francesco has noted that the distribution of the number of diagonal steps in the top row of the TSSCPP-NILP corresponds to the refined enumeration of ASMs. So one might hope to begin a general bijection by determining the $(n-1)$st row of the monotone triangle from the top row of the NILP (or the bottom row of the boolean triangle) by left-justifying all the vertical steps and then bijecting in the same way as in the permutation case. After that, though, it is unclear how to proceed. See Figure 5 for a summary of the various statistics which are preserved in the permutation case DPP-ASM-TSSCPP bijections and which should correspond in full generality.

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DPP	ASM	TSSCPP boolean triangle
no special parts*	no -1's	rows weakly decrease
number of parts*	number of inversions	number of zeros
number of n's*	position of 1 in last column	position of lowest 1 in last diagonal
largest part value that does not appear	position of 1 in last row	number of zeros in last row*

FIGURE 11. This table show the statistics preserved by the permutation-TSSCPP-DPP bijections. There is a star by the DPP and TSSCPP statistics that have the same distribution as the ASM statistic in the general case.

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