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d-Complete Posets Generalize Young Diagrams for the Hook Product Formula: Partial Presentation of Proof

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1. Introduction

This paper presents results that were jointly obtained with Dale Peterson; here most of the details for our original 1997 proofs are belatedly presented, since the completion of [Pr9] has been long delayed. After three sections of definitions, the full statements of these results are given in Section 5:

Theorem 1. d-Complete posets are hook length posets.

Corollary 1. There is a hook length enumeration formula for the number of extensions of d-complete posets that generalizes the FRT hook formula for the number of standard Young tableaux.

Theorem 2. Colored d-complete posets are colored hook length posets.

Corollary 2. There is a product of root heights enumeration formula for the number of reduced decompositions of a λ-minuscule element of a simply laced Kac-Moody Weyl group.

Peterson already obtained Corollary 2 (without the "simply laced" assumption) by 1989 using a different approach [Car]. The "forget the colors" specialization produces the right hand side for Theorem 1 from the right hand side for Theorem 2 once our two definitions of hook length are reconciled. But it is not the case that Theorem 1 is a quick consequence of Theorem 2: it took significant effort in [Pr5] to see that each d-complete poset can be colored to produce a colored d-complete poset. Given this fact, the left hand side for Theorem 1 can be transformed to the left hand side for Theorem 2. Then Theorem 1 can be deduced from Theorem 2.

Section 6 prepares for the proofs of Theorems 1 and 2. Presentations of the four parts of these proofs are given in Sections 7-10 in varying levels of detail; the two corollaries are proved in Section 11. Sections 12 and 13 contain mathematical and developmental remarks.

2. General Combinatorial Definitions

Fix some $n \geq 1$ throughout the paper and set $[n] := \{1,2,\ldots,n\}$. The elements of $[n]$ are *colors*. Here $\mathbb{N} := \{0,1,2,\ldots\}$ and $\mathbb{P} := \{1,2,3,\ldots\}$. For $(g_1,\ldots,g_n) := \gamma \in \mathbb{N}^n$, set $x^\gamma := x_1^{g_1}\ldots x_n^{g_n}$.

Let P be a poset and $x,y \in P$. Set $p := |P|$. Consult [St4] for the following common concepts: (Hasse) diagram of P, dual poset P^* of P, connected P, x covers y, filter, ideal, interval, chain, and (linear) extension of P. For $b \in P$, define the *principal ideal* $(b) := \{c \in P\colon\ c \leq b\}$ and the *principal filter* $\langle b\rangle := \{c \in P\colon\ c \geq b\}$. A subset $S \subseteq P$ is *convex* if whenever $x,y \in S$, then $x \leq z \leq y$ implies $z \in S$. A multiset M based upon P is a *multichain* if its underlying set is a chain in P.

A P-*partition on* P is a function $\psi\colon P \to \mathbb{N}$ such that $b \leq c$ implies $\psi(b) \geq \psi(c)$. Set $|\psi| := \Sigma_{b\in P}\psi(b)$. The P-*partition generating function* for P is $F_P(x) := \Sigma_\psi x^{|\psi|}$, summation over P-partitions ψ of P. We say P is a *hook length poset* if there exists some function $h\colon P \to \mathbb{P}$ such that $F_P(x) = \Pi_{b\in P}(1 - x^{h(b)})^{-1}$. Given a fixed $m \geq 1$, an m-*bounded* P-*partition* is a P-partition ψ satisfying $\psi(b) \leq m$, for $b \in P$.

The poset $J(P)$ is the set of ideals of P ordered by inclusion. It is a distributive lattice. If L is a distributive lattice, let P_L be the subposet of its elements that each cover exactly one element. Then it is known [§3.4, St4] that $J(P_L) \cong L$ and $P_{J(P)} \cong P$.

We say P is *colored* if it has been equipped with a surjective function $\kappa\colon P \to [n]$. Let ψ be a P-partition for P. For each $i \in [n]$, set $m_i := \Sigma_{\kappa(b)=i}\psi(b)$. The n-*variate weight* of ψ is $\mathbf{x}^\psi := x_1^{m_1}\ldots x_n^{m_n}$. The *colored* P-*partition generating function for* P is $F_P(\mathbf{x}) := \Sigma_\psi \mathbf{x}^\psi$. We say P is a *colored hook length poset* if there exists some function $\mathbf{h}\colon P \to \mathbb{N}^n$ such that $F_P(\mathbf{x}) = \Pi_{b\in P}(1 - \mathbf{x}^{\mathbf{h}(b)})^{-1}$. Denote the specialization $x_i \mapsto x$ for $1 \leq i \leq n$ by $\mathbf{x} \mapsto x$. Note that $F_P(\mathbf{x}{\mapsto}x) = F_P(x)$. The *colored* m-*bounded* P-*partition generating function* for P is $F_P(m;\mathbf{x}) := \Sigma_\psi \mathbf{x}^{|\psi|}$, sum over all m-bounded P-partitions ψ.

3. d-Complete Poset and Hook Definitions

Let $k \geq 3$. The *double-tailed diamond poset* $d_k(1)$ is the poset with $2k-2$ elements that has exactly two elements that are incomparable and that has $k-2$ elements greater than each of those two elements. The poset $d_k(1)^-$ is the poset produced by removing the maximum element from $d_k(1)$. It has one maximal element when $k \geq 4$, but has two maximal elements when $k = 3$.

Let P be a poset. Let $k \geq 3$. A convex subset $S \subseteq P$ is a d_k^--*convex set* if is isomorphic to $d_k(1)^-$.

Definition. A poset P is d-*complete* if for every d_k^--convex set S, $k \geq 3$, there exists an element of P which covers exactly the maximal element(s) of S and which does not cover the maximal element(s) of any other d_k^--convex set.

Let P be a d-complete poset. An element $c \in P$ is a *neck* element if there exists $b \in P$ such that the closed interval $[b,c] \cong d_k(1)$ for some $k \geq 3$. Since such an element b must be unique, to each neck element c there corresponds a *tail* element $b_c := b$ and two *elbow* elements d_c and e_c of $[b_c,c]$ that are the only two incomparable elements of $[b,c]$. The *hook function* h: $P \to \mathbb{P}$ for P is recursively determined by:

(i) If $c \in P$ is not a neck element, set $h(c) := |(c)|$.

(ii) If $c \in P$ is a neck element, set $h(c) := h(d_c) + h(e_c) - h(b_c)$.

Let P be a poset. The *top forest* Γ of P is both a subposet and a simple graph: The nodes of Γ are the $b \in P$ such that $\langle b \rangle$ is a chain in P, and the edges of Γ are the unordered pairs $\{b,c\}$ such that b covers c in the poset Γ. If P is connected, then Γ is a tree. Nodes b and c of Γ are *adjacent* if $\{b,c\}$ is an edge for Γ; they are *weakly adjacent* if they are adjacent or $b = c$. Suppose P is colored by some κ. Suppose all of the occurences of a color γ in P are comparable. Here $b,c \in P$ such that $b < c$ are *consecutive occurences of* γ if there does not exist $b < d < c$ such that $\kappa(d) = \gamma$. Within the following definition, Axiom (1) is to be used to identify the nodes in the top forest graph Γ with the colors from $[n]$:

Definition. A colored poset P is *colored d-complete* if the following are satisfied:

(1) Every color from $[n]$ occurs once in the top forest Γ of P.

(2) If b is covered by c in P, then $\kappa(c)$ is adjacent to $\kappa(b)$ in Γ.

(3) If the colors $\kappa(b)$ and $\kappa(c)$ of $b,c \in P$ are weakly adjacent in Γ, then b and c are comparable in P.

(4) If $b < c$ are consecutive occurences in P of some color k, then in the open interval (b,c) there are exactly two elements whose colors are adjacent to k in Γ.

If $b < c$ are consecutive occurences of some color in a colored d-complete poset, then it can be seen that there are exactly two elements in (b,c) whose colors occur only once there. Denote these $d_{b,c}$ and $e_{b,c}$. The axis basis vectors ε_i, $1 \leq i \leq n$, for $\mathbb{N}^n$ are indexed by the colors from $[n]$. The *multi-hook function* $\mathbf{h}$: $P \to \mathbb{N}^n$ for P is recursively determined by:

(i) If $b \in P$ is the minimal element in P of its color, set $\mathbf{h}(b) := \Sigma_{c \leq b} \varepsilon_{\kappa(c)}$.

(ii) If $b < c$ are consecutive occurences of some color, require $\mathbf{h}(b) + \mathbf{h}(c) = \mathbf{h}(d_{b,c}) + \mathbf{h}(e_{b,c})$.

It can be seen that each colored d-complete poset is a d-complete poset, and the elements of a d-complete poset can be colored in essentially only one way to produce a colored d-complete poset [Prop. 8.6, Pr5].

4. Lie Theoretic Definitions and Constructions

Let Γ be a simple graph whose nodes are bijectively colored with $[n]$. Use Γ as a Dynkin diagram and create the corresponding *simply laced* generalized Cartan matrix A: Here for $i,j \in [n]$ we have $a_{ii} = 2$ and when $j \neq i$ we also have $a_{ij} = a_{ji} \in \{0,-1\}$. Set $\ell := \mathit{nullity}$ (A). Create [Kac] a corresponding complex vector space $\mathfrak{h}$ of dimension $n + \ell$ and sets of simple roots $\{\alpha_i\}_{i\in[n]} \subseteq \mathfrak{h}^*$ and simple coroots $\{\alpha_j^\vee\}_{j\in\{n\}} \subseteq \mathfrak{h}$ such that $\alpha_i(\alpha_j^\vee) = a_{ij}$. Let W denote the corresponding Kac-Moody Weyl group; its generating reflections on $\mathfrak{h}^*$ are $s_i.\gamma := \gamma - \gamma(\alpha_i^\vee)\alpha_i$ for $i \in [n]$ and $\gamma \in \mathfrak{h}^*$. The *length* of $w \in W$ is denoted $\ell(w)$. For $i \in [n]$, in $\mathfrak{h}^*$ choose ω_i such that $\omega_i(\alpha_j^\vee) = \delta_{ij}$ for all $j \in [n]$. Let $\mathfrak{h}'$ denote $\mathbb{C}\{\alpha_i^\vee\}_{i\in\{n\}} \subseteq \mathfrak{h}$. Let $\mathfrak{h}'^{\perp} \subseteq \mathfrak{h}^*$ denote the annihilator of $\mathfrak{h}'$. For $\gamma,\delta \in \mathfrak{h}^*$ we write $\gamma \equiv \delta$ to indicate $\gamma + \mathfrak{h}'^{\perp} = \delta + \mathfrak{h}'^{\perp}$ in $\mathfrak{h}^*/\mathfrak{h}'^{\perp}$. For $i \in [n]$ note that $\alpha_i \equiv 2\omega_i - \Sigma_j\omega_j$, sum over $j \in [n]$ that are adjacent to i in G. Re-using the symbols s_i, the induced reflections on $\mathfrak{h}^*/\mathfrak{h}'^{\perp}$ are such that $s_i.\omega_j \equiv \omega_j$ when $i \neq j$ and $s_i.\omega_j \equiv -\omega_j + \Sigma_k\omega_k$ when $i = j$, sum over $k \in [n]$ adjacent to j.

Set $\Delta := \{\alpha_i\}_{i\in[n]}$ and construct the real roots $\Phi := W\Delta$. Split these into positives and negatives, $\Phi =: \Phi^+ \cup \Phi^-$. For $w \in W$ define $\Phi(w) := \Phi^+ \cap w\Phi^-$; this is Φ_w in [Lemma 1.3.14, Ku2]. Set $p := p_w := \ell(w) = |\Phi(w)|$. Set $\Lambda := \{\lambda \in \mathfrak{h}^*\colon \lambda(\alpha_i^\vee) \in \mathbb{Z}$ for $1 \le i \le n\}$ and $\Lambda^+ := \{\lambda \in \mathfrak{h}^*\colon \lambda(\alpha_i^\vee) \in \mathbb{N}$ for $1 \le i \le n\}$. Fix $\lambda \in \Lambda^+$. Set $W_\lambda := \{w\in W\colon w\lambda = \lambda\}$ and let $W^\lambda \subseteq W$ denote the set of the minimal length representatives of the cosets in W/W_λ. Following Peterson [Car], an element $w \in W$ is *λ-minuscule* if there exists an expression $s_{i_p}s_{i_{p-1}}\ldots s_{i_1}$ for w such that $s_{i_j}.[(s_{i_{j-1}}\ldots s_{i_1}).\lambda] = (s_{i_{j-1}}\ldots s_{i_1}).\lambda - \alpha_{i_j}$ for $1 \le j \le p$. Such an element is in W^λ.

For each $\gamma \in \Lambda$ create a formal exponential e^γ. Let Q^+ denote the set of nonnegative sums of simple roots. For $\gamma \in Q^+$, and write $\gamma =: \Sigma c_i\alpha_i$ for some $c_i \in \mathbb{N}$. Define $ht(\gamma) := \Sigma c_i$. Re-use the variables $x_1, \ldots, x_n$ from Section 2. Define the coordinatized formal exponential $\mathbf{x}^\gamma := x_1^{c_1}\ldots x_n^{c_n}$. Under the "principal" specialization $\mathbf{x} \mapsto x$, note that $\mathbf{x}^\gamma \mapsto x^{ht(\gamma)}$.

Construct the symmetrizable Kac-Moody algebra $\mathfrak{g}$ and Borel subalgebra $\mathfrak{b}$ corresponding to A, $\mathfrak{h}$, and Δ. Let $\mathcal{U}(\mathfrak{b})$ denote the universal enveloping algebra of $\mathfrak{b}$. Denote the maximal integrable highest weight $\mathfrak{g}$-module $L^{max}(\lambda)$ with highest weight λ of [Dfn. 2.1.5, Ku2] by V_λ. Let $w \in W^\lambda$. Moving down the page by w, let $v_{w\lambda} \neq 0$ be a weight vector in V_λ of weight $w\lambda$. Denote the Demazure module $L^{max}(\lambda)_w$ of

[Lemma 8.1.23, Ku2] by $V_\lambda(w)$; this is the $\mathscr{b}$-submodule $\mathcal{U}(\mathscr{b}).v_{w\lambda}$ of V_λ created by moving up the page from $v_{w\lambda}$. The lowest weight of this module is $w\lambda$. Its weight spaces are subspaces of the weight spaces of V_λ. For $\mu \in \Lambda$, let $d_\lambda(w,\mu)$ denote the dimension of the weight space of $V_\lambda(w)$ of weight μ. The formal character $\chi_\lambda(w)$ of $V_\lambda(w)$ is $\Sigma d_\lambda(w,\mu)e^\mu$, summation over the $\mu \in \Lambda$ for which $d_\lambda(w,\mu) \neq 0$. Define its *adjusted character* by $\xi_\lambda(w) := e^{-w\lambda}\chi_\lambda(w)$. If μ is a weight of $V_\lambda(w)$, then $\mu - w\lambda \in Q^+$. The coordinatized adjusted character is denoted $\xi_\lambda(w;\mathbf{x})$; it is a polynomial in $x_1, \ldots, x_n$.

5. Fully Detailed Main Results

Theorem 1. Let P be a d-complete poset. Then $h(b) > 0$ for every $b \in P$ and

$$F_P(x) = \Pi_{b\in P}(1 - x^{h(b)})^{-1}.$$

Corollary 1. The number of extensions of P is $p!/[\Pi_{b\in P}h(b))]$.

Theorem 2. Let P be a colored d-complete poset. Then $\mathbf{h}(b) \in \mathbb{N}^n$ for every $b \in P$ and

$$F_P(\mathbf{x}) = \Pi_{b\in P}(1 - \mathbf{x}^{\mathbf{h}(b)})^{-1}.$$

Corollary 2. Let w be a λ-minuscule element of length p in a simply laced Kac-Moody Weyl group. The number of reduced decompositions of w is $p!/[\Pi_{\alpha\in\Phi(w)}ht(\alpha)]$.

6. Ordering and Viewing Conventions, Connected Reduction

Several ordering and up/down viewing conventions need to be established. For various reasons, it is impossible to establish an entirely satisfying sequence of choices. For example, when creating the orbit $W\lambda$, the Bruhat order convention of viewing the minimal $e \in W$ at the bottom conflicts with the Lie convention of viewing the maximal λ at the top. The definition of d-complete poset used in [Pr5] is the order dual of the definition used elsewhere; here we assume that the reader will dualize quotes of those statements as needed. The most fundamental conflict is: Both the highest weight λ and the lowest weight $w\lambda$ for a Demazure module are important, and both the processes of working down "from the top" when creating w and of working up "from the bottom" when acting with $\mathscr{b}$ are important.

Here we use the more-common up/down convention for d-complete posets (as in [Pr6]), and we do not reverse any other existing order definitions. However, we ask that some posets be viewed upside-down to respect the Lie tradition of viewing the highest weight at the top of the page. Here connected d-complete posets have unique maximal elements, Bruhat orders are to be drawn upside-down, in Case A the Young/Ferrers diagrams need to be rotated by only 45°, and there the identity element of W^J corresponds to the empty Young diagram.

If the d-complete poset given in Theorem 1 or 2 is not connected, express it as a direct sum of its connected components. All of the structures considered and all of the methods used in the proofs of Theorems 1 and 2 are well-behaved under direct sums/direct products. For example, $J(P_1 \oplus P_2) \cong J(P_1) \times J(P_2)$. Here are a few more of these aspects: the P-partition generating function of a direct sum of posets is the product of the P-partition generating functions, the notion of "top tree" of a poset is replaced by that of "top forest", and the representation constructed is replaced by the tensor product of the representations corresponding to the connected components. To avoid added verbiage and length, for the sake of brevity and clarity we will henceforth assume that P is connected and we will omit the details for passing from and back to the non-connected case.

In each of Parts I, II, III, and IV of the proof of Theorem 2, the foremost result from an external perspective is called a "Proposition" and its application to the proof of the theorem is called a "Lemma".

7. Part I: Combinatorial Front End

The claims made in this section come from [Pr5] or may be confirmed using it. The order duals of the d-complete definitions in that paper are equivalent via structural arguments to the d-complete definitions in this paper.

To launch the proof of Theorem 1, consider a connected d-complete poset P whose top tree Γ has n elements. Fix a coloring of Γ with [n]. Let z be the color of the unique maximal element of Γ. As in [Prop. 8.6, Pr5], color $P - \Gamma$ with [n] so that P becomes colored d-complete.

The proof of Theorem 2 begins here. Construct the dual poset P*, but view it upside-down to re-use the diagram of P. Ideals $I \subseteq P^*$ bijectively correspond to ideals $H \subseteq P$ via $H := P^* - I$. Construct the lattice $L^* := J(P^*)$. Also view it upside-down, since it will become a Bruhat order.

Now use Γ as a Dynkin diagram and create the corresponding Kac-Moody algebra $\mathfrak{g}$. Note that this set-up is equivalent to the set-up in Section 10 of [Pr5]. Set $\lambda := \omega_z$. Fix an extension of P*, from the top of the page. Using the sequence of colors determined by this extension, construct an element $w := w_P$ of the Weyl group W of $\mathfrak{g}$ by composing the corresponding simple reflections from right to left. This is a reduced decomposition of w, which is λ-minuscule, and all of the reduced decompositions of w correspond to the extensions of P* in this manner [Cor. 5.5, Pr5]. Combining Proposition 9.1 of [Pr5] with remarks in Section 10 of [Pr5] yields the transition from the given colored d-complete poset P to the Kac-Moody Weyl group element w defined in the Kac-Moody context established above; this is essentially one direction of [Thm. B, Pr5]:

Proposition I. The lattice L* is isomorphic to the principal ideal (w) in the Bruhat order on W^λ: Given $I \in L^*$, the corresponding $u \in (w)$ can be formed by choosing any extension of I.

To any ideal $H \subseteq P$ there corresponds the indicator P-partition ψ_H such that $\psi_H(b)$ is 1 or 0 depending upon whether $b \in H$ or $b \notin H$. Fix an ideal $I \subseteq P^*$ and set $H := P^* - I$. We are considering six versions of essentially the same entity: $I \subseteq P^*$, $I \in L^*$, $u \in W^\lambda$, $u\lambda \in W\lambda$, $H \subseteq P$, and ψ_H. If $u \in (w)$ corresponds to I in Proposition I, in its proof the difference $\lambda - u.\lambda$ of weights is $\Sigma_{b\in I}\alpha_{\kappa(b)}$. Therefore the difference $u\lambda - w\lambda$ of weights is $\Sigma_{b\in H}\alpha_{\kappa(b)}$. For each $i \in [n]$, let m_i be the number of elements of H of color i. Then the contribution to the colored P-partition generating function for P for ψ_H is $x_1^{m_1}\ldots x_n^{m_n} =: \mathbf{x}^H$, the *weight monomial* for H. Hence the coordinatization of the formal exponential $e^{-w\lambda+u\lambda}$ is equal to this combinatorial weight monomial $\mathbf{x}^H$.

Fix $m \geq 1$. There are bijections such that: An m-bounded P-partition ψ is taken to a weakly decreasing sequence $H_1 \supseteq \ldots \supseteq H_m$ of ideals of P. This is taken to a weakly increasing sequence $I_1 \subseteq \ldots \subseteq I_m$ of ideals in P^*, which is also a multichain in L^*. This is taken to a multichain $u_1 \leq \ldots \leq u_m$ in (w). By summing the color censuses in the m layers H_k, it can be seen that the coordinatization of $e^{-mw\lambda}e^{mu_1\lambda}\ldots e^{mu_m\lambda}$ is $\mathbf{x}^\psi$.

Summarizing Part I:

Lemma I. The top tree of the colored d-complete poset P has been denoted Γ. The Kac-Moody entities W, λ, and w determined by Γ and P have been constructed and $m \geq 1$ has been fixed. Here the colored m-bounded P-partition generating function $F_P(m;\mathbf{x})$ is equal to the sum of the coordinatizations of $e^{-mw\lambda}e^{u_1\lambda}\ldots e^{u_m\lambda}$ over all multichains $u_1 \leq \ldots \leq u_m$ in the ideal (w) of P^* of the Bruhat order W^J.

8. Part II: Lakshmibai-Seshadri-Littelmann Character Description

In Part I we fixed $\lambda = \omega_z$, a λ-minuscule $w = w_P$ of length p, and $m \geq 1$. Via Proposition I, extensions of P^* correspond to maximal chains $e = v_0 < v_1 < \ldots < v_p = w$ in (w).

To describe the weights of the Demazure module $V_{m\lambda}(w)$, we adapt the material of [§3, La2] to this special case: Take her r to be our p. Our maximal chains $e = v_0 < v_1 < \ldots < v_p = w$ are her "λ-chains". Presumably her "$m_i = (\mu_i,\beta_i^*)$" should read "$m_i = (\mu_i\lambda,\beta_i^*)$". At each reflection stage here one has $[v_{k-1}.m\lambda](\alpha_{i_k}^\vee) = m$. To obtain an "admissible weighted λ-chain" here one must associate some sequence $m \geq n_1 \geq \ldots \geq n_p \geq 0$ to such a chain. Tracking the strict descents in this sequence leads to her $D_{c,n}$. Once she imposes her $\phi(\pi) \leq$ w condition, using reasoning as in Section 7 above it can be seen that the $D_{c,n}$ bijectively correspond to the m-bounded P-partitions of P^*. Such $D_{c,n}$ also exactly index the equivalence classes that are collected in her set $I_w(\lambda)$. These m-bounded P-partitions of P^* can be viewed as sequences $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_m$ of ideals of P^*. These correspond to m-multichains $e \leq u_1 \leq \ldots \leq u_m \leq w$. Understanding $a_0 = m$ and $a_{s+1} = 0$, her definition of $\nu(\pi)$ yields the product of formal exponentials $e^{u_1\lambda} \ldots e^{u_m\lambda}$ for the summand in [Eqn. 3.6.2, La2]. The sum in that equation runs over $I_w(\lambda)$.

Here is our version of [Eqn. 3.6.3, La2] for our case:

Proposition II. Since w is λ-minuscule, the Demazure character $\chi_{m\lambda}(w)$ is $\Sigma e^{u_1\lambda} \ldots e^{u_m\lambda}$, sum over the m-multichains $e \leq u_1 \leq \ldots \leq u_m \leq w$ in the Bruhat ideal (w).

Adjust and coordinatize these characters and apply this to Lemma I:

Lemma II. The colored m-bounded P-partition generating function $F_P(m;\mathbf{x})$ is equal to the coordinatized adjusted Demazure character $\xi_{m\lambda}(w;\mathbf{x})$.

9. Part III: Kumar-Peterson Identity for a Limit of Demazure Characters

The following identity was independently obtained by Shrawan Kumar and Peterson; it does not explicitly appear in a published source:

Proposition III. For any Kac-Moody algebra $\mathfrak{g}$, let $\lambda \in \Lambda^+$ and $w \in W^\lambda$. Then in the ring of formal power series on Q^+ the direct limit $\lim_{m\to\infty} \xi_{m\lambda}(w)$ of adjusted Demazure characters is equal to the product $\Pi_{\alpha\in\Phi(w)}(1 - e^\alpha)^{-1}$ over the roots made positive by w.

An outline for deriving this result from statements in the book [Ku2] appears in Section 12 below.

For any colored poset P, in the ring of formal power series the direct limit $\lim_{m\to\infty} F_P(m;x)$ of the generating functions for the m-bounded P-partitions is the generating function $F_P(\mathbf{x})$ for all P-partitions. Apply this observation to Lemma II to obtain $F_P(\mathbf{x}) = \lim_{m\to\infty} \xi_{m\lambda}(w;\mathbf{x})$. Combine this statement with the coordinatization of Proposition III:

Lemma III. The colored P-partition generating function $F_P(\mathbf{x})$ is equal to the product $\Pi_{\alpha\in\Phi(w)}(1 - \mathbf{x}^\alpha)^{-1}$ over the roots made positive by w.

10. Part IV: Combinatorial Back End

We return to the context of Part I and of [Pr5]: we are considering the colored d-complete poset P whose filters correspond to the elements of the ideal (w) in the Bruhat order on the Kac-Moody group W. All of the claims below come from [Pr5] or can be verified using the techniques of [Pr5], especially those used in the proof of Proposition 9.1. To complete the proof of Theorem 2, we need to show that the elements c of P can be bijectively mapped to the roots α in $\Phi(w)$ in a way such that $\mathbf{x}^{\mathbf{h}(c)} = \mathbf{x}^\alpha$. Although the recursive definition of $\mathbf{h}(c)$ refers to the order on P, the Weyl group facts are expressed in terms of the order of P^*.

Let $H \subseteq P^*$ be a convex set. Extensions of H specify sequences of simple reflections. To see that each of these is a reduced decomposition of a unique element v of W determined by H, use any ideal $I \subseteq P^*$ such that such that $I \cap H = \varnothing$ and $I \cup H$ is an ideal and find the elements of W^λ corresponding to I and to $I \cup H$.

Let $y \in P$. Set $k := \kappa(y)$. View y as an element of P^* and form $\langle y \rangle \subseteq P^*$. Let $u \in (w)$ correspond to the ideal $P^* - \langle y \rangle$ of P^*. Let $v \in W$ correspond to the filter $\langle y \rangle - \{y\}$ of P^*. Set $\beta_y := u^{-1}.(-\alpha_k)$ and $\alpha_y := v.\alpha_k$. (Here the not-necessarily-simple root α_y is indexed by an element of P, while simple roots are indexed by colors from $[n]$.) Note that $w = vs_k u$ and $\alpha_y = w.\beta_y$. As in the proof of Lemma 1.3.14 of [Ku2], it can be seen that $\beta_y \in \Phi^-$ and $\alpha_y \in \Phi^+$. Hence $\alpha_y \in \Phi(w)$. Given any reduced decomposition of any $w' \in W$, the recipe in the proof of that lemma produces all of the roots in $\Phi(w')$. Using properties for the coloring of P from the definition of colored d-complete in this paper, by manipulating reduced decompositions it can be seen that the root in $\Phi(w)$ that that recipe associates to the simple reflection "at y" in the extension of P corresponding to any reduced decomposition of w is always the root α_y we have associated to y above. So the bijectivity of that recipe implies that our assignment of α_y to y describes a bijection from P to $\Phi(w)$.

It remains to show that $\mathbf{x}^{\mathbf{h}(y)} = \mathbf{x}^{\alpha_y}$. For the reflection calculations performed below, it will be sufficient to work within Λ^+ modulo $\mathfrak{h}'^{\perp}$: Any discrepancy within $\mathfrak{h}'^{\perp}$ will evaluate to zero on any coroot. By the equivalence between d-complete posets and colored d-complete posets, we can use the properties listed in either definition. Keep in mind that the graph Γ of colors is acyclic.

Suppose y is the minimal element in P of color k. We use the "wave" (numbers game) viewpoint of [Pr5] for the succession of "node firings" that arise during the computation of $w.\lambda$ for reduced decompositions of a λ-minuscule w. Note that u is λ-minuscule and y is the unique minimal element in P^* of $\langle y \rangle$. Since all of the firings corresponding to the elements in $P^* - \langle y \rangle$ have been executed, it can be seen that all of the firings arising in the application of v to $s_k u.\lambda$ are eventual consequences of the application of s_k to $u.\lambda$. It can also be seen that $s_k u.\lambda \equiv -\omega_k + \Sigma_{j \neq k} b_j \omega_j$ with $b_j = +1$ implying that j is adjacent to k or that the color j is not present in $\langle y \rangle - \{y\}$. So all of the firings arising in the application of v to $s_k u.\lambda$ are eventual consequences of firings at some of the $h \in \Gamma$ adjacent to k for which $b_h = +1$.

Rather than computing $v.\alpha_k$, we first compute $v.(-\alpha_k)$. Recall that $-\alpha_k \equiv -2\omega_k + \Sigma_j \omega_j$, sum over $j \in [n]$ that are adjacent to i in Γ. So each $j \in \Gamma$ that has a coefficient of $+1$ in the expansion of $s_k u.\lambda$ and is present in $\langle y \rangle - \{y\}$ also appears in the expansion of $-\alpha_k$ with a coefficient of $+1$. This similarity of the linear combination for $s_k u.\lambda$ to that for $-\alpha_k$ can be exploited by using the computation of $v.(s_k u.\lambda)$ to "guide" the computation of $v.(-\alpha_k)$. (Since in this case the simple reflection s_k does not appear in any reduced decomposition for v, the difference between the contribution $-2\omega_k$ and the contribution $-\omega_k$ does not matter.) It can be seen that each of the firing sequences used to produce $w.\lambda$ from $s_k u.\lambda$ can be analogously applied within the computation of $v.(-\alpha_k)$ to subtract the same sum $s_k u.\omega_k - w.\omega_k$ of simple roots from $-\alpha_k$ as from $s_k u.\omega_k$. Hence $\alpha_y := -v.(-\alpha_k)$ is the sum of the multiset of simple roots corresponding to the colors in the ideal (y) of P, or $\alpha_y = \Sigma_{a \le y} \alpha_{\kappa(a)}$. This agrees with the multi-hook definition $\mathbf{h}(y) := \Sigma_{a \le y} \varepsilon_{\kappa(a)}$.

Otherwise, the element y is not the minimal element of color k in P. Let b denote the element of P of color k that is maximal with respect to $b < y$ and let d and e denote the elements in the open interval (b,y) of P whose colors occur only once. This is equivalent to y being a neck element with tail b and elbows d and e. Here $\kappa(b) = k$; set $f := \kappa(d)$ and $g := \kappa(e)$. The interval [b,y] in P is isomorphic to $d_{t+2}(1)$ for some $t \ge 1$. Denote the colors of its t neck elements by $k =: k_1$ and $k_2, \ldots, k_t$. Note that k_t is adjacent to at least f, g, and (if $t \ge 2$) k_{t-1} in Γ. Let $q \in P$ be the unique element covered by d and e. If H is a convex subset of P* with $v' \in W$ corresponding to H, rather than writing $v'.\gamma$ for $\gamma \in \Lambda$ we write $H.\gamma$.

Let Y, D, E, B respectively denote the filters $\langle y \rangle - \{y\}$, $\langle d \rangle - \{d\}$, $\langle e \rangle - \{e\}$, and $\langle b \rangle - \{b\}$ of P*. Then $\alpha_y = Y.\alpha_k$, $\alpha_d = D.\alpha_f$, $\alpha_e = E.\alpha_g$, and $\alpha_b = B.\alpha_k$. Consider the convex subsets $[b,y) =: M$ and $[b,q] =: Q$ in P. Here $[b,y) \cong d_{t+2}(1)^-$ and $[b,q]$ is a chain of t elements. Construct the filters $Y' := Y - M$, $D' := D - Q$, and $E' := E - Q$. So $\alpha_y = Y'.M.\alpha_k$, $\alpha_d = D'.Q.\alpha_f$, and $\alpha_e = E'.Q.\alpha_g$. Computing within the simply laced Γ yields $M.\alpha_k = \alpha_k + 2\Sigma_{2 \le i \le t} \alpha_{k_i} + \alpha_f + \alpha_g$. Also one computes $Q.\alpha_f = \alpha_f + \Sigma_{t \ge i \ge 1} \alpha_{k_i}$ and $Q.\alpha_g = \alpha_g + \Sigma_{t \ge i \ge 1} \alpha_{k_i}$. Hence $M.\alpha_k + \alpha_k = Q.\alpha_f + Q.\alpha_g$. Applying Y' to both sides produces $\alpha_y + Y'.\alpha_k = Y'.Q.\alpha_f + Y'.Q.\alpha_g$. Note that $Y' - D'$ is an ideal in $Y' \subseteq P^*$. So we can write $Y'.Q.\alpha_f = D'.(Y'-D').Q.\alpha_f$. Within the simply laced diagram Γ one has $\alpha_f + \Sigma_{t \ge i \ge 1} \alpha_{k_i} \equiv \omega_f + \omega_k - \omega_g - \Sigma_{r \in R} \omega_r$, where R is the set of nodes in Γ adjacent to f or k other than q and k_2. So $Q.\alpha_f \equiv \omega_f + \omega_k - \omega_g - \Sigma_{r \in R} \omega_r$. Using d-complete properties, it can be seen that no color in $\{f,k,g\} \cup R$ appears in $Y' - D'$. So $(Y'-D').(\omega_f + \omega_k - \omega_g - \Sigma_{r \in R} \omega_r) = \omega_f + \omega_k - \omega_g - \Sigma_{r \in R} \omega_r$, which implies that $(Y'-D').Q.\alpha_f = Q.\alpha_f$. Hence $Y'.Q.\alpha_f = D'.Q.\alpha_f = \alpha_d$. Similarly one obtains $Y'.Q.\alpha_g = E'.Q.\alpha_g = \alpha_e$. We have $Y'.\alpha_k = B.(Y'-B).\alpha_k$. Using d-complete properties, it can be seen that none of the colors appearing in $Y' - B$ are weakly adjacent to the color k. Hence $(Y'-B).\alpha_k = \alpha_k$, and so $Y'.\alpha_k = B.\alpha_k = \alpha_b$.

We have obtained:

Proposition IV. If two elements y and b in a colored d-complete poset P are such that $[b,y] \cong d_k(1)$ for some $k \geq 3$ and d and e are the elbow elements of $[b,y]$, then in $\Phi(w)$ the associated roots satisfy $\alpha_y = \alpha_d + \alpha_e - \alpha_b$.

This root fact agrees with the multi-hook recursion $\mathbf{h}(y) = \mathbf{h}(d) + \mathbf{h}(e) - \mathbf{h}(b)$. Since α_y and $\mathbf{h}(y)$ agree on the initializing color-minimal elements of P and satisfy the same recurrence that effectively and uniquely determines these two quantities for the other elements of P, we can conclude:

Lemma IV. The mapping $y \mapsto \alpha_y$ is a bijection from P to $\Phi(w)$ such that $\mathbf{x}^{\mathbf{h}(y)} = \mathbf{x}^{\alpha_y}$. Hence $\mathbf{h}(y) \in \mathbb{N}^n$ for all $y \in P$.

Combining this lemma with Lemma III completes the proof of Theorem 2.

Since it is known that the notion of "neck element" becomes equivalent to the notion of "non-minimal element of a given color" when a d-complete poset is colored, it can be seen that the weight monomial $\mathbf{x}^{\mathbf{h}(c)}$ for a colored d-complete poset becomes the weight $x^{h(c)}$ for a d-complete poset under the specialization $\mathbf{x} \mapsto x$. Hence $h(y) > 0$ for all $y \in P$. So as the coloring of the elements of P is forgotten, specializing $\mathbf{x} \mapsto x$ converts the right side of Theorem 2 to the right side of Theorem 1 and completes its proof.

11. Proofs of Corollaries

For Corollary 1, combine Theorem 1 with Theorem 3.15.7 of [St4] for a natural labelling ω to produce $W_{P,\omega}(x) = [\prod_{1\leq i\leq p}(1 - x^i)]/[\prod_{y\in P}(1 - x^{h(y)})]$. Here $W_{P,\omega}(x)$ is a polynomial such that $W_{P,\omega}(1)$ is the number of extensions of P. Divide top and bottom by $(1 - x)^p$ and set $x = 1$.

For Corollary 2, specialize $\mathbf{x} \mapsto x$ in Lemma III to obtain $F_P(x) = \prod_{\alpha\in\Phi(w)}(1 - x^{ht(\alpha)})^{-1}$. Apply the proof of Corollary 1 to see that the number of extensions of P is $p!/[\prod_{\alpha\in\Phi(w)}ht(\alpha)]$. By [Cor. 5.5, Pr5] the number of extensions of P^* is the number of reduced decompositions of w.

12. Mathematical Remarks

Filters of (colored) d-complete posets are (colored) d-complete posets A poset is (colored) d-complete if and only its connected components are (colored) d-complete.

Connected d-complete posets were classified [Pr6] using their top trees These posets are tree-like "slant sums" formed from 15 classes of "slant irreducibles" whose top trees are

"Y-shaped" (most often of Type E_n for $n \geq 6$) Rooted trees are slant sums of one element posets, and conversely. The first two slant irreducible classes consist of shapes and of shifted shapes Since the (colored) d-complete posets form a tightly constrained Dynkin diagram-indexed class of posets, it should not be surprising that there are varying axiomatic formulations of the (colored) d-complete property Further comments on d-complete posets appear in [Pr7], Sections 1 and 10 of [Pr5], Sections 1 and 15 of [Pr6], and Section 1 of [Pr8] It can be seen that our two hook length definitions subsume the historic hook length definitions for rooted trees, shapes, and shifted shapes Hence Theorems 1 and 2 and Corollary 1 generalize the hook product identities for the (colored) P-partition generating functions and the counts of standard Young tableaux for such posets.

A converse [Thm. B, Pr5] to Proposition I is: Let $\lambda \in \Lambda^+$ for a simply laced Kac-Moody algebra $\mathcal{g}$. If $w \in W$ is λ-minuscule, then the ideal (w) in the Bruhat order is a distributive lattice L' and there is a colored d-complete poset P such that $J(P^*) \cong L'$.

The prototypical d-complete posets were the "minuscule" posets; these arose in the more familiar context of [Hum] (rather than of [Kac]) as follows: Let X_n denote the simple Lie algebra of type X and rank n For $\lambda \in \Lambda^+$, let $X_n(\lambda)$ denote its irreducible highest weight representation V_λ Let $\lambda := \omega_j$ be one of the "minuscule" dominant integral weights for X_n, as listed in [Exer. 13.13, Hum] Here the longest element $w^\lambda{}_0$ of W^λ is λ-minuscule It was shown [Pr1] that the Bruhat poset $W^\lambda = (w^\lambda{}_0)$ is a distributive lattice; then it was denoted as the *irreducible minuscule* lattice $X_n(j)$ The corresponding poset P of join irreducible elements was denoted $x_n(j)$ Applying this convention to the list of minuscule representations led to the following list of *irreducible minuscule* posets: $a_n(j)$ for $1 \leq j \leq n$ (rectangular shapes); $b_n(n)$, $d_n(n-1)$, and $d_n(n)$ (staircases); $d_n(1)$ (double-tailed diamonds); $c_n(1)$ (chains); and $e_6(1)$, $e_6(6)$, $e_7(1)$ (exceptionals) The elements of these posets were colored in Section 11 of [Pr1]; for the cases with $X \in \{A,D,E\}$ they are now known to be colored d-complete A precise translation of the work of [Pr1] to the setting of this paper would actually state that it is the order duals of these posets that are d-complete But ignoring this detail is harmless, since each minuscule poset is self-dual In fact, the irreducible minuscule posets constitute [§14, Pr6] all self-dual connected d-complete posets. Filters of the minuscule posets $a_n(n)$ and $d_n(n)$ are respectively shapes and shifted shapes, and conversely.

We have stated Proposition II in a way so that it generalizes the character description that follows from Seshadri's main theorem [Ses]; that theorem described the semisimple Lie algebra character of the highest weight module $X_n(m\lambda) := V_{m\lambda}$ with "Bruhat m-multichains" for minuscule weights λ Lakshmibai and Seshadri proposed a vast generalization of that prototypical theorem in Section 4 of [LS] by conjecturing a description of the character for the

Demazure module $V_\lambda(w)$ of a symmetrizable Kac-Moody algebra Littelmann confirmed that conjecture [Lit] after using paths to reformulate it: Combine the second statement of his Theorem 5.2 with the Demazure character formula Lakshmibai reconverted Littelmann's "Lakshmibai-Seshadri paths" to "admissible weighted λ-chains" [Eqn. 3.6.3, La2].

To discuss Proposition III, first consider a connected simply connected semisimple algebraic group G over $\mathbb{C}$ with a torus T, a Borel subgroup $B \supseteq T$, and Weyl group W. Let $\lambda \in \Lambda^+$. Form the stabilizer W_λ and then W^λ. Let $m \geq 1$. Let $V_{m\lambda}$ be a highest weight irreducible representation of G. Let P be the parabolic subgroup $BW_\lambda B$. To warm up, we start with the torus characters of the homogeneous coordinate rings of some *projective* varieties: Let $\mathcal{L}_{m\lambda}$ denote the homogeneous line bundle of weight $-m\lambda$ over the flag manifold G/P. Let $\Gamma(G/P,\mathcal{L}_{m\lambda})$ denote the space of global sections of this line bundle. Denoting the dualization of a module with $*$, the Borel-Weil theorem may be combined with a rewritten Weyl character formula to obtain a "BWW" character identity:

$$\mathrm{char}[\,\Gamma(G/P,\mathcal{L}_{m\lambda})^*\,] = [\,\Sigma_{w\in W}(-1)^{\ell(w)}e^{ww_0(m\lambda+\delta)-w_0\delta}\,] \,/\, \Pi_{\alpha\in\Phi^+}(1-\mathbf{x}^{\alpha})\,.$$

Let $w \in W^\lambda$. Let X_w denote the Schubert variety that is the Zariski closure of the Bruhat cell BwP/P. The generalization of the Borel-Weil theorem to Schubert varieties may similarly be combined with the Demazure character formula to obtain a more general "BWD" identity [Thm. 8.2.9, Ku2] for the torus character of $\Gamma(X_w,\mathcal{L}_{m\lambda})^*$.

Now consider a (not necessarily symmetrizable) Kac-Moody group $\mathcal{G}$ over $\mathbb{C}$ with $\mathcal{T}$, $\mathcal{B}$, W, λ, $\mathcal{P}$ denoting the Kac-Moody analogs of the entities chosen or constructed above. Theorem 8.2.9 of [Ku2] is actually stated at this level of generality, for Schubert varieties in the Kac-Moody flag manifold $\mathcal{G}/\mathcal{P}$. The Kumar-Peterson identity, Proposition III above, gives an analog of the BWD identity for the "*affine* cousins" of the Schubert varieties: it describes the duals of the characters of the coordinate rings of the Bruhat cells in $\mathcal{G}/\mathcal{P}$ when they are viewed as $\mathcal{T}$-modules. By 1996 Kumar had discovered this identity when studying singularities in Kac-Moody Schubert varieties [Ku1]; he could have stated it in that paper (after Proposition 2.9) with little further work. In 1997 Peterson also developed this identity, but instead with the motivation of helping this author prove Theorem 2 above. Theorem 12.1.3 of [Ku2] finds the torus character of the graded algebra of the local ring at any point of any Schubert variety in $\mathcal{G}/\mathcal{B}$. A proof of Proposition III can be based upon the first half of the proof of that theorem. The setting of his Theorem 12.1.3 is both more general and less general than the setting needed here: it is concerned with pairs $(v,w) \in W \times W$ such that $v \leq w$, but limits its attention to the case $\mathcal{P} = \mathcal{B}$. Here we take $v := w$. According to Kumar [personal communication], this theorem could have readily been developed for any $\mathcal{P} \supseteq \mathcal{B}$. Using

Lemma 7.3.10 it can be shown that his $X_w \cap w\mathcal{B}^e$ is the Bruhat cell $\mathcal{B}w\mathcal{P}/\mathcal{P}$. Let $\mathbb{C}_{mw\lambda}$ denote the one dimensional $\mathcal{T}$-module of weight $mw\lambda$ for each $m \geq 1$. The coordinate ring for $\mathcal{B}w\mathcal{P}/\mathcal{P}$ arises as a $\mathcal{T}$-module from a direct limit [Eqn. 12.1.3.2, Ku2]:

$$\lim_{m\to\infty}[\,\Gamma(X_w,\mathcal{L}_{m\lambda}) \otimes \mathbb{C}_{mw\lambda}\,] \cong \mathbb{C}[\mathcal{B}w\mathcal{P}/\mathcal{P}]\,.$$

But it is well known that $\mathcal{B}w\mathcal{P}/\mathcal{P} \cong \prod_\alpha \mathcal{U}_\alpha$, product over $\alpha \in \Phi(w)$, where $\mathcal{U}_\alpha$ is the unipotent subgroup for $\alpha \in \Phi^+$. It is easy to see that the torus character of $\mathbb{C}[\mathcal{U}_\alpha]$ is $(1 - e^{-\alpha})^{-1}$. The generalization of the Borel-Weil theorem to Schubert varieties appears as Corollary 8.1.26 of [Ku2]:

$$\Gamma(X_w,\mathcal{L}_{m\lambda}) \cong V_{m\lambda}(w)^*\,.$$

Substitute the two isomorphisms into the direct limit, form the $\mathcal{T}$-characters, and dualize to obtain Proposition III.

In the geometric context just presented return to considering only the finite flag manifolds G/P, as above for the BWW identity. Take $w := w^\lambda{}_0$, the longest element in W^λ. Then $X_w = G/P$. Multiply both sides of the BWW identity above by $e^{-wm\lambda}$. Some cancellation in the right hand side that occurs in the limit $m \to \infty$ may be routinely computed; consequently the limit of that side is $\prod_{\alpha\in\Phi(w)}(1 - e^{\alpha})^{-1}$. This is the dual of the character of the coordinate ring of the "big" Bruhat cell $Bw^\lambda{}_0P/P$ of the flag manifold G/P. Here Proposition III has been obtained from the Weyl character formula without using algebraic geometric reasoning (as was used in the proof of Theorem 12.1.3).

Let W be a Kac-Moody Weyl group. Let $w \in W$. In addition to $\Phi(w) := \Phi^+ \cap w(\Phi^-)$, also define $\Psi(w) := \Phi^+ \cap w^{-1}.\Phi^-$. Computing $\beta := w.\alpha$ for each $\alpha \in \Psi(w)$ specifies a bijection from $\Psi(w)$ to $\Phi(w)$. Here $\Psi(w)$ consists of some roots in their "original" positions and $-\Phi(w)$ consists of the "flipped" images of those roots. With respect to a fixed reduced decomposition of w, for each $\alpha \in \Psi(w)$ there is a unique $\alpha_k \in \Delta$ which is the "last" positive image of α as that reduced decomposition is successively applied; then the application of the next simple reflection in the decomposition, which is s_k, "flips" that image.

Let P be a colored d-complete poset. Let W, λ, w be the simply-laced Weyl structure determined by P as in Proposition I; this w is λ-minuscule. To each element y in P of color k, Part IV of the proof of Theorem 2 associated three roots: $\alpha_k \in \Delta$, $u^{-1}.\alpha_k \in \Psi(w)$, and $v.\alpha_k \in -\Phi(w)$. The last two of these associations are bijections from P to sets of roots. Here set $\alpha_y := u^{-1}.\alpha_k$ and $\beta_y := v.\alpha_k$. So $\beta_y = vs_ku.\alpha_y = w.\alpha_y$. As a consequence of w being λ-minuscule, the color k at which α_y is "flipped" now does not depend upon the choice of a reduced decomposition. Let ${}^v\Phi^+$ be the set of positive coroots ${}^v\alpha$ and set ${}^v\Psi(w) := {}^v\Phi^+ \cap w^{-1}({}^v\Phi^-)$. Each of the three theorems below respectively emphasize one of α_k, α_y (actually ${}^v\alpha_y$), and β_y (actually $-\beta_y$): First, Proposition I together with its converse is roughly Theorem B of [Pr5]. The proof in [Pr5] of Theorem B refers only to the $\alpha_k \in \Delta$, and not to the α_y or the β_y. Stembridge's Proposition 3.1(c) generalized Theorem B by loosening the simply laced requirement on W to requiring symmetrizable instead [Ste]. The notion of "heap" was used to state that result. Second, Stembridge's Theorem 5.5 then generalized Theorem 11 of [Pr1]. It used the coroots ${}^{\vee}\alpha_y := u^{-1}.{}^{\vee}\alpha_k$ to represent P as the set ${}^{\vee}\Psi(w)$ of coroots ordered by the simple coroots. Third, Proposition IV above is stated in terms of the roots $-\beta_y := -v.\alpha_k$ in $\Phi(w)$.

Joseph Seaborn has recently developed [Sea] a combinatorial version of Proposition III in Type A. To state this result he used Willis' description [Wil] of the "right key" to obtain a notion of "limiting Demazure tableau" for $m \to \infty$. Applying this notion to the Lascoux-Schützenberger description of the Demazure character [e.g. Thm. 3.1, PW] produced the left hand side. The right hand side is a product over the inversions of the given permutation. When λ has r different column lengths, his combinatorial proof of the identity uses r parallel invocations of Gansner's colored Hillman-Grassl algorithm [Thm. 5.1, Gan] for shapes.

13. Development of the Notion of d-Complete and of Theorems 1 and 2

The following classic results can now be viewed as the cases of Corollary 1 and of Theorems 1 and 2 for the filters of the minuscule posets $a_n(j)$ and $d_n(n)$: Frame-Robinson-Thrall and Knuth found hook product counting formulas for ordinary and shifted standard Young tableaux. In his thesis [St1] Stanley introduced the notion of P-partition and posed [St2] the problem of finding "hook length" posets. (This terminology was later introduced by Sagan.) Stanley and Gansner [Gan] obtained hook product identities for (eventually colored) P-partition generating functions for shapes and shifted shapes. Here the P-partitions were reverse (shifted) plane partitions on the given (shifted) shape, with the entry of '0' allowed. A detailed listing of the predescessor results to Theorems 1 and 2 will appear in [Pr9]; for now consult [Pr7]. Here we describe the development of our viewpoints and techniques for Theorems 1 and 2 from 1978 through 1997, including the origins of the notion of d-complete poset. Consideration of the generating function identities for the (shifted) plane partitions bounded by m in [Pr1] can help to motivate the definition of P-partition. That paper also provides an entree to the approach of this paper, by working in the context of [Hum] (rather than of [Kac]).

This author began his doctoral research under Stanley in October 1978. The results attributed here to [Pr0] later appeared in [Pr1]. Interested in unimodal sequences, Stanley had learned about Dynkin's unimodality result for the dimensions of the weight spaces of the principal specialization of an irreducible finite dimensional representation of a semisimple Lie algebra [St3]. He was aware of the quotient-of-products formula for these specializations, and knew that for two families of characters these formulas gave the right hand sides of known identities (MacMahon and Bender-Knuth-Gordon-Andrews) for the generating functions of the m-bounded P-partitions on the posets that this author later called the minuscule posets $a_n(j)$ and $b_n(n)$. In Problem 3 of [St3] he posed the problem for any irreducible representation of specifying a ranked poset whose rank generating function was the principal specialization of the given representation. To this author he posed the finer problem of finding a basis for the irreducible representation whose elements are indexed by the elements of the poset. For any such representation of Type A, this could be done by forming a distributive lattice from the semistandard tableaux for the Young symmetrizer basis. This author's first thesis result was a classification of the finite Bruhat orders W^J that are (distributive) lattices [Pr0]. This result was obtained with a computational method that was later named the "numbers game" by Mozes. Let $m \geq 1$. Stanley knew that his problems could be solved for the irreducible representations $A_n(m\omega_j)$ and $B_n(m\omega_n)$ using the elements of the distributive lattices $J(m \times a_n(\omega_j))$ and $J(m \times b_n(\omega_n))$: To obtain these Type A cases the semistandard tableaux description could be reformulated, and by 1978 Macdonald had found solutions [Exmps. I.5.16, I.5.19, Mac] for these Type B cases. This motivated a search for irreducible representations all of whose weights appeared in $W\lambda$, as was true for $A_n(\omega_j)$ and $B_n(\omega_n)$. This author recognized that his classification of Bruhat lattices had also classified such representations, and soon found the pre-existing list of irreducible "minimal" (minuscule) representations in [Exer. 13.13, Hum]. (To reproduce this list here, replace each $x_n(j)$ in the list of Section 12 with $X_n(\omega_j)$.) This led to the introduction [Pr0] of the minuscule lattices $X_n(j)$ and the minuscule posets $x_n(j)$. By January 1979 we had arrived at the conjecture that when ω_j is a minuscule weight, for each irreducible representation $X_n(m\omega_j)$ there existed a basis that was indexed by multivariately weighted m-multichains in the Bruhat lattice $W^\lambda \cong J(x_n(j))$. The Hasse diagrams [Pr0] for the minuscule posets $x_n(j)$ were first published in [Pr2].

Let v be a highest weight vector for the minuscule representation $X_n(\omega_j)$. This author soon proposed constructing a basis for $V_{m\lambda} =: X_n(m\omega_j)$ by acting on $s^m v \in S^m[V_\lambda]$ with sequences of the Lie algebra generators y_i, $1 \leq i \leq n$, that are indexed by m-multichains in W^J. But he was unable to prove that the results of these actions were linearly independent and spanned. In March 1979 Stanley learned of Seshadri's first "standard monomial" result [Ses]. The Lie algebraic version of that result could be seen to confirm our basis conjecture for the $X_n(m\omega_j)$. (Two weeks later, this author met Dale Peterson.) In [Pr0], Seshadri's theorem was used [Thm. 6, Pr1] to re-prove some plane partition identities corresponding to $A_n(m\omega_j)$ and $B_n(m\omega_n)$, to obtain such identities for $e_6(1)$ and $e_7(1)$, and to thereby uniformly show that all minuscule posets were "Gaussian". Sections 8, 9, 11, and 12 of [Pr1] were developed in 1981-82. A version of Theorem 1 above for the minuscule cases was stated [p. 347, Pr1] as an immediate consequence of that thesis theorem. The precursor minuscule poset cases of Corollaries 1 and 2 above were also derived [pp. 345-348, Pr1] from Seshadri's theorem, without restriction to Type ADE.

Upon reading this author's thesis in 1981, Robert Steinberg raised the question of whether minuscule posets could be described as sets of roots ordered by the simple roots. This author answered this question uniformly in all types using negative *coroots;* see Theorem 11 of [Pr1] and its supporting lemmas. There the elements of the posets of coroots were colored with simple roots. To relate those results to the statements in Section 7 above, now order the coroots in that U_j^- by simple coroots (rather than by negative simple coroots). In Proposition I above, the starting poset is an axiomatically defined colored d-complete poset, whereas the $k = 1$ case of Lemmas 11.2 and 11.3 of [Pr1] start with a filter of the minuscule poset U_j^- of some negative coroots. Since those results in Section 11 of [Pr1] pertain to filters of minuscule posets as well as to entire minuscule posets, they provide early examples of a fully "root-explained" J(.) process from some d-complete posets to Bruhat ideals (w) for λ-minuscule w's: Consider the $k = 1$ cases of Lemmas 11.2 and 11.3. There the coloring by simple roots is used to explicitly describe an order isomorphism from $J(I_1)$ to (w) when I_1 is a filter of a minuscule poset P. As in Proposition I, this isomorphism is defined with the subtraction of simple roots from λ. These Section 11 statements were the semisimple precursor to Stembridge's Theorem 5.5 [Ste]; they held for any λ-minuscule element w of any finite Weyl group when λ is a minuscule weight.

The hook length assigned to an element of an irreducible minuscule poset in [Pr1] was its rank, where the rank of the unique minimal element was 1. The proof of the Gaussian poset result Theorem 6 verified case-by-case that the rank census of each minuscule poset agreed with the data used to compute the product principal specialization of the Weyl character formula. Summarizing, the following parts of [Pr1] were precursors for the minuscule posets to Parts I - IV of this paper: Theorem 11 and its lemmas were somewhat analogous to Part I, reference to Seshadri's [Ses] has been generalized by the reference here in Part II to Lakshmibai's Eqn. 3.6.3 of [La2], the production of the right hand side from the product principal specialization of Weyl's character formula has been replaced by use of the Kumar-Peterson limiting character identity in Part III, and the verification that the rank counts agree with the Weyl product data has been replaced with the Part IV proof that our general recursive definition of hook length models an assignment of the roots in $\Phi(w)$ to the elements of P.

By 1989, Peterson had developed the notion of a λ-minuscule element of a Kac-Moody Weyl group. His more-general version [Sect. V.3, Car] of Corollary 2 above was a generalization of the hook length formula derived on pp. 345-348 of [Pr1] for the number of reduced decompositions of the minuscule elements w^J_0 of all finite Weyl groups. This result was a consequence of two more detailed identities that he had developed, his "additive" and "product" identities for each λ-minuscule element w. The left hand side of each was a sum over the reduced decompositions of w and the right hand side of each was a product over $\Phi(w)$. (A general version of an identity related to the additive identity was later independently discovered and proved by Nakada [Eqn. 1.4, Nak].)

Here is how the notion of d-complete poset arose: In 1993 we returned to our 1979 proposal for producing bases of the representations $X_n(m\omega_j)$. Execution of that proposal would imply Proposition II above when $\mathfrak{g}$ is semisimple and $w = w^\lambda_0$. However, as described in [Pr3], we soon began to work in the following non-Lie context of $\mathbb{Z}$-modules constructed from colored posets: A colored poset is *properly colored* if any two elements of the same color must be comparable, but one cannot cover the other. A colored poset is *simply colored* if the colors within an interval that is a chain must be distinct. (Every colored d-complete poset is simply properly colored.) Fix a simply properly colored poset P. Let V denote the $\mathbb{Z}$-module generated by the ideals of P and let v denote the generator for $\varnothing$. Let $m \geq 1$ and scatter m identical bins around a table. Place one copy of P in each bin, converting its elements to colored sockets. Suspend a pipe full of colored marbles above the bins and then release the marbles. A *landing pattern* results if each marble lands in a socket of its own color, and if at every intermediate time the filled sockets within each bin form in ideal in P. (Consecutive marbles of the same color are to be released and to land simultaneously; i.e. they are indistinguishable.) This is a multiset of ideals of P. Emptying a pipe can be viewed as a sequence of actions on $S^m V$ according to the Leibniz rule, beginning with acting upon $s^m v$. The *result* of a pipe is the sum of its landing patterns. Let Ξ denote the $\mathbb{Z}$-submodule of $S^m V$ spanned by all pipe results. If one fixes an extension of P, then there is one "stackwise" pipe filling for each m-bounded P-partition. Using no external facts or methods, by September 1993 we had proved [Thm. 4, Pr3] that the pipe results for the stackwise fillings form bases for Ξ over $\mathbb{Z}$ for all extensions of P if and only if P is colored d-complete. (After this project was already underway, we learned of [La1].)

The development of the axioms for colored d-complete posets was entirely driven by the desire to characterize the existence of bases in the marble pouring context. While doing so we were empirically aware that a good way to form a distributive lattice L whose join irreducible poset P "worked" for the marble pouring basis problem was to form the Bruhat ideal (w) for an element w of a simply laced Kac-Moody Weyl group that acted on a fundamental weight λ in a manner like acting on a minuscule weight of a simple Lie algebra. Peterson visited Kumar in Chapel Hill in October 1993; then we learned that he had already formalized this notion with his definition of "λ-minuscule" (without the simply laced assumption). Thus arose the problem of characterizing the colored posets P that arise as posets of join irreducible elements of such Bruhat ideals. Theorem B of [Pr5] stated that the colored d-complete posets were exactly these posets, in the simply laced cases. So by early 1997 it had become easy to see that when working in the context of Section 7 above, the $m = 1$ case of marble pouring describes a basis and $\mathfrak{b}^+$-actions for any Demazure module $V_{w\lambda}$ of the simply laced Kac-Moody algebra $\mathfrak{g}$ when w is λ-minuscule. It then could be seen that the 1993 marble pouring theorem had constructed a weight basis for the Demazure module $V_{mw\lambda}$ that is indexed by the m-multichains $e \leq u_1 \leq \ldots \leq u_m \leq w$ in the Bruhat ideal (w). Therefore the references to [Lit] and [La2] for Proposition II above could be replaced by a reference to [Pr3]. The spanning argument for the 1993 theorem could be viewed as giving effectively (albeit recursively) specified descriptions of the actions of the generators of $\mathfrak{b}^+$ on these modules in a uniform fashion that was independent of the "type" of the Dynkin diagram. At that time it appeared that this was the first such uniform-across-type description of non-trivial actions for semisimple or Kac-Moody algebras, and this author has not heard of any other such descriptions since then.

Peterson informed this author in October 1993 of his additive identity; he also noted that every minuscule poset has the jeu de taquin property. This jeu de taquin remark inspired this author to conduct computer calculations to find that small d-complete posets also possessed the jeu de taquin property. This raised the question of which other combinatorial properties possessed by shapes, shifted shapes, and rooted trees might be possessed by d-complete posets; calculations then showed that small d-complete posets were hook length posets. The name "d-complete" was chosen for these posets to indicate the foremost requirement that their local structures had to satisfy: Any interval that was nearly isomorphic to the minuscule poset $d_n(1)$ had to be "completable" to an interval isomorphic to all of $d_n(1)$. In June 1994 this author conjectured that every d-complete poset is a hook length poset; this was announced in November 1994 at the Richmond AMS meeting [Pr4], along with the conjecture that every d-complete poset has the jeu de taquin property. By December 1996 this author had conjectured Theorem 1's recursive hook length rule: It appeared in David Behrman's Master's project [Beh], which extended the original calculations and checked the recursive rule. (These computations were later confirmed and then published by Cheryl Gann and this author on the internet [GP].) Michael Kart observed that the statement of the hook recursion could be simplified. By October 1995 this author had classified the d-complete posets [Pr6], and by October 1996 he had proved [Thm. B, Pr5] that the colored d-complete posets were exactly the posets of join irreducibles of the Bruhat lattices (w) for the λ-minuscule w in the simply laced case.

By 1996 this author was aware of [La2] and suspected that there was a representation-theoretic proof of the hook product conjecture that would refer to that paper, and that perhaps a finer colored version of the identity would result. In March 1997 Peterson visited Kumar in Chapel Hill again. He reminded this author of his additive identity and additionally stated his product identity. Once Peterson learned of the hook length conjecture, within three days he produced Proposition III and indicated how to connect that statement to Proposition I via Proposition II. At the end of 1997 this author obtained Proposition IV, thereby finishing the proof of the colored Theorem 2.

Theorem 1 and Corollary 1 above and the confirmation of the jeu de taquin conjecture were announced at the October 1998 RIMS conference on the interactions of combinatorics and representation theory. At that time these results were described on this author's website [Pr7], which alluded to the existence of Theorem 2. A proof of the jeu de taquin property for d-complete posets was distributed in 2000 [Pr8].

The definition of uncolored d-complete poset has evolved since 1994. Theorem 2 was originally proved with respect to the definition of colored d-complete poset in [Pr5]; the definition of "colored d-complete poset" used here was inspired by Stembridge's Conditions H1-H4 of Proposition 3.1(c) of [Ste].

14. Subsequent Developments and Acknowledgments

Following the announcements of Theorems 1 and 2 and Corollary 1 in 1998, many intriguing related results have been obtained by the attendees of that conference and their coworkers. Conjectured and proven generalizations of those results, alternate general or class-by-class proofs of those results, plus some completely new results for d-complete posets have been developed by Noriaki Kawanaka, Shuji Okamura, Masao Ishikawa, Hiroyuki Tagawa, Bridget Tenner, Kento Nakada, and Soichi Okada.

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